



## ACTION-BASED DYNAMICAL MODELING FOR THE MILKY WAY DISK

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## ABSTRACT

We present *RoadMapping*, a full-likelihood dynamical modeling machinery that aims to recover the Milky Way’s (MW) gravitational potential from large samples of stars in the Galactic disk. *RoadMapping* models the observed positions and velocities of stars with a parameterized, three-integral distribution function (DF) in a parameterized axisymmetric potential. We investigate through differential test cases with idealized mock data how the breakdown of model assumptions and data properties affect constraints on the potential and DF. Our key results are: (i) If the MW’s true potential is not included in the assumed model potential family, we can—in the axisymmetric case—still find a robust estimate for the potential, with only  $\lesssim 10\%$  difference in surface density within  $|z| \leq 1.1$  kpc inside the observed volume. (ii) Modest systematic differences between the true and model DF are inconsequential. E.g., when binning stars to define sub-populations with simple DFs, binning errors do not affect the modeling as long as the DF parameters of neighboring bins differ by  $< 20\%$ . In addition, *RoadMapping* ensures unbiased potential estimates for either (iii) small misjudgements of the spatial selection function (i.e.,  $\lesssim 15\%$  at the survey volume’s edge), (iv) if distances are known to within  $10\%$ , or (v) if proper motion uncertainties are known within  $10\%$  or are smaller than  $\delta\mu \lesssim 1$  mas yr<sup>−1</sup>. Challenges are the rapidly increasing computational costs for large sample sizes. Overall, *RoadMapping* is well suited to making precise new measurements of the MW’s potential with data from the upcoming *Gaia* releases.

**Key words:** Galaxy: disk – Galaxy: fundamental parameters – Galaxy: kinematics and dynamics – Galaxy: structure

## 1. INTRODUCTION

Through dynamical modeling we can infer the Milky Way’s (MW) gravitational potential from stellar motions (Binney & Tremaine 2008; Binney 2011; Rix & Bovy 2013). Observational information on the 6D phase-space coordinates of stars is currently growing at a rapid pace, and will be taken to a whole new level in quantity and precision by the upcoming data from the *Gaia* mission (Perryman et al. 2001). Yet, rigorous and practical modeling tools that turn position–velocity data of individual stars into constraints both on the gravitational potential and on the distribution function (DF) of stellar orbits are scarce (Rix & Bovy 2013).

The Galactic gravitational potential is fundamental for understanding the MW’s dark matter and baryonic structure (Famaey 2012; Rix & Bovy 2013; Strigari 2013; Read 2014) and the stellar-population-dependent orbit DF is a basic constraint on the Galaxy’s formation history (Binney 2013; Sanders & Binney 2015).

There are a variety of practical approaches to the dynamical modeling of discrete collisionless tracers, such as the stars in the MW, e.g., Jeans modeling (Kuijken & Gilmore 1989; Bovy & Tremaine 2012; Garbari et al. 2012; Zhang et al. 2013; Büdenbender et al. 2015), action-based DF modeling (with parametric DFs: Bovy & Rix 2013; Piffl et al. 2014; Sanders & Binney 2015; Das & Binney 2016; with marginalization over non-parametric DFs: Magorrian 2014), torus modeling (McMillan & Binney 2008, 2012, 2013), or made-to-measure modeling (Syer & Tremaine 1996; de Lorenzi et al. 2007; Hunt & Kawata 2014). Most of them—explicitly or implicitly—describe the stellar distribution through a DF. Not all of them avoid binning to exploit the full discrete information content of the data.

Recently, Binney (2012b) and Bovy & Rix (2013) proposed constraining the MW’s gravitational potential by combining parameterized axisymmetric potential models with DFs that are simple analytic functions of the three orbital actions to model discrete data.

Bovy & Rix (2013) (BR13 hereafter) put this in practice by implementing a rigorous modeling approach for so-called mono-abundance populations (MAPs), i.e., sub-sets of stars with similar [Fe/H] and [ $\alpha$ /Fe] within the Galactic disk, which seem to follow simple DFs (Bovy et al. 2012a, 2012b, 2012c). Given an assumed (axisymmetric) model for the Galactic potential and action-based DF (Binney 2010; Binney & McMillan 2011; Ting et al. 2013) they calculated the likelihood of the observed  $(\mathbf{x}, \mathbf{v})$  for each MAP, using SEGUE G-dwarf stars (Yanny et al. 2009). They also accounted for the complex but known selection function of the kinematic tracers (Bovy et al. 2012c). For each MAP the modeling resulted in an independent estimate of the same gravitational potential. Taken as an ensemble, they constrained the disk surface density over a wide range of radii ( $\sim 4$ – $9$  kpc), and powerfully constrained the disk mass scale length and the stellar-disk-to-dark-matter ratio at the Solar radius.

BR13 made, however, a number of quite severe and idealizing assumptions about the potential, the DF, and the knowledge of observational effects. These idealizations could plausibly translate into systematic errors on the inferred potential, well above the formal error bars of the upcoming surveys with their wealth and quality of data.

In this work we present *RoadMapping* (“Recovery of the Orbit Action Distribution of Mono-Abundance Populations and Potential Inference for our Galaxy”), an improved, refined, flexible, robust, and well-tested version of the original dynamical modeling machinery by BR13. Our goal is to

explore which of the assumptions [BR13](#) made and which other aspects of data, model, and machinery limit *RoadMapping*'s recovery of the true gravitational potential.

We investigate the following aspects of the *RoadMapping* machinery that become especially important for a large number of stars. (i) Numerical inaccuracies must not be an important source of systematics (Section 2.6). (ii) As parameter estimates become much more precise, we need more flexibility in the potential and DF model, and efficient strategies to find the best fit parameters. The improvements made in *RoadMapping* as compared to the machinery used in [BR13](#) are presented in Section 2.8. (iii) *RoadMapping* should be an unbiased estimator (Section 3.1).

We also explore how different aspects of the observational experiment design impact the parameter recovery: (i) We consider the importance of the survey volume geometry, size, shape, and position within the MW to constrain the potential (Section 3.2). (ii) We ask what happens if our knowledge of the sample selection function is imperfect, and potentially biased (Section 3.3). (iii) We investigate how to best account for individual, and possibly misjudged, measurement uncertainties (Section 3.4). (iv) We determine which of several stellar sub-populations is best for constraining the potential (Section 3.7).

One of the strongest assumptions is restricting the dynamical modeling to a certain family of parameterized functions for the gravitational potential and the DF. We investigate how well we can hope to recover the true potential, when our models do not encompass the true DF (Section 3.5) and potential (Section 3.6).

For all of the above aspects we show some plausible and illustrative examples on the basis of investigating mock data. The mock data is generated from galaxy models outlined in Sections 2.1–2.4 following the procedure in Appendices A–B and analyzed according to the description of the *RoadMapping* machinery in Sections 2.5–2.8. Section 3 compiles our results on the investigated modeling aspects. In particular, our key results about the systematics introduced by using wrong DF or potential models are presented in the Sections 3.5 and 3.6. Section 4 finally summarizes and discusses our findings.

## 2. DYNAMICAL MODELING

In this section we summarize the basic elements of *RoadMapping*, the dynamical modeling machinery presented in this work, which in many respects follows [BR13](#) and makes extensive use of the *galpy* Python package for galactic dynamics<sup>3</sup> ([Bovy 2015](#)).

### 2.1. Coordinate System

Our modeling takes place in the Galactocentric rest-frame with cylindrical coordinates  $\mathbf{x} \equiv (R, \phi, z)$  and corresponding velocity components  $\mathbf{v} \equiv (v_R, v_T, v_z)$ . If the stellar phase-space data is given in observed heliocentric coordinates, position  $\tilde{\mathbf{x}} \equiv (\text{R.A.}, \text{decl.}, m - M)$  in right ascension R.A., declination decl., and distance modulus  $(m - M)$ , and velocity  $\tilde{\mathbf{v}} \equiv (\mu_{\text{R.A.}} \cdot \cos(\text{decl.}), \mu_{\text{decl.}}, v_{\text{los}})$  as proper motions and line-of-sight velocity, the data  $(\tilde{\mathbf{x}}, \tilde{\mathbf{v}})$  has to be converted into the Galactocentric rest-frame coordinates  $(\mathbf{x}, \mathbf{v})$  using the Sun's

position and velocity. We assume for the Sun

$$(R_\odot, \phi_\odot, z_\odot) = (8 \text{ kpc}, 0^\circ, 0 \text{ kpc})$$

$$(v_{R_\odot}, v_{T_\odot}, v_{z_\odot}) = (0, 230, 0) \text{ km s}^{-1}.$$

### 2.2. Actions

Stellar orbits in (axisymmetric) gravitational potentials are best described and fully specified by the three actions  $\mathbf{J} \equiv (J_R, J_\phi = L_z, J_z)$ , defined as

$$J_i \equiv \frac{1}{2\pi} \oint_{\text{orbit}} p_i dx_i, \quad (1)$$

which is evaluated along the orbit with position,  $\mathbf{x}(t)$ , and momentum,  $\mathbf{p}(t)$ , in a given potential,  $\Phi$ . Actions have several convenient properties which make them excellent orbit labels and ideal as arguments for orbit DFs: actions are integrals of motion; actions have an intuitive physical meaning as they quantify the amount of oscillation of the orbit in each coordinate direction; actions—together with a set of angle coordinates  $\boldsymbol{\theta}$ —form canonical conjugate phase-space coordinates, i.e., the Jacobian determinant  $|\partial(\mathbf{J}, \boldsymbol{\theta})/\partial(\mathbf{x}, \mathbf{v})| = 1$  (with Cartesian  $\mathbf{x}$  and  $\mathbf{v}$ ). The angles  $\boldsymbol{\theta}(t) \propto t$  evolve linearly in time and specify the position of the star along the orbit. (For a full introduction to angle-action variables see [Binney & Tremaine 2008](#), Section 3.5.)

Action calculation from a star's phase-space coordinates,  $(\mathbf{x}, \mathbf{v}) \xrightarrow{\Phi} \mathbf{J}$ , is typically very computationally expensive. Only for some special, separable potentials does Equation (1) simplify significantly. The triaxial Stäckel potentials ([de Zeeuw 1985](#)) are the most general potentials, that allow exact action calculations using a single quadrature. Some flattened axisymmetric Stäckel potentials are quite similar to our Galaxy's potential ([Batsleer & Dejonghe 1994](#); [Famaey & Dejonghe 2003](#); [Binney & Tremaine 2008](#), Section 3.5.3). The spherical isochrone potential ([Henon 1959](#); [Binney & Tremaine 2008](#), Section 3.5.3) is the most general special case for which the action calculation is analytic without any integration. In all other potentials, actions have to be numerically estimated; see [Sanders & Binney \(2016\)](#) for a recent review of action estimation methods. According to [Sanders & Binney \(2016\)](#) the best compromise of speed and accuracy for the Galactic disk is the *Stäckel fudge* by [Binney \(2012a\)](#) for axisymmetric potentials. In addition we use action interpolation grids ([Binney 2012a](#); [Bovy 2015](#)) to speed up the calculation. The latter is one of the improvements employed by *RoadMapping*, which was not used in [BR13](#).

### 2.3. Potential Models

For the gravitational potential in our modeling we assume a family of parameterized models. We use: a MW-like potential with disk, halo, and bulge (DHB-Pot); the spherical isochrone potential (ISO-Pot); and the 2-component Kuzmin-Kutuzov Stäckel potential ([Batsleer & Dejonghe 1994](#); KKS-Pot), which also displays a disk and halo structure. Table 1 summarizes all reference potentials used in this work together with their free parameters,  $p_\Phi$ . The true circular velocity at the Sun was chosen to be  $v_{\text{circ}}(R_\odot) = 230 \text{ km s}^{-1}$  for all potential models. The ISO-Pot allows both accurate and particularly fast action

<sup>3</sup> *galpy* is an open-source code that is being developed on <http://github.com/jobovy/galpy>. The latest documentation can be found at <http://galpy.readthedocs.org/en/latest/>.

**Table 1**  
Axisymmetric Gravitational Potential Models used Throughout this Work

| Name     | Potential Model  | Parameters $p_\Phi$   | Action Calculation   |
|----------|--|---|--|
| Iso-Pot  | isochrone potential <sup>a</sup><br>(Henon 1959)   | $b$ 0.9 kpc   | <i>analytic and exact</i><br>(Binney & Tremaine 2008, Section 3.5.2)                                   |
| KKS-Pot  | 2-component<br>Kuzmin-Kutuzov-<br>Stäckel potential <sup>b</sup> :   | $\Delta$ 0.3<br>$\left(\frac{a}{c}\right)_{\text{Disk}}$ 20<br>$\left(\frac{a}{c}\right)_{\text{Halo}}$ 1.07<br>$k$ 0.28  | <i>exact</i><br>using interpolation on action grid<br><br>(Binney 2012a; Bovy 2015)                    |
| DHB-Pot  | Disk+Halo+Bulge potential <sup>c</sup> :<br>Miyamoto-Nagai disk,<br>NFW halo,<br>Hernquist bulge<br>(same as MW14-Pot,<br>except of bulge) | $a_{\text{disk}}$ 3 kpc<br>$b_{\text{disk}}$ 0.28 kpc (fixed)<br>$f_{\text{halo}}$ 0.35/0.95<br>$a_{\text{halo}}$ 16 kpc (fixed)<br>$f_{\text{bulge}}$ 0.05/1.0 (fixed)<br>$a_{\text{bulge}}$ 0.6 kpc (fixed) | <i>approximate</i><br>using <i>Stäckel fudge</i><br>(Binney 2012a)<br>and interpolation on action grid |
| MW14-Pot | MW-like potential <sup>d</sup> :<br>Miyamoto-Nagai disk,<br>NFW halo,<br>cut-off power-law bulge<br>(Bovy 2015)                            |   | <i>approximate</i><br>using <i>Stäckel fudge</i>   |

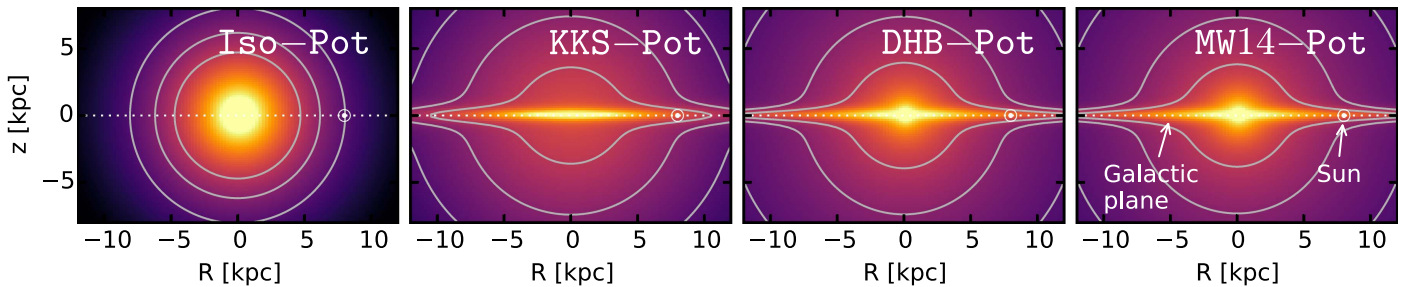
**Notes.** The potential parameters are fixed for the mock data creation at the values given in this table, which we subsequently aim to recover with *RoadMapping*. The parameters of DHB-Pot and KKS-Pot were chosen to resemble the MW14-Pot (see Figure 1). We use  $v_{\text{circ}}(R_\odot) = 230 \text{ km s}^{-1}$  as the circular velocity at the Sun for all potentials in this work.

<sup>a</sup> The free parameter of the spherical Iso-Pot is the isochrone scale length  $b$ .

<sup>b</sup> The coordinate system of each of the two Stäckel-potential components of the KKS-Pot is  $R^2/(\tau_{i,p} + \alpha_p) + z^2/(\tau_{i,p} + \gamma_p) = 1$  with  $p \in \{\text{Disk, Halo}\}$  and  $\tau_{i,p} \in \{\lambda_p, \nu_p\}$ . Both components have the same focal distance  $\Delta \equiv \sqrt{\gamma_p - \alpha_p}$ , to ensure that the superposition itself is a Stäckel potential. The axis ratio of the coordinate surfaces,  $(a/c)_p \equiv \sqrt{\alpha_p/\gamma_p}$ , describes the flatness of each component.  $k$  is the relative contribution of the disk mass to the total mass.

<sup>c</sup> The parameters of the DHB-Pot are the Miyamoto-Nagai disk scale length  $a_{\text{disk}}$  and height  $b_{\text{disk}}$ , the NFW halo scale length  $a_{\text{halo}}$  and its relative contribution to  $v_{\text{circ}}^2(R_\odot)$  with respect to the total disk+halo contribution,  $f_{\text{halo}}$ , and the Hernquist bulge scale length,  $a_{\text{bulge}}$ , and its contribution to the total  $v_{\text{circ}}^2(R_\odot)$ ,  $f_{\text{bulge}}$ . We keep all except  $v_{\text{circ}}(R_\odot)$ ,  $a_{\text{disk}}$  and  $f_{\text{halo}}$  fixed to their true values in the analysis.

<sup>d</sup> The MWPotential2014 by Bovy (2015) (see their Table 1) has  $v_{\text{circ}}(R_\odot) = 220 \text{ km s}^{-1}$ . We use, however,  $v_{\text{circ}}(R_\odot) = 230 \text{ km s}^{-1}$ .



**Figure 1.** Density distribution of the four reference galaxy potentials in Table 1. These potentials are used throughout this work to create and model mock data with *RoadMapping*.

calculations; we therefore use it for tests requiring a large number of analyses. The KKS-Pot and DHB-Pot were chosen for their more realistic galaxy shape and because their closed-form expression for  $\Phi(R, z)$  makes the computation of forces and densities fast and easy. The KKS-Pot also allows for exact action calculations, while the DHB-Pot has physically more intuitive potential parameters but requires the *Stäckel fudge* to estimate actions. The parameter values of KKS-Pot and DHB-Pot in Table 1 were chosen to resemble

the MW potential from Bovy (2015) (MW14-Pot). The density distribution of all these potentials is illustrated in Figure 1.

#### 2.4. Stellar Distribution Functions

A stellar distribution function  $DF(\mathbf{x}, \mathbf{v})$  can be considered as the probability of a star to be found at  $(\mathbf{x}, \mathbf{v})$ . Using orbit DFs in terms of  $(\mathbf{J}, \boldsymbol{\theta})$  instead has the advantage that the distribution of stars in  $\boldsymbol{\theta}$  is uniform and the orbit DF reduces

**Table 2**  
Reference Parameters for the qDF in Equations (2)–(7)

| Name   | qDF Parameters $p_{\text{DF}}$ |                                      |                                      |                      |                      |
|--------|--------------------------------|--------------------------------------|--------------------------------------|----------------------|----------------------|
|        | $h_R$ (kpc)                    | $\sigma_{R,0}$ (km s <sup>-1</sup> ) | $\sigma_{z,0}$ (km s <sup>-1</sup> ) | $h_{\sigma,R}$ (kpc) | $h_{\sigma,z}$ (kpc) |
| hot    | 2                              | 55                                   | 66                                   | 8                    | 7                    |
| cool   | 3.5                            | 42                                   | 32                                   | 8                    | 7                    |
| cooler | 3                              | 27.5                                 | 33                                   | 8                    | 7                    |
| colder | 2 + $X\%$                      | 55 - $X\%$                           | 66 - $X\%$                           | 8                    | 7                    |
| warmer | 3.5 - $X\%$                    | 42 + $X\%$                           | 32 + $X\%$                           | 8                    | 7                    |

**Note.** These parameters are used to create 6D phase-space mock data sets for stellar populations of different kinematic temperature. The parameter  $X$  is explained in Section 2.4.

effectively to a function of the actions  $\mathbf{J}$  only. As  $|\partial(\mathbf{J}, \boldsymbol{\theta})/\partial(\mathbf{x}, \mathbf{v})| = 1$ , the function  $\text{DF}(\mathbf{J})$  can still be thought of as a probability in  $(\mathbf{x}, \mathbf{v})$ .

The action-based quasi-isothermal distribution function (qDF) by Binney (2010) and Binney & McMillan (2011) is a simple DF which we will employ as a specific example throughout this work to describe individual stellar subpopulations. This is motivated by the findings of Bovy et al. (2012a, 2012b, 2012c) and Ting et al. (2013) on the simple phase-space structure of stellar MAPs and BR13’s successful application. The qDF has the form

$$\begin{aligned} \text{qDF}(\mathbf{J}|p_{\text{DF}}) \\ = f_{\sigma_R}(J_R, L_z|p_{\text{DF}}) \times f_{\sigma_z}(J_z, L_z|p_{\text{DF}}) \end{aligned} \quad (2)$$

with some free parameters,  $p_{\text{DF}}$ , and

$$\begin{aligned} f_{\sigma_R}(J_R, L_z|p_{\text{DF}}) = n \times \frac{\Omega}{\pi\sigma_R^2(R_g)\kappa} \exp\left(-\frac{\kappa J_R}{\sigma_R^2(R_g)}\right) \\ \times [1 + \tanh(L_z/L_0)] \end{aligned} \quad (3)$$

$$f_{\sigma_z}(J_z, L_z|p_{\text{DF}}) = \frac{\nu}{2\pi\sigma_z^2(R_g)} \exp\left(-\frac{\nu J_z}{\sigma_z^2(R_g)}\right) \quad (4)$$

(Binney & McMillan 2011). Here  $R_g$ ,  $\Omega$ ,  $\kappa$  and  $\nu$  are functions of  $L_z$ , and denote, respectively, the guiding-center radius, circular, radial/epicycle, and vertical frequency of the near-circular orbit with angular momentum  $L_z$  in a given potential. The term  $[1 + \tanh(L_z/L_0)]$  suppresses counter-rotation for orbits in the disk with  $L_z < L_0$  (with  $L_0 \sim 10 \text{ km s}^{-1} \text{ kpc}$ ).

Following BR13, we choose the functional forms

$$n(R_g|p_{\text{DF}}) \propto \exp\left(-\frac{R_g}{h_R}\right) \quad (5)$$

$$\sigma_R(R_g|p_{\text{DF}}) = \sigma_{R,0} \times \exp\left(-\frac{R_g - R_\odot}{h_{\sigma,R}}\right) \quad (6)$$

$$\sigma_z(R_g|p_{\text{DF}}) = \sigma_{z,0} \times \exp\left(-\frac{R_g - R_\odot}{h_{\sigma,z}}\right), \quad (7)$$

which indirectly set the stellar number density, and radial and vertical velocity dispersion profiles. The qDF has therefore a set of five free parameters  $p_{\text{DF}}$ : the density scale length of the tracers  $h_R$ , the radial and vertical velocity dispersion at the Solar position  $R_\odot$ ,  $\sigma_{R,0}$  and  $\sigma_{z,0}$ , and the scale lengths  $h_{\sigma,R}$  and

$h_{\sigma,z}$ , that describe the radial decrease of the velocity dispersion. *RoadMapping* allows to fit any number of DF parameters simultaneously, while BR13 kept  $\{\sigma_{R,0}, h_{\sigma,R}\}$  fixed. Throughout this work we make use of a few example stellar populations whose qDF parameters are given in Table 2; most tests use the `hot` and `cool` qDFs, which correspond to kinematically hot and cool populations, respectively. The `warmer` (`cooler` and `colder`) qDFs in Table 2 were chosen to have the same anisotropy  $\sigma_{R,0}/\sigma_{z,0}$  as the `cool` (`hot`) qDF, with  $X$  being a free parameter describing the temperature difference. Hotter populations have shorter tracer scale lengths (Bovy et al. 2012c) and the velocity dispersion scale lengths were fixed according to Bovy et al. (2012b).

One indispensable step in our dynamical modeling technique (Sections 2.5–2.6), as well as in creating mock data (Appendix A), is to calculate the (axisymmetric) spatial tracer density  $\rho_{\text{DF}}(\mathbf{x}|p_\Phi, p_{\text{DF}})$  for a given DF and potential. Analogously to BR13,

$$\begin{aligned} \rho_{\text{DF}}(R, |z| |p_\Phi, p_{\text{DF}}) \\ = \int_{-\infty}^{\infty} \text{DF}(\mathbf{J}[R, z, \mathbf{v}|p_\Phi] |p_{\text{DF}}) d^3\mathbf{v} \\ \approx \int_{-n_\sigma\sigma_R(R|p_{\text{DF}})}^{n_\sigma\sigma_R(R|p_{\text{DF}})} \int_{-n_\sigma\sigma_z(R|p_{\text{DF}})}^{n_\sigma\sigma_z(R|p_{\text{DF}})} \int_0^{1.5v_{\text{circ}}(R_\odot)} \\ \text{DF}(\mathbf{J}[R, z, \mathbf{v}|p_\Phi] |p_{\text{DF}}) dv_T dv_z dv_R, \end{aligned} \quad (8)$$

where  $\sigma_R(R|p_{\text{DF}})$  and  $\sigma_z(R|p_{\text{DF}})$  are given by Equations (6) and (7).<sup>4</sup> Each integral is evaluated using a  $N_v$ th order Gauss–Legendre quadrature. For a given  $p_\Phi$  and  $p_{\text{DF}}$  we explicitly calculate the density on  $N_x \times N_x$  regular grid points in the  $(R, z)$  plane and interpolate  $\ln \rho_{\text{DF}}$  in between using bivariate spline interpolation. The grid is chosen to cover the extent of the observations (for  $|z| \geq 0$ , because the model is symmetric in  $z$  by construction). The total number of actions to be calculated to set up the density interpolation grid is  $N_x^2 \times N_v^3$ , which is one of the factors limiting the computation speed. To complement the work by BR13, we will specifically work out in Section 2.6 and Figure 2 how large  $N_x$ ,  $N_v$  and  $n_\sigma$  have to be chosen to get the density with a sufficiently high numerical accuracy.

## 2.5. Data Likelihood

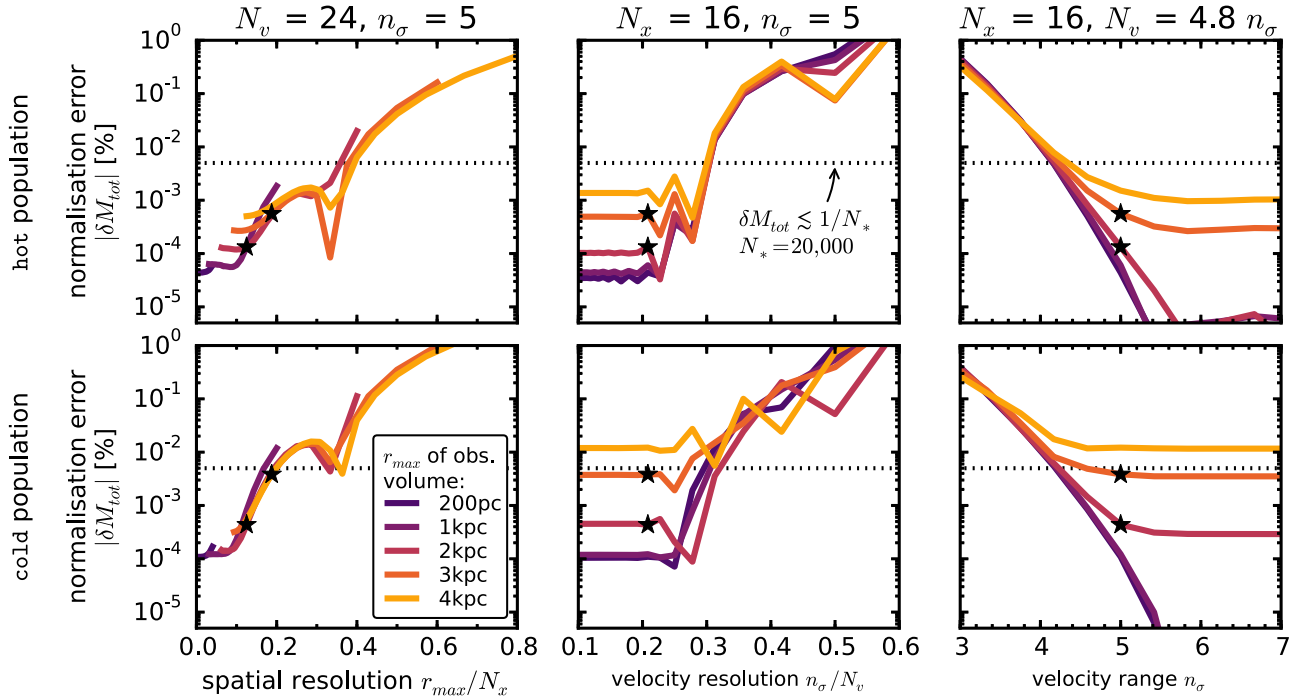
As data,  $D$ , we consider here the positions and velocities of a sub-population of stars within a given survey selection function,  $\text{SF}(\mathbf{x})$ ,

$$\begin{aligned} D \equiv \{\mathbf{x}_i, \mathbf{v}_i | \text{ (star } i \text{ in given sub-population)} \\ \wedge (\text{SF}(\mathbf{x}_i) > 0)\}. \end{aligned}$$

For simplicity, we assume in most tests of this study contiguous, spherical SFs centered on the Sun, which are functions of  $\mathbf{x}$  only and which we motivate in Appendix B. The maximum radius of this spherical observed volume is denoted by  $r_{\text{max}}$ .

<sup>4</sup> The integration ranges over the velocities are motivated by Figure 19, and  $n_\sigma$  should be chosen as  $n_\sigma \sim 5$  (see Figure 2). The integration range  $[0, 1.5v_{\text{circ}}(R_\odot)]$  over  $v_T$  is, in general, sufficient, only for observation volumes with larger mean stellar  $v_T$  does this upper limit need to be increased.





**Figure 2.** Relative error of the likelihood normalization,  $\delta M_{\text{tot}}$ , in Equation (12), depending on the accuracy of the grid-based density calculation in Equation (8) (and surrounding text) in five spherical observation volumes with different radius,  $r_{\text{max}}$ , and depending on the kinematic temperature of the population in the DHB-POT. (Test 1 in Table C1 summarizes the model parameters.) The tracer density in Equation (8) is calculated on  $N_x \times N_x$  spatial grid points in  $R \in [R_\odot \pm r_{\text{max}}]$  and  $|z| \in [0, r_{\text{max}}]$ . The integration over the velocities is performed with Gauss–Legendre quadratures of order  $N_v$  within an integration range of  $\pm n_\sigma$  times the dispersion  $\sigma_R(R)$  and  $\sigma_z(R)$  (and  $[0, 1.5v_{\text{circ}}]$  in  $v_T$ ). (We vary  $N_x$ ,  $N_v$  and  $n_\sigma$  separately, and keep the other two fixed at the values indicated above each panel.) We calculate the “true” normalization  $M_{\text{tot}}$  in Equation (12) with high accuracy as  $M_{\text{tot}} \equiv M_{\text{tot,approx}}(N_x = 32, N_v = 68, n_\sigma = 7)$ . The black stars indicate the accuracy used in analyses with the DHB-POT, Tests 5 and 7. It is better than 0.005% (dotted line), which is required for  $N_* = 20,000$  stars. We find that the spatial resolution of the grid is important and depends on the kinematic temperature of the population, as cooler populations have a steeper density gradient in the  $z$ -direction, which has to be sampled sufficiently.

We fit a model potential and DF (here, the qDF) which are specified by a number of fixed and free model parameters,

$$p_M \equiv \{p_{\text{DF}}, p_\Phi\}.$$

The orbit of the  $i$ th star in a potential with  $p_\Phi$  is labeled by the actions  $\mathbf{J}_i \equiv \mathbf{J}[\mathbf{x}_i, \mathbf{v}_i | p_\Phi]$  and the DF evaluated for the  $i$ th star is then  $\text{DF}(\mathbf{J}_i | p_M) \equiv \text{DF}(\mathbf{J}[\mathbf{x}_i, \mathbf{v}_i | p_\Phi] | p_{\text{DF}})$ .

The likelihood of the data given the model is, following BR13 and McMillan & Binney (2013),

$$\begin{aligned} \mathcal{L}(D | p_M) &= \prod_{i=1}^{N_*} p(\mathbf{x}_i, \mathbf{v}_i | p_M) \\ &= \prod_{i=1}^{N_*} \frac{\text{DF}(\mathbf{J}_i | p_M) \cdot \text{SF}(\mathbf{x}_i)}{\int \text{DF}(\mathbf{J} | p_M) \cdot \text{SF}(\mathbf{x}) d^3x d^3v} \\ &\propto \prod_{i=1}^{N_*} \frac{\text{DF}(\mathbf{J}_i | p_M)}{\int \rho_{\text{DF}}(R, |z| | p_M) \cdot \text{SF}(\mathbf{x}) d^3x}, \end{aligned} \quad (9)$$

where  $N_*$  is the number of stars in  $D$ , and in the last step we used Equation (8).<sup>5</sup>  $\prod_i^{N_*} \text{SF}(\mathbf{x}_i)$  is independent of  $p_M$ , so we treat it as unimportant proportionality factor. We find the best fitting  $p_M$  by maximizing the posterior probability

distribution,  $\text{pdf}(p_M | D)$ , which is, according to Bayes’ theorem

$$\text{pdf}(p_M | D) \propto \mathcal{L}(D | p_M) \cdot p(p_M),$$

where  $p(p_M)$  is some prior probability distribution on the model parameters. We assume flat priors in both  $p_\Phi$  and

$$p_{\text{DF}} := \{\ln h_R, \ln \sigma_{R,0}, \ln \sigma_{z,0}, \ln h_{\sigma,R}, \ln h_{\sigma,z}\} \quad (10)$$

(see Section 2.4) throughout this work. Then  $\text{pdf}$  and likelihood are proportional to each other and differ only in units.

In this case, where we use uninformative priors, a maximum likelihood estimation procedure (e.g., via the expectation-maximization algorithm and parameter uncertainty estimates from the Fisher information matrix) would lead to the same result as the Bayesian inferential procedure described in this work (see Section 2.8). We expect, however, that in due course increasingly informative priors will become available (like, e.g., rotation curve measurements from maser sources by Reid et al. 2009) and Bayesian inference is therefore the preferred framework.

## 2.6. Likelihood Normalization

The normalization in Equation (9) is a measure for the total number of tracers inside the survey volume,

$$M_{\text{tot}} \equiv \int \rho_{\text{DF}}(R, |z| | p_M) \cdot \text{SF}(\mathbf{x}) d^3x. \quad (11)$$

<sup>5</sup> Because  $|\partial(\mathbf{J}, \boldsymbol{\theta})/\partial(\mathbf{x}, \mathbf{v})| = 1$ , the integration over phase-space in the normalization term can be performed either over  $(\mathbf{J}, \boldsymbol{\theta})$  or Cartesian  $(\mathbf{x}, \mathbf{v})$ .

In the case of an axisymmetric Galaxy model and  $SF(\mathbf{x}) = 1$  within the observation volume (as in most tests in this work), the normalization is essentially a two-dimensional integral in the  $(R, z)$  plane over  $\rho_{DF}$  with finite integration limits. We evaluate the integrals using Gauss–Legendre quadratures of order 40. The integral over the azimuthal direction can be solved analytically.

It turns out that a sufficiently accurate evaluation of the likelihood is computationally expensive, even for only one set of model parameters. This expense is dominated by the number of action calculations required, which in turn depends on  $N_*$  and the numerical accuracy of the tracer density interpolation grid with  $N_x^2 \times N_v^3$  grid points in Equation (8) needed for the likelihood normalization in Equation (11). The accuracy of the normalization has to be chosen high enough, such that the resulting numerical error

$$\delta M_{\text{tot}} \equiv \frac{M_{\text{tot,approx}}(N_x, N_v, n_\sigma) - M_{\text{tot}}}{M_{\text{tot}}} \quad (12)$$

does not dominate the numerically calculated log-likelihood, i.e.,

$$\begin{aligned} \ln \mathcal{L}_{\text{approx}}(D|p_M) &= \sum_i^{N_*} \ln DF(J_i|p_M) - N_* \ln(M_{\text{tot}}) \\ &\quad - N_* \ln(1 + \delta M_{\text{tot}}), \end{aligned} \quad (13)$$

with

$$\ln(1 + \delta M_{\text{tot}}) \leq \frac{1}{N_*}, \quad (14)$$

and therefore  $\delta M_{\text{tot}} \lesssim 1/N_*$ . Otherwise, numerical inaccuracies could lead to systematic biases in the potential and DF recovery. For data sets as large as  $N_* = 20,000$  stars, which in the age of *Gaia* could very well be the case, one needs a numerical accuracy of 0.005% in the normalization. We made sure that this is satisfied for all analyses in this work. Figure 2 demonstrates how the numerical accuracy for analyses with the DHB-POT depends on the spatial and velocity resolution of the grid and that the accuracy we use,  $N_x = 16$ ,  $N_v = 24$  and  $n_\sigma = 5$ , is sufficient.<sup>6</sup> It has to be noted however, that the optimal values for  $N_x$ ,  $N_v$ , and  $n_\sigma$  depend not only on  $N_*$ , but also on the kinematic temperature of the population (and to a certain degree even on the choice of potential<sup>7</sup>) and it has to be checked on a case-by-case basis what the optimal accuracy is.

McMillan & Binney (2013), who use a similar modeling approach and likelihood normalization, argued that the required accuracy for the normalization scales as  $\log_{10}(1 + \delta M_{\text{tot}}) \leq 1/N_* \Rightarrow \delta M_{\text{tot}} \lesssim 2.3/N_*$ , which is satisfied for our tests as well. They evaluate the integrals in the normalization via Monte Carlo integration with  $\sim 10^9$  sample points in action space. Our

approach uses a tracer density interpolation grid for which the resolution needs to be optimized by hand, but it has the advantage that it then only requires the calculation of  $N_x^2 \times N_v^3 \sim 4 \cdot 10^6 - 10^7$  actions per normalization.

## 2.7. Measurement Uncertainties

Measurement uncertainties of the data have to be incorporated in the likelihood. We assume Gaussian uncertainties in the observable space  $\mathbf{y} \equiv (\tilde{\mathbf{x}}, \tilde{\mathbf{v}}) = (\text{R.A., decl., } (m - M), \mu_{\text{R.A.}}, \cos(\text{decl.}), \mu_{\text{decl.}}, v_{\text{los}})$ , i.e., the  $i$ th star's observed  $\mathbf{y}_i$  is drawn from the normal distribution  $N[\mathbf{y}'_i, \delta \mathbf{y}_i] \equiv \prod_k^6 N[\mathbf{y}_{i,k}', \delta \mathbf{y}_{i,k}] = \prod_k^6 \exp\{-(y_{i,k} - y_{i,k}')^2 / (2\delta y_{i,k}^2)\} / \sqrt{2\pi\delta y_{i,k}^2}$ , with  $\mathbf{y}'_i$  being the star's true phase-space position,  $\delta \mathbf{y}_i$  its uncertainty, and  $y_k$  the  $k$ th coordinate component of  $\mathbf{y}$ . Stars follow the DF( $J[\mathbf{y}'|p_\Phi]$ ) ( $\equiv DF(\mathbf{y}')$  for short) convolved with the measurement uncertainties  $N[0, \delta \mathbf{y}_i]$ . The selection function  $SF(\mathbf{y})$  acts on the space of (uncertainty affected) observables. Then the probability of one star becomes

$$\begin{aligned} \tilde{p}(\mathbf{y}_i|p_\Phi, p_{DF}, \delta \mathbf{y}_i) &= \frac{SF(\mathbf{y}_i) \cdot \int DF(\mathbf{y}') \cdot N[\mathbf{y}_i, \delta \mathbf{y}_i] d^6 \mathbf{y}'}{\int \left( DF(\mathbf{y}') \cdot \int SF(\mathbf{y}) \cdot N[\mathbf{y}', \delta \mathbf{y}_i] d^6 \mathbf{y} \right) d^6 \mathbf{y}'} \end{aligned} \quad (15)$$

In the case of uncertainties in distance and/or (R.A., decl.) the evaluation of this is very expensive computationally, especially if the stars have heteroscedastic  $\delta \mathbf{y}_i$ , which is the case for realistic data sets, and the normalization needs to be calculated for each star separately. In practice, we compute the convolution using Monte Carlo (MC) integration with  $N_{\text{samples}}$  samples,

$$\begin{aligned} \tilde{p}_{\text{approx}}(\mathbf{y}_i|p_\Phi, p_{DF}, \delta \mathbf{y}_i) &\approx \frac{SF(\tilde{\mathbf{x}}_i)}{M_{\text{tot}}} \cdot \frac{1}{N_{\text{samples}}} \sum_n^{N_{\text{samples}}} DF(\tilde{\mathbf{x}}_i, \mathbf{v}[\mathbf{y}'_{i,n}]) \end{aligned} \quad (16)$$

with

$$\mathbf{y}'_{i,n} \sim N[\mathbf{y}_i, \delta \mathbf{y}_i].$$

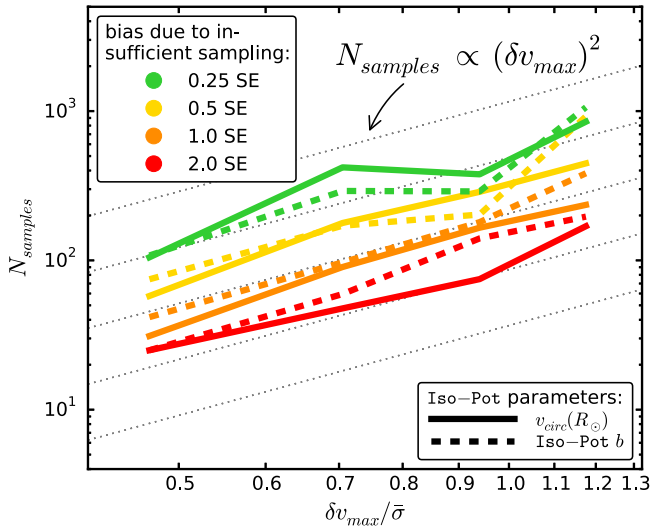
In addition, this approximation assumes that the star's position,  $\tilde{\mathbf{x}}_i$ , is perfectly measured. As the SF is also velocity independent, this simplifies the normalization drastically to  $M_{\text{tot}}$  in Equation (11). Measurement uncertainties in R. A. and decl. are often negligible anyway. The uncertainties in the Galactocentric velocities  $\mathbf{v}_i = (v_{R,i}, v_{T,i}, v_{z,i})$  depend not only on  $\delta \mu$  and  $\delta v_{\text{los}}$  but also on the distance and its uncertainty, which we do *not* neglect when drawing MC samples,  $\mathbf{y}'_{i,n}$ , from the full uncertainty distribution,  $N[\mathbf{y}_i, \delta \mathbf{y}_i]$ .

An analogous but one-dimensional treatment of measurement uncertainties in only  $v_z$  was already applied by BR13. Similar approaches ignoring measurement uncertainties in the likelihood normalization and using MC sampling of the error ellipses were also used by McMillan & Binney (2013) and Das & Binney (2016). In Section 3.4, Figure 9 (Test 6.2 in Table C1), we will investigate the breakdown of our approximation for non-negligible distance uncertainties.

Figure 3 demonstrates that in the absence of position uncertainties, the  $N$  needed for the convolution integral to

<sup>6</sup> The accuracy used in this work's analyses is slightly higher than in BR13, where  $N_*$  was only a few  $\sim 100$ .

<sup>7</sup> In Figure 17 we will show a comparison for the qDF parameters of two very similar mock data distributions in two different potentials, the MW14-POT and a best fit potential of the parametric form of the KKS-POT. As some of the qDF parameters in both potentials are very different, and even more different from the actual physical scale lengths and velocity dispersions, an optimal  $n_\sigma$  has to be estimated first for a given potential model before running the RoadMapping analysis.



**Figure 3.** Number of MC samples,  $N_{\text{samples}}$ , needed for the numerical convolution of the model probability with the measurement uncertainties in Equation (16), given the maximum velocity uncertainty,  $\delta v_{\text{max}}$ , within the stellar sample with respect to the sample’s kinematic temperature,  $\bar{\sigma}$ . Insufficient sampling introduces systematic biases in the parameter recovery; the size of the bias (in units of the standard error (SE) on the parameter estimate) is indicated in the legend. The relation found here,  $N_{\text{samples}} \propto \delta v_{\text{max}}^2$ , was distilled from analyses (with different  $N_{\text{samples}}$ ) of mock data sets with different proper motion uncertainties  $\delta\mu \in [2, 5]$  mas yr $^{-1}$  (see Test 2 in Table C1). As the reference for the converged convolution integral, we used  $N_{\text{samples}} = 800$  and  $1200$  for  $\delta\mu \leq 3$  mas yr $^{-1}$  and  $\delta\mu > 3$  mas yr $^{-1}$ , respectively (see also left panels in Figure 9). We plot  $\delta v_{\text{max}}$  in units of the sample temperature, which we quantify by  $\bar{\sigma} \equiv (\sigma_{R,0} + \sigma_{z,0})/2$  (see Table 2 for the hot qDF).

converge depends as

$$N_{\text{samples}} \propto (\delta v)^2$$

on the uncertainties in the (1D) velocities. Figure 3 is based on analyses of mock data sets with different proper motion uncertainties,  $\delta\mu$  (see Test 2 in Table C1 for all parameters). The proper motion uncertainty,  $\delta\mu$ , translates to heteroscedastic velocity uncertainties according to

$$\delta v [\text{km s}^{-1}] \equiv 4.74047 \cdot r [\text{kpc}] \cdot \delta\mu [\text{mas yr}^{-1}],$$

with  $r$  being the distance of the star to the Sun. Stars with larger  $\delta v$  require more  $N_{\text{samples}}$  for the integral over their measurement uncertainties to converge; Figure 3 therefore shows how the  $N_{\text{samples}}$ —needed for the *pdf* of the *whole* data set to be converged—depends on the *largest* velocity error,  $\delta v_{\text{max}} \equiv \delta v(r_{\text{max}})$ , within the data set.

These mock data sets each contained  $N_* = 10,000$  stars. We found that for  $N_* = 5000$  the required  $N_{\text{samples}}$  to reach a given accuracy becomes smaller for  $v_{\text{circ}}(R_{\odot})$ , but remains similar for  $b$ . The former is consistent with our expectation that we need higher accuracy and therefore more  $N_{\text{samples}}$  for larger data sets. The latter seems to be a special property of the Iso-Pot (see also the discussion in Section 3.3).

### 2.8. Fitting Procedure

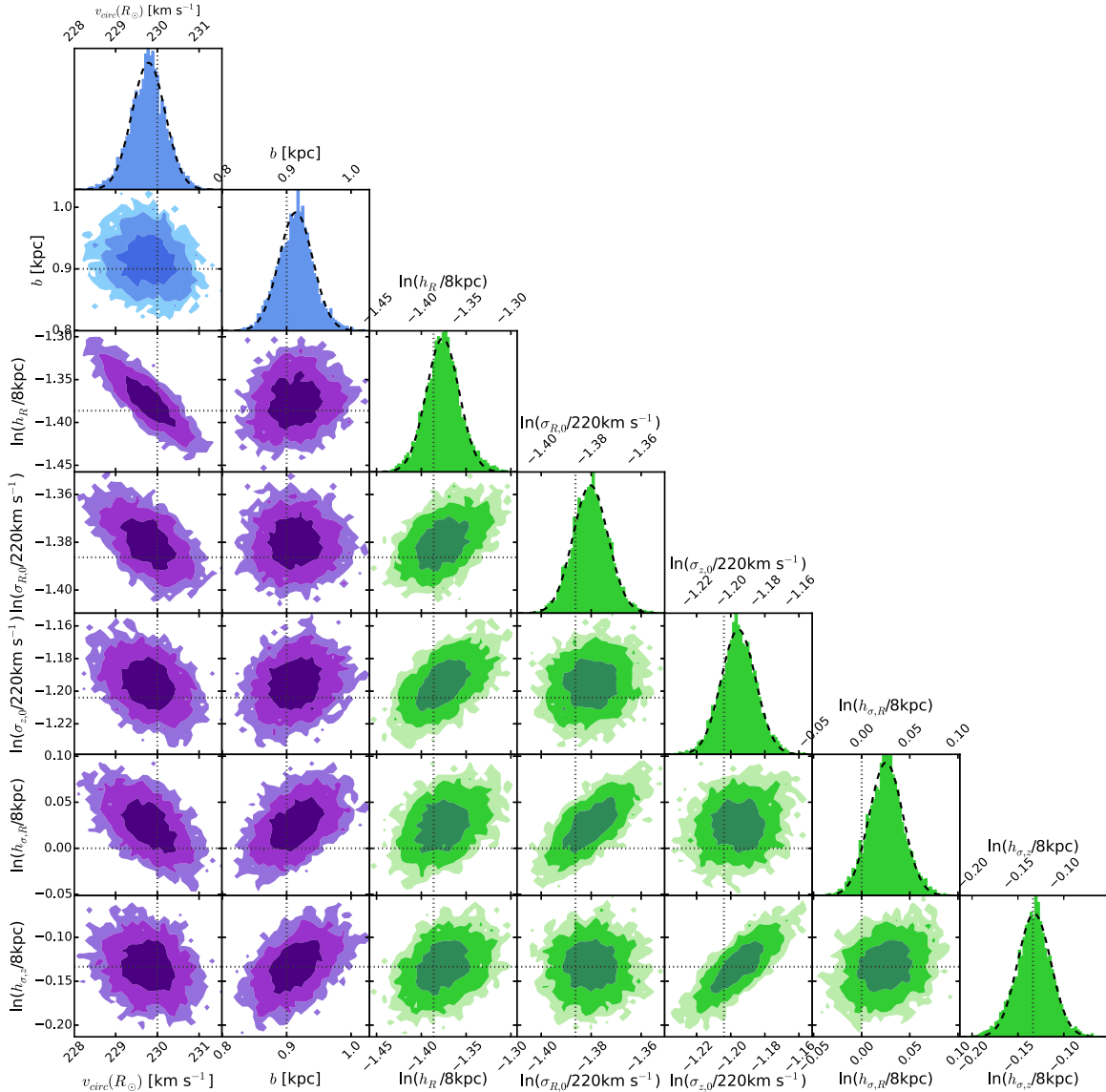
To search the  $(p_{\Phi}, p_{\text{DF}})$  parameter space for the maximum of the *pdf* in Equation (9), we go beyond the single fixed grid

search by BR13 and employ an efficient two-step procedure: Nested-grid search and Monte-Carlo Markov Chain (MCMC).

The first step employs a nested-grid search to find the approximate peak and width of the *pdf* in the high-dimensional  $p_M$  space with a low number of likelihood evaluations:

- *Initialization.* For  $N_p$  free model parameters,  $p_M$ , we start with a sufficiently large grid with  $3^{N_p}$  regular points.
- *Evaluation.* We evaluate the *pdf* at each grid-point similar to BR13 (their Figure 9). An outer loop iterates over the potential parameters,  $p_{\Phi}$ , and pre-calculates all  $N_* \times N_{\text{samples}} + N_x^2 \times N_v^3$  actions required for the likelihood calculation (see Equations (8), (9) and (16)). Then an inner loop evaluates Equation (9) (or (16)) for all DF parameters,  $p_{\text{DF}}$ , in the given potential.
- *Iteration.* For each of the model parameters,  $p_M$ , we marginalize the *pdf*. A Gaussian is fitted to the marginalized *pdf* and the peak  $\pm 4\sigma$  become the boundaries of the next grid with  $3^{N_p}$  grid points. The grid might still be too coarse or badly positioned to fit Gaussians. In that case we either zoom into the grid point with the highest probability or shift the current range to find new approximate grid boundaries. We proceed with iteratively evaluating the *pdf* on finer and finer grids, until we have found a reliable  $4\sigma$  fit range in each of the  $p_M$  dimensions. The central grid point is then very close to the best fit  $p_M$ , and the grid range is of the order of the *pdf* width.
- *The fiducial qDF.* To save time by pre-calculating actions, they have to be independent of the choice of  $p_{\text{DF}}$ . However, the normalization in Equation (11) requires actions on a  $N_x^2 \times N_v^3$  grid and the grid ranges in velocity space *do* depend on the current  $p_{\text{DF}}$  (see Equation (8)). To relax this, we follow BR13 and use a fixed set of qDF parameters (the *fiducial qDF*) to set the velocity grid boundaries in Equation (8) globally for a given  $p_{\Phi}$ . Choosing a fiducial qDF that is very different from the true DF can, however, lead to large biases in the  $p_M$  recovery. BR13 did not account for that. *RoadMapping* avoids this as follows: to successively get closer to the optimal fiducial qDF—with the (yet unknown) best fit  $p_{\text{DF}}$ —we use in each iteration step of the nested-grid search the central grid point of the current  $p_M$  grid as the fiducial qDF’s  $p_{\text{DF}}$ . As the nested-grid search approaches the best fit values, the fiducial qDF approaches its optimum as well.
- *Computational expense.* Overall, the computation speed of this nested-grid approach is dominated (in descending order of importance) by a) the complexity of potential and action calculation, b) the  $N_* \times N_{\text{samples}} + N_x^2 \times N_v^3$  actions required to be calculated per  $p_{\Phi}$ , c) the number of potential parameters, and d) the number of DF parameters.

The second step samples the shape of the *pdf* using MCMC. Formally, calculating the *pdf* on a fine grid like BR13 (e.g., with  $K = 11$  grid points in each dimension) would provide the same information. However the number of expensive *pdf* evaluations scales as  $K^{N_p}$ . For a high-dimensional  $p_M$  ( $N_p > 4$ ), an MCMC approach might sample the *pdf* much faster: we use *emcee* by Foreman-Mackey et al. (2013) and release the walkers very close to the best fit  $p_M$  found by the nested-grid search, which assures fast convergence in much less than  $K^{N_p}$



**Figure 4.** The  $pdf$  in the parameter space  $p_M = \{p_\phi, p_{DF}\}$  for one example mock data set (see Test 3.1 in Table C1). Blue indicates the  $pdf$  for the potential parameters,  $p_\phi$ , and green indicates the qDF parameters,  $p_{DF}$ . The true parameters are marked by dotted lines. The dark, medium, and bright contours in the 2D distributions represent  $1\sigma$ ,  $2\sigma$ , and  $3\sigma$  confidence regions, respectively. The parameters are weakly to moderately covariant, but their level of covariance depends on the actual choice of the mock data's  $p_M$ . The  $pdf$  here was sampled using MCMC. The dashed lines in the 1D distributions are Gaussian fits to the histogram of MCMC samples. This demonstrates that, for such a large number of stars, the  $pdf$  approaches the shape of a multivariate Gaussian that also projects into Gaussians when considering the marginalized  $pdf$  for all the individual  $p_M$ , as expected for a maximum likelihood estimator.

$pdf$  evaluations. We also use the best fit  $p_M$  of the grid search as fiducial qDF for the whole MCMC. In doing so, the normalization varies smoothly with different  $p_M$  and is less sensitive to the accuracy in Equation (8).

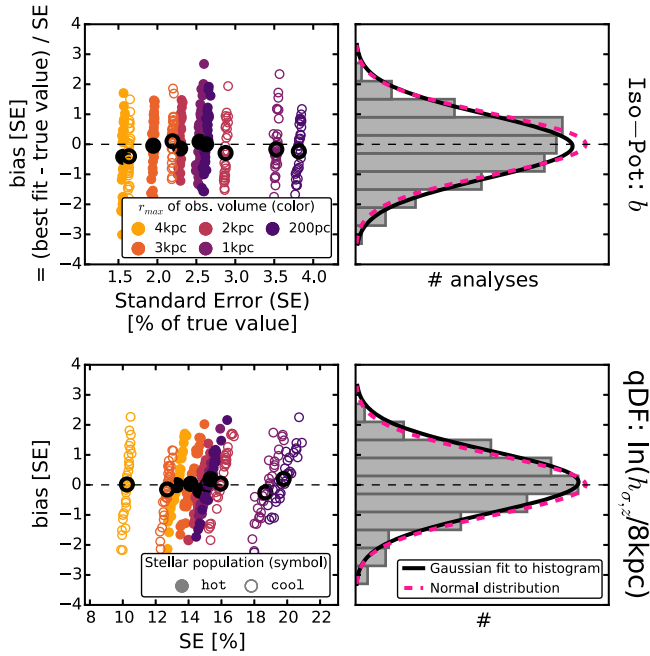
### 3. RESULTS

We are now in a position to examine the limitations of action-based modeling posed in the introduction using our *RoadMapping* machinery. We explore: (i) whether the parameter estimates are unbiased, (ii) the role of the survey volume, (iii) imperfect selection functions, (iv) measurement uncertainties, and what happens if the true (v) DF or (vi) potential are not included in the space of models.

We will rely on mock data as input to explore the limitations of the modeling. The mock data is generated directly from the fiducial potential and DF models introduced in Sections 2.3 and 2.4, following the procedure described in Appendix A. With the exception of the test suite on measurement uncertainties in Section 3.4, we assume that phase-space uncertainties are negligible. All tests are also summarized in Table C1.

We do not explore the breakdown of the assumption that the system is axisymmetric and in steady state nor the impact of resonances, which is not possible in the current setup using mock data drawn from axisymmetric galaxy models. We plan, however, to investigate this in a future paper, where we will apply *RoadMapping* to  $N$ -body simulations of disk galaxies.





**Figure 5.** Lack of bias in the parameter estimates. Maximum likelihood estimators converge to the true parameter values for large numbers of data points and have a Gaussian spread—if the model assumptions are fulfilled. To test that these conditions are satisfied for *RoadMapping*, we create 320 mock data sets, which come from two different stellar populations and five spherical observation volumes (see legends). (All model parameters are summarized in Table C1 as Test 3.2.) Bias and relative standard error (SE) are derived from the marginalized *pdf* for two model parameters (isochrone scale length  $b$  in the first row, and qDF parameter  $h_{\sigma,z}$  in the second row). The second column displays a histogram of the 320 bias offsets. As it closely follows a normal distribution, our modeling method is therefore well-behaved and unbiased. The black dots denote the *pdf* expectation value for the 32 analyses belonging to the same  $p_M$ .

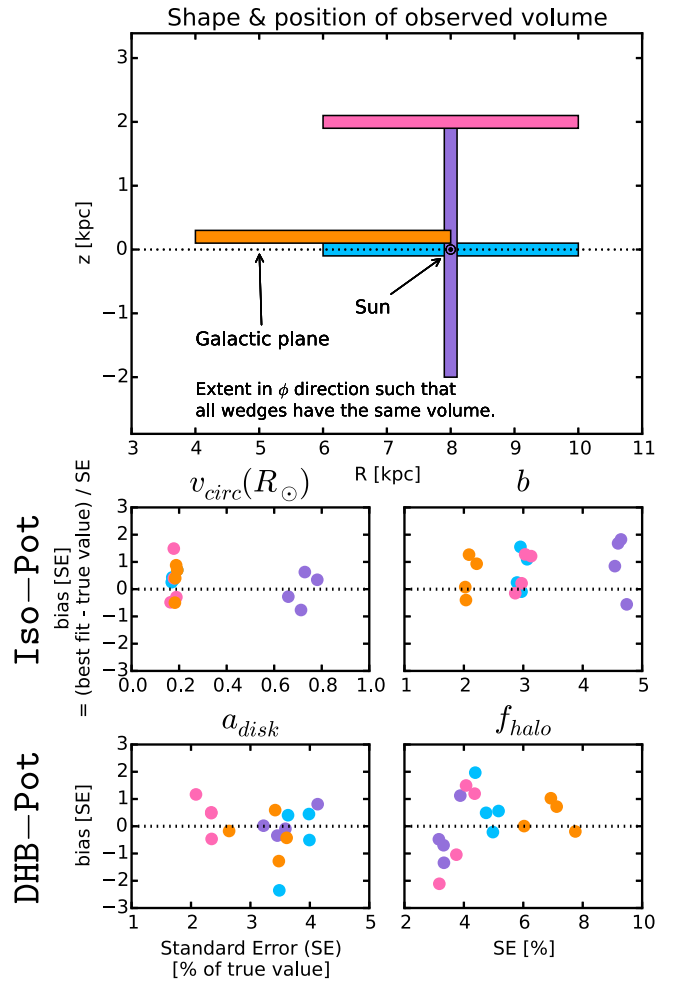
### 3.1. Model Parameter Estimates in the Limit of Large Data Sets

The MAPs in BR13 contained between 100 and 800 objects, which implied broad *pdf*s for the model parameters,  $p_M$ . Several consequences arise in the limit of much larger samples, say  $N_* = 20,000$ . (i) As outlined in Section 2.6 and investigated in Figure 2 (Test 1 in Table C1), higher numerical accuracy is needed due to the likelihood normalization requirement,  $\delta M_{\text{tot}} \lesssim 1/N_*$  (see Equation (14)), which drives the computing time. (ii) The *pdf*(s) of the  $p_M$  become Gaussian, with a *pdf* width (i.e., the standard error (SE) on the parameter estimate) that scales as  $1/\sqrt{N_*}$ . The former is demonstrated in Figure 4 (Test 3.1 in Table C1) and we also verified that the latter is true. (iii) Any bias in the *pdf* expectation value has to be considerably less than the SE. Figure 5 (Test 3.2 in Table C1) illustrates that *RoadMapping* behaves like an unbiased maximum likelihood estimator; the average parameter estimates from many mock data sets are very close to the input  $p_M$ , and the distribution of the actual parameter estimates are Gaussian around it.

### 3.2. The Role of the Survey Volume Geometry

To explore the role of the survey volume at a given sample size, we devise two suites of mock data sets.

The first suite draws mock data for two different potentials (Iso-Pot and DHB-Pot) and four volume wedges (see



**Figure 6.** Bias vs. standard error in recovering the potential parameters for mock data sets drawn from four different wedge-shaped test observation volumes within the Galaxy (illustrated in the upper panel; the corresponding analyses are color-coded) and two different potentials (Iso-Pot and DHB-Pot from Table 1; see also Test 4 in Table C1 for all model parameters used). Standard error and offset were determined from a Gaussian fit to the marginalized *pdf*. The angular extent of each wedge-shaped observation volume was adapted such that all have a volume of  $4.5 \text{ kpc}^3$ , even though their extent in  $(R, z)$  is different. (The recovery of the free potential parameter  $v_{\text{circ}}(R_\odot)$  in the different wedges is very similar for both potentials and therefore only shown for the Iso-Pot). Minor expected differences can be seen (e.g.,  $v_{\text{circ}}(R_\odot)$  and tracer density scale lengths requiring larger radial extent), but overall there is no clear trend that an observation volume around the Sun, above the disk, or at smaller Galactocentric radii should give remarkably better constraints on the potential than the other volumes.

Appendix B) with different extent and at different positions within the Galaxy, illustrated in the upper panel of Figure 6. Otherwise the data sets are generated from the same  $p_M$  (see Test 4 in Table C1). To isolate the role of the survey volume geometry and position, the mock data sets all have the same number of stars ( $N_* = 20,000$ ), and are drawn from identical total survey volumes ( $4.5 \text{ kpc}^3$ , achieved by adjusting the angular width of the wedges). We investigate these rather unrealistic survey volumes to test (i) if there are regions in the Galaxy where stars are on intrinsically more informative orbits and (ii) if spatial cuts applied to the survey volume (e.g., to avoid regions of large dust extinction or measurement uncertainties) would therefore strongly affect the precision of the potential constraints. To make this effect—

if it exists—noticeable, we choose some extreme but illustrative examples.

The results are shown in Figure 6. The wedges all have the same volume and all give results of similar precision. There are some minor and expected differences, e.g.,  $v_{\text{circ}}(R_{\odot})$  and radial scale lengths ( $b$  and  $a_{\text{disk}}$ ) are slightly better recovered for a large radial extent and the halo fraction at the Sun,  $f_{\text{halo}}$ , for volumes centered around  $R_{\odot}$ . In the case of an axisymmetric model galaxy, the extent in  $\phi$  direction is not expected to matter. Overall radial extent and vertical extent seem to be equally important to constrain the potential. Figure 6 implies, therefore, that volume offsets or spatial cuts of the survey volume in the radial or vertical direction have at most a modest impact—even in case of the very large sample size at hand.

The second suite of mock data sets was already introduced in Section 3.1 (see also Test 3.2 in Table C1), where mock data sets were drawn from five spherical volumes around the Sun with different  $r_{\text{max}}$ , for two different stellar populations. The results of this second suite are shown in Figure 5 and exemplify the effect of the size of the survey volume.

Figure 5 demonstrates that, given a choice of  $p_{\text{DF}}$ , a larger volume always results in tighter constraints. There is no obvious trend that a hotter or cooler population will always give better results; it depends on the survey volume and the model parameter in question.

While it appears that the argument for significant radial and vertical extent is generic, we have not done a full exploration of all combinations of  $p_M$  and volumes.

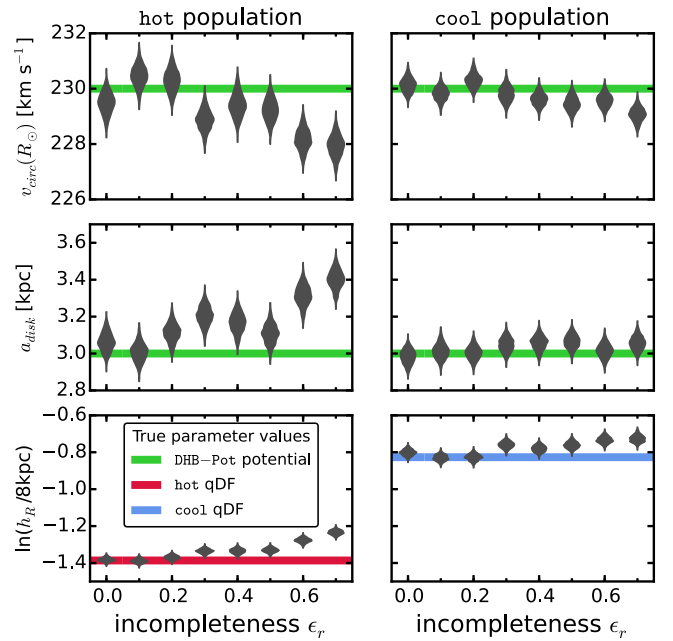
In reality, different regions in the Galaxy have different stellar number densities and different measurement uncertainties, which should therefore be the major factor to drive the precision of the potential recovery when choosing a survey volume.

### 3.3. Impact of Misjudging the Selection Function of the Data Set

The survey SF (see also Appendix B) is sometimes not perfectly determined. While the pattern of the survey area on the sky may be complex, it is usually precisely known. It is the uncertainties (e.g., in completeness) in the line-of-sight directions that is most prone to systematic misassessment. We therefore focus here on completeness misjudgements (with a simplified angular pattern) and investigate how much this could affect the recovery of the potential. We do this by creating mock data in the DHB-Pot within a spherical survey volume with radius  $r_{\text{max}}$  around the Sun (see Test 5 in Table C1) and a spatially varying completeness function

$$\text{completeness}(r) \equiv 1 - \epsilon_r \frac{r}{r_{\text{max}}}, \quad (17)$$

which drops linearly with the distance,  $r$ , from the Sun. The completeness function can be understood as the probability of a star at distance  $r$  to be detected (see also Equation (18)). In the *RoadMapping* analysis on the other hand, we assume constant completeness ( $\epsilon_r = 0$ ). The incompleteness parameter  $\epsilon_r$  of the mock data therefore quantifies how much we misjudge the SF. This mock test captures the relevant case of stars being less



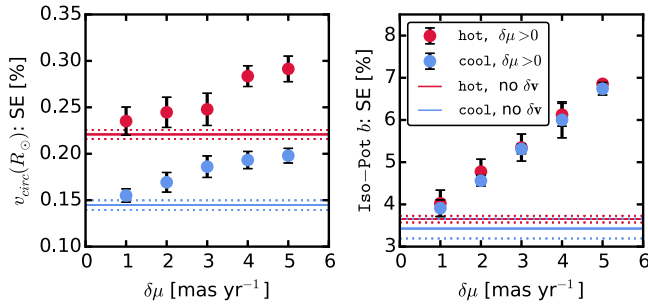
**Figure 7.** Impact of misjudging the completeness of the data on the parameter recovery with *RoadMapping*. Each mock data set was created with a different incompleteness parameter,  $\epsilon_r$ , (shown on the x-axis, see Equation (17)). (The model parameters are given as Test 5 in Table C1.) The analysis, however, assumed that all data sets had constant completeness within the survey volume ( $\epsilon_r = 0$ ). The violins show the full shape of the projected *pdfs* for each model parameter, and the solid lines show their true values. The *RoadMapping* method seems to be robust against modest deviations between the true and the assumed data incompleteness. (The potential parameter  $f_{\text{halo}}$ , and the other qDF parameters are recovered to a comparable accuracy and are therefore not shown here.)

likely to be observed (than assumed) the further away they are (e.g., due to unknown dust obscuration).

Figure 7 demonstrates that the potential recovery with *RoadMapping* is quite robust against somewhat wrong assumptions about the completeness of the data, i.e.,  $\epsilon_r \lesssim 0.15$  for the hot and  $\epsilon_r \lesssim 0.2$  for the cool population. The cool population is more robust, because it is less affected by the SF misjudgement at high  $|z|$  than the hot population. Our simple model SF affects stars at large and small radii in equal proportion. As long as the misjudgement is small, the tracer scale length parameter,  $h_R$ , can still be reliably recovered, and, with it, the potential.

We have also investigated several test suites using the Iso-Pot. The recovery of  $v_{\text{circ}}(R_{\odot})$  and the qDF parameters at different  $\epsilon_r$  is qualitatively and quantitatively similar to Figure 7 for the DHB-Pot. The isochrone scale length,  $b$ , however, is recovered independently of  $\epsilon_r$ —probably because rotation curve measurements in the plane alone, which are not affected by the SF cuts, give reliable constraints on  $b$ . When not including tangential velocity measurements in the analysis (which is done by marginalizing the likelihood in Equation (9) over  $v_T$ ), the parameters are well recovered only for  $\epsilon_r \lesssim 0.15$  and  $\epsilon_r \lesssim 0.2$  for the hot and cool population, respectively. As this is in concordance with our findings for the DHB-Pot, this result seems to be valid for different choices of potentials.

For spatial completeness functions varying with the distance from the plane  $|z|$  only, the Iso-Pot potential



**Figure 8.** Effect of proper motion uncertainties,  $\delta\mu$ , on the precision of potential parameter recovery for two stellar populations of different kinematic temperature (see Test 6.1 in Table C1 for all model parameters). The relative standard error (SE) derived from the marginalized *pdf* for each model parameter was determined for precise data sets without measurement uncertainties (solid lines, with dotted lines indicating the error) and for data sets affected by different proper motion uncertainties  $\delta\mu$  and  $\delta v_{\text{los}} = 2 \text{ km s}^{-1}$  (data points with error bars), but no uncertainties in position. The errors come from taking the mean over several data sets.

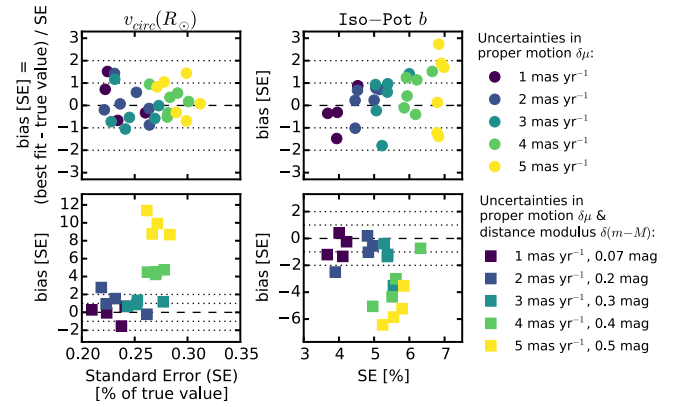
recovery is similarly robust as long as  $v_T$  measurements are included.

### 3.4. Measurement Uncertainties and their Effect on the Parameter Recovery

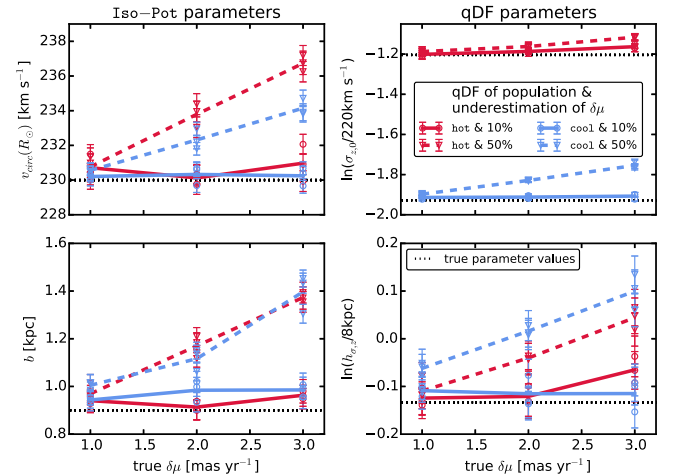
Measurement uncertainties in proper motions and distance dominate over uncertainties in position on the sky (R.A., decl.) and line-of-sight velocity, which can be more accurately determined.

The range of proper motion uncertainties in this section,  $1\text{--}5 \text{ mas yr}^{-1}$ , is the approximate measurement accuracy that can be achieved by combining catalogs from ground-based surveys like the Sloan Digital Sky Survey (SDSS; Abazajian et al. 2003), the USNO-B catalog (Monet et al. 2003), 2MASS (Skrutskie et al. 2006), and the Pan-STARRS1 photometric catalog (PS1; Kaiser et al. 2010).<sup>8</sup> Space-based surveys can do even better. The *Hipparcos* (ESA 1997) and *Tycho-2* (Høg et al. 2000) catalogs achieve  $\delta\mu \sim 2.5 \text{ mas yr}^{-1}$  (and even  $\delta\mu \lesssim 1 \text{ mas yr}^{-1}$  for all stars with  $V < 12$ ), which will be soon superseded by *Gaia* with only  $\delta\mu \sim 0.3 \text{ mas yr}^{-1}$  at its faint end at magnitude  $G \sim 20$  (de Bruijne et al. 2014).

We first investigate the impact of (perfectly known) proper motion uncertainties on the precision of the potential parameter recovery (see Test 6.1 in Table C1). Figure 8 demonstrates that for data sets with  $\delta\mu$  as high as  $5 \text{ mas yr}^{-1}$  the precision degrades by a factor of no more than  $\sim 2$  as compared to a data set without measurement uncertainties. The precision gets monotonically better for smaller  $\delta\mu$ , being larger only by a factor of  $\sim 1.15$  at  $\delta\mu = 1 \text{ mas yr}^{-1}$ . With relative standard errors on the recovered parameters of only a few percent at most for 10,000 stars, this means we still get quite precise



**Figure 9.** Potential parameter recovery using the approximation for the model probability convolved with measurement uncertainties in Equation (16). We show the *pdf* offset and relative width (i.e., standard error SE) for potential parameters recovered from mock data sets (which were created according to Test 6.2 in Table C1). The data sets in the upper panels are affected only by proper motion uncertainties,  $\delta\mu$ , (and  $\delta v_{\text{los}} = 2 \text{ mas yr}^{-1}$ ), while the data sets in the lower panels also have distance (modulus) uncertainties,  $\delta(m - M)$ , as indicated in the legend. For data sets with  $\delta\mu \leq 3 \text{ mas yr}^{-1}$ , Equation (16) was evaluated with  $N_{\text{samples}} = 800$ , for  $\delta\mu > 3 \text{ mas yr}^{-1}$  we used  $N_{\text{samples}} = 1200$ . In the absence of distance uncertainties, Equation (16) gives unbiased results. For  $\delta(m - M) > 0.2 \text{ mag}$  (i.e.,  $\delta r/r > 0.1$ ; for  $r \sim 3 \text{ kpc}$ ), however, biases of several  $\sigma$  are introduced, as Equation (16) is only an approximation for the true likelihood in this case.



**Figure 10.** Effect of a systematic underestimation of proper motion uncertainties,  $\delta\mu$ , on the recovery of the model parameters. (The true model parameters used to create the mock data are summarized as Test 6.3 in Table C1, four of them are indicated as black dotted lines in this figure.) The mock data was perturbed according to proper motion uncertainties,  $\delta\mu = \delta\mu_{\text{decl.}} = \delta\mu_{\text{R.A.}}$ , as indicated on the x-axis. In the *RoadMapping* analysis (see likelihood in Equation (16)), however, we underestimated the true  $\delta\mu$  by 10% (circles, solid lines) and 50% (triangles, dashed lines). The symbols denote the best fit parameters with  $1\sigma$  error bars of several mock data sets. The lines connect the mean of corresponding data realizations to guide the eye.

constraints on the potential, as long as we know the proper motion uncertainties perfectly.

We also note that, in this case, the relative and absolute difference in recovered precision between the precise and the uncertainty affected data sets does not seem to depend strongly on the kinematic temperature of the stellar population.

Secondly, we investigate the impact of additional measurement uncertainties in distance (modulus). In the absence of

<sup>8</sup> Combining observations from the SDSS Data Release 1 with the USNO-B catalog based on the Palomar Observatory Sky Survey's (POSS) photographic plates from the 1950s lead to proper motion measurements precise to  $\delta\mu \sim 3$  or  $5 \text{ mas yr}^{-1}$  depending on magnitude  $r < 18$  or  $r < 20$ , respectively (Gould & Kollmeier 2004; Munn et al. 2004, 2008). The same accuracy can be achieved when using the four years of measurements by the PS1 only. By careful calibration of USNO-B and 2MASS with PS1, Sesar et al. (2015) got proper motions as accurate as  $\delta\mu \sim 1.5 \text{ mas yr}^{-1}$  for  $r \lesssim 18$ . The Large Synoptic Survey Telescope (LSST, Ivezić et al. 2008b) planned for 2021 might even achieve  $\delta\mu \lesssim 1 \text{ mas yr}^{-1}$  during its 10 years of scanning the sky (Ivezić et al. 2008a).

distance uncertainties, the uncertainty-convolved model probability given in Equation (16) is unbiased (see upper left panel in Figure 9). When including distance (modulus) uncertainties, Equation (16) is just an approximation for the true likelihood; the systematic bias thus introduced in the parameter recovery gets larger with the size of  $\delta(m - M)$ , as demonstrated in Figure 9, lower panels (see also Test 6.2 in Table C1). We find, however, that in the case of  $\delta(m - M) \lesssim 0.2$  mag (if also  $\delta\mu \lesssim 2$  mas yr<sup>-1</sup> and a maximum distance of  $r_{\max} = 3$  kpc, see Test 6.2 in Table C1) the potential parameters can still be recovered within  $2\sigma$ . This corresponds to a relative distance uncertainty of  $\sim 10\%$ . The overall precision of the potential recovery is also not degraded much by introducing distance uncertainties of less than  $10\%$ .

How does this compare with the distance uncertainties expected for *Gaia*? For a typical red clump giant star with  $M_I \sim 0$  mag and  $(V - I) \sim 1$  mag at a distance of  $r = 3$  kpc we estimate (using the magnitude transformation by Jordi et al. (2010) and the uncertainty parameterization by de Bruijne et al. (2014)) a parallax uncertainty of  $\delta\pi \sim 11 \mu\text{as}$ , which is consistent with a distance uncertainty of less than  $5\%$ , and a proper motion uncertainty of  $\delta\mu \sim 6 \mu\text{as yr}^{-1}$ , which is negligible. For the case that the modeling is not restricted to giant stars only, a quick investigation by Rene Andrae (private communication) of stars at  $r = 3 \text{ kpc} \pm 5 \text{ pc}$  from the *Gaia* universe model snapshot catalog (GUMS; Robin et al. 2012) revealed that a magnitude cut at  $G \sim 15$  mag in the overall *Gaia* data set should keep all distance uncertainties within 3 kpc below  $\sim 10\%$ , while also preserving *Gaia*'s simple SF.

We therefore found that in case we perfectly know the measurement uncertainties (and the distance uncertainty is negligible or of the order of the uncertainties expected from *Gaia* within  $\sim 3$  kpc), the convolution of the model probability with the measurement uncertainties gives *precise and accurate* constraints on the model parameters—even if the measurement uncertainty itself is quite large.

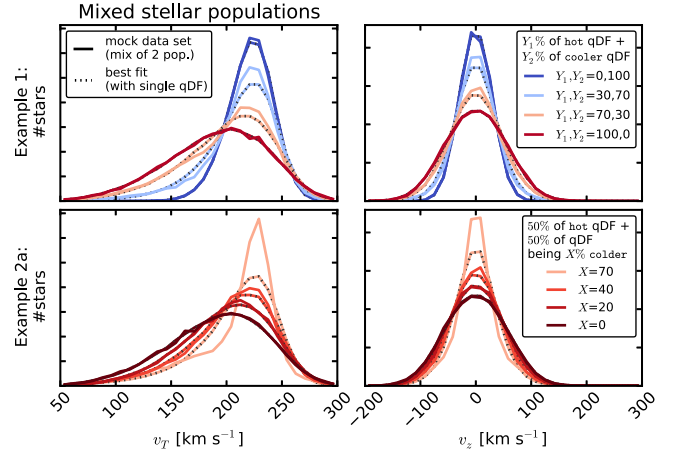
Lastly, Figure 10 investigates the effect of a systematic underestimation of the true proper motion uncertainties,  $\delta\mu$ , by  $10\%$  and  $50\%$  (see also Test 6.3 in Table C1). We find that this causes a bias in the parameter recovery that grows seemingly linear with  $\delta\mu$ . For an underestimation of only  $10\%$ , however, the bias becomes  $\lesssim 2\sigma$  for 10,000 stars—even for  $\delta\mu \sim 3$  mas yr<sup>-1</sup>.

The size of the bias also depends on the kinematic temperature of the stellar population and the model parameter considered (see Figure 10). The qDF parameters are, for example, better recovered by hotter populations. This is because the relative difference between the *true*  $\sigma_i(R)$  (with  $i \in \{R, z\}$ ) and *measured*  $\sigma_i(R)$  (which comes from the deconvolution with an underestimated velocity uncertainty) is smaller for hotter populations.

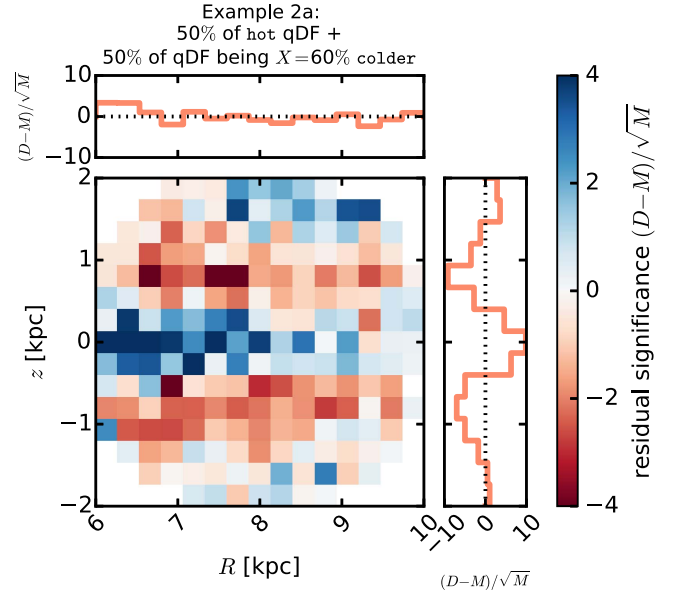
### 3.5. The Impact of Deviations of the Data from the Idealized Distribution Function

Our modeling approach assumes that each stellar population follows a simple DF; here we use the qDF. In this section we explore what happens if this idealization does not hold. We investigate this issue by creating mock data sets that are drawn from *two* distinct qDFs of different temperatures<sup>9</sup> (see Table 2 and Test 7 in Table C1) in the DHB-Pot, and analyze

<sup>9</sup> Following the observational evidence, our mock data populations with cooler qDFs also have longer tracer scale lengths.



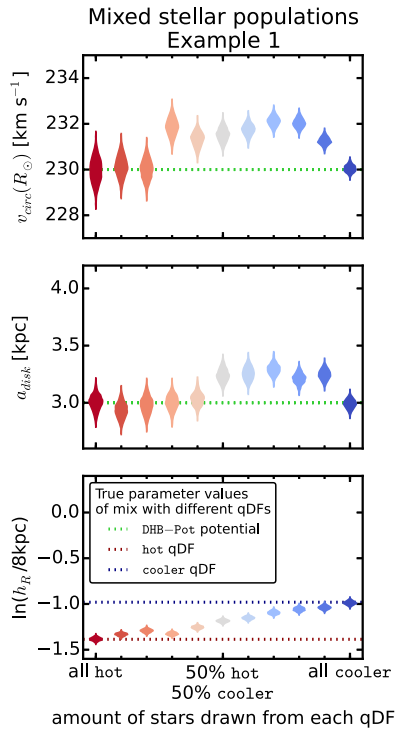
**Figure 11.** Distribution of mock data  $v_T$  and  $v_z$  created by mixing stars drawn from two different qDFs (solid lines), and the distribution predicted by the best fit of a single qDF and potential to the data (dotted lines). (The model parameters used to create the mock data are given in Table C1 as Test 7, Examples 1 and 2a, with the qDF parameters referred to in the legend given in Table 2.) The corresponding single qDF best fit curves were derived from the best fit parameters found in Figures 13 and 14. (The data sets were color-coded in the same way as the corresponding analyses in Figures 13 and 14.) We use the mixtures of two qDFs to demonstrate how *RoadMapping* behaves for data sets following DFs with shapes differing slightly from a single qDF. For large deviations it might already become visible from directly comparing the mock data and best fit distribution, that a single qDF is a bad assumption for the stars' true DF.



**Figure 12.** Residual significance,  $(D - M) / \sqrt{M}$ , of one example mock data set,  $D$ , and its best fit single qDF model,  $M$ , in the  $(R, z)$  plane. The mock data set,  $D$ , was created by mixing a hot and an  $X\%$  colder population in equal proportion (see also Table C1, Test 7, Example 2a, with  $X = 60\%$ ). The best fit distribution,  $M$ , was derived analogously to the ones shown for the velocity components in Figure 11. This is an extreme example where the best fit single qDF is not a good fit anymore (see Figure 14, Example 2a,  $X = 60\%$ ), but it illustrates how we constructed mock data distributions with radial and vertical density profiles differing from a single qDF by mixing two different qDF populations for the Test suite 7.

the composite mock data set by fitting a *single* qDF to it. The velocity distributions of some mock data sets and their best fit qDFs are illustrated in Figure 11, and Figure 12 shows the tracer density residuals between data and best fit in the  $(R, z)$





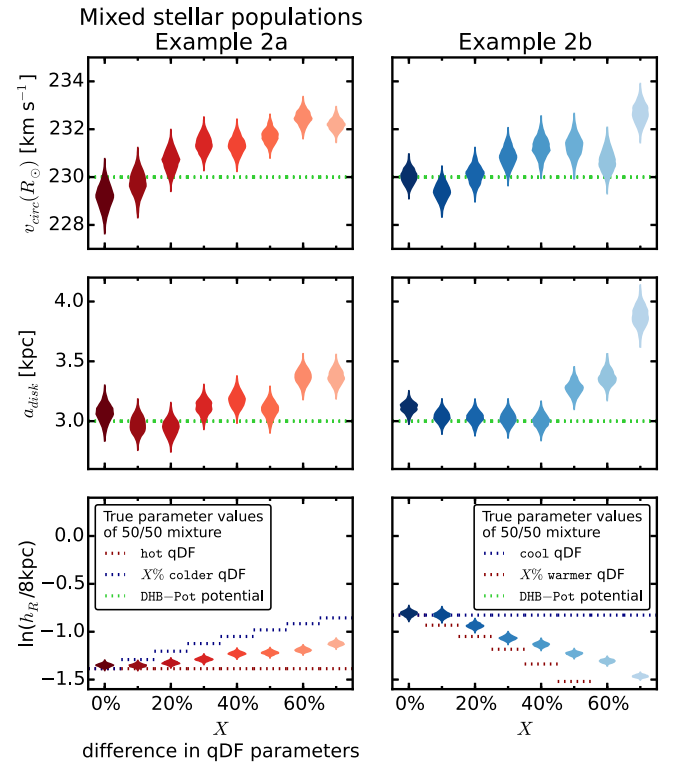
**Figure 13.** The dependence of the parameter recovery on the degree of pollution and temperature of the stellar population. We mix (i.e., “pollute”) varying amounts of stars from a *hot* stellar population with stars from a very different *cooler* population (see Table 2), as indicated on the x-axis. (All model parameters used to create the mock data are given as Test 7, Example 1, in Table C1.) The composite polluted mock data set follows a true DF that has a slightly different shape than the qDF. We then analyze it using *RoadMapping* and fit a *single* qDF only. The violins represent the marginalized *pdfs* for the best fit model parameters. Some mock data sets are shown in Figure 11, first row, in the same colors as the violins here. We find that a hot population is much less affected by pollution with stars from a cooler population than vice versa. (The potential parameter  $f_{\text{halo}}$  is recovered to a similar or even slightly better accuracy than  $a_{\text{disk}}$  at each given mixing rate and is therefore not shown here.)

plane. Figures 13 and 14 compare the input and best fit parameters. In Example 1 we choose qDFs of widely different temperatures and vary their relative fraction of stars in the composite mock data set (Figure 13); in *Example 2* we always mix mock data stars from two different qDFs in equal proportion, but vary by how much the qDFs’ temperatures differ (Figure 14).

The first set of tests mimics a DF that has wider wings or a sharper core in velocity space than a qDF (see Figure 11), and slightly different radial and vertical tracer density profiles (similar to Figure 12). The second test could be understood as mixing neighboring MAPs in the  $[\alpha/\text{Fe}]$ -versus- $[\text{Fe}/\text{H}]$  plane due to large bin sizes or abundance measurement errors (cf. BR13).

We consider the impact of the DF deviations on the recovery of the potential and the qDF parameters separately.

We find from Example 1 that the potential parameters can be more robustly recovered if a mock data population is polluted by a modest fraction ( $\lesssim 30\%$ ) of stars drawn from a much cooler qDF, as opposed to the same pollution of stars from a hotter qDF. When considering the case of a 50/50 mix of contributions from different qDFs in Example 2 there is a systematic, but mostly small, bias in recovering the potential parameters, monotonically increasing with the qDF parameter difference. In particular, for fractional differences

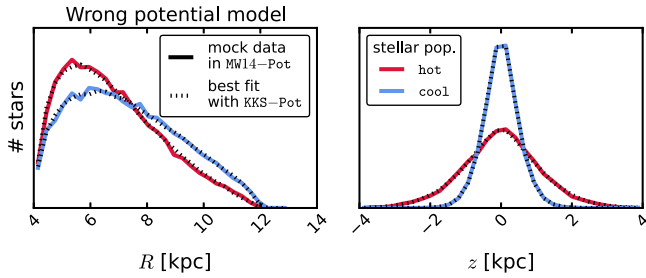


**Figure 14.** The dependence of the parameter recovery on the difference in qDF parameters of a 50/50 mixture of two stellar populations and their temperature. The two qDFs from which the stars in each mock data set were drawn are indicated in the legend, with the qDF parameters  $\sigma_{R,0}$ ,  $\sigma_{z,0}$ , and  $h_R$  differing by  $X\%$  (see also Table 2 and Section 2.4) as indicated on the x-axis. (The model parameters used for the mock data creation are given as Test 7, Examples 2a and 2b, in Table C1.) Each composite mock data set is fitted with a *single* qDF, and the marginalized *pdfs* are shown as violins. Some mock data sets of Example 2a and their best fit distributions are shown in Figure 11, last row (color-coded analogous to the violins here), and Figure 12 shows the corresponding residuals in the  $(R, z)$  plane. By mixing populations with varying difference in their qDF parameters, we model the effect of finite bin size or abundance errors when sorting stars into different MAPs in the  $[\alpha/\text{Fe}]$ -vs.- $[\text{Fe}/\text{H}]$  plane and assuming that they follow single qDFs (cf. BR13). We find that the bin sizes should be chosen such that the difference in qDF parameters between neighboring MAPs is less than 20%. (The potential parameter  $f_{\text{halo}}$  is recovered to a similar or even slightly better accuracy than  $a_{\text{disk}}$  at each given  $X$  and is therefore not shown here.)

in the qDF parameters of  $\lesssim 20\%$  the systematics are insignificant even for sample sizes of  $N_* = 20,000$ , as used in the mock data.

Overall, the circular velocity at the Sun is very reliably recovered to within 2% in all these tests. But the best fit  $v_{\text{circ}}(R_\odot)$  is not always unbiased at the implied precision.

The recovery of the effective qDF parameters, in light of non-qDF mock data, is quite intuitive (in Figures 13 and 14 we therefore show only  $h_R$ ): the effective qDF temperature lies between the two temperatures from which the mixed DF of the mock data was drawn; in all cases the scale lengths of the velocity dispersion fall-off,  $h_{\sigma,R}$  and  $h_{\sigma,z}$ , are shorter than the true scale lengths because the stars drawn from the hotter qDF dominate at small radii, while stars from the cooler qDF (with its longer tracer scale length) dominate at large radii; the recovered tracer scale lengths,  $h_R$ , vary smoothly between the input values of the two qDFs that entered the mix of mock data. The latter is also demonstrated in Figure 12: the radial tracer



**Figure 15.** Comparison of the spatial distribution of mock data in  $R$  and  $z$  created in the MW14-Pot potential and with two different stellar populations (see Test 8 in Table C1 for all mock data model parameters), and the best fit distribution recovered by fitting the family of KKS-Pot potentials to the data. The best fit potentials are shown in Figure 16 and the corresponding best fit qDF parameters in Figure 17. The data is very well recovered, even though the fitted potential family did not incorporate the true potential.

density profile of the mock data is steeper than a single qDF in the mid-plane and more shallow at higher  $|z|$ ; overall the best fit  $h_R$  therefore lies in between.

We note that in the cases where the systematic bias in the potential parameter recovery becomes as big as several  $\sigma$ , a direct comparison of the true mock data set and best fit distribution (see Figure 11) can sometimes already reveal that the assumed DF is not a good model for the data.

We also performed the same tests for the spherical ISO-Pot instead of the galaxy-like DHB-Pot and for a much higher sampling of the mixing rate and qDF difference,  $X$ . The results are qualitatively and quantitatively very similar and therefore independent of the exact choice of potential.

Overall, we find that the potential inference is quite robust to modest deviations of the data from the assumed DF.

### 3.6. The Implications of a Gravitational Potential not from the Space of Model Potentials

We now explore what happens when the mock data were drawn from one axisymmetric potential family, here MW14-Pot, and are then modeled considering potentials from another axisymmetric family, here KKS-Pot (see Table 1 and Figure 1). In the analysis we assume the circular velocity at the Sun to be fixed and known, and we only fit the parametric potential form.<sup>10</sup>

We analyze a mock data set from a `hot` and `cool` stellar population each (see Test 8 in Table C1) with high numerical accuracy. The distributions generated from the best fit parameters reproduce the data in configuration space very well (see Figure 15 for the spatial distribution and the circles in Figure 17 for the velocity distribution).

The comparison between true and best fit potentials are shown in Figure 16. We find that the potential recovered by *RoadMapping* is in good agreement with the true potential inside of the observed volume of mock tracers. Outside of it we can make predictions, at least to a certain extent. Especially the potential forces, to which the stellar orbits are sensitive, are recovered and tightly constrained. This robust recovery of the radial and vertical forces leads to small errors on the estimated circular velocity curve ( $\lesssim 5\%$ ) and surface density within  $|z| = 1.1$  kpc ( $\lesssim 10\%$ ), respectively. We

get the best results for the local density, the surface density, and disk-to-halo ratio between  $R \sim 4$  kpc and  $R \sim 8$  kpc, i.e., where most of the tracer stars used in the analysis are located (see Figure 15).

The local density distribution is, in general, less reliably constrained than the forces, but we still capture the essentials. Exceptions are the inner regions  $R \lesssim 3$  kpc, where the KKS-Pot model is missing a bulge by construction, and the local radial density profile, which is somewhat misjudged by the KKS-Pot model. The `cool` population, where most stars are confined to regions close to the mid-plane, recovers the flatness of the disk better than the `hot` population, but overall the best fit disk is slightly less dense in the mid-plane than the true disk. While it is generally possible to generate very flattened density distributions from Stäckel potentials, it might be difficult to simultaneously have a roundish halo and to require that both Stäckel components have the same focal distance (see Table 1).

The disk-to-halo surface density fraction within  $|z| = 1.1$  kpc is not tightly constrained ( $\gtrsim 20\%$ ), but recovered within the errors inside of the survey volume. Using a wrong potential model does therefore not necessarily lead to biases in local dark matter measurements.

Overplotted in Figure 16 is also the KKS-Pot with the parameters from Table 1, which were fixed based on a (by-eye) fit directly to the force field (within  $r_{\max} = 4$  kpc from the Sun) and the rotation curve of the MW14-Pot. The potential found with the *RoadMapping* analysis is an even better fit. This demonstrates that *RoadMapping* fitting infers a potential that, in its actual properties, resembles the input potential for the mock data in regions of large tracer density as closely as possible, given the differences in functional forms.

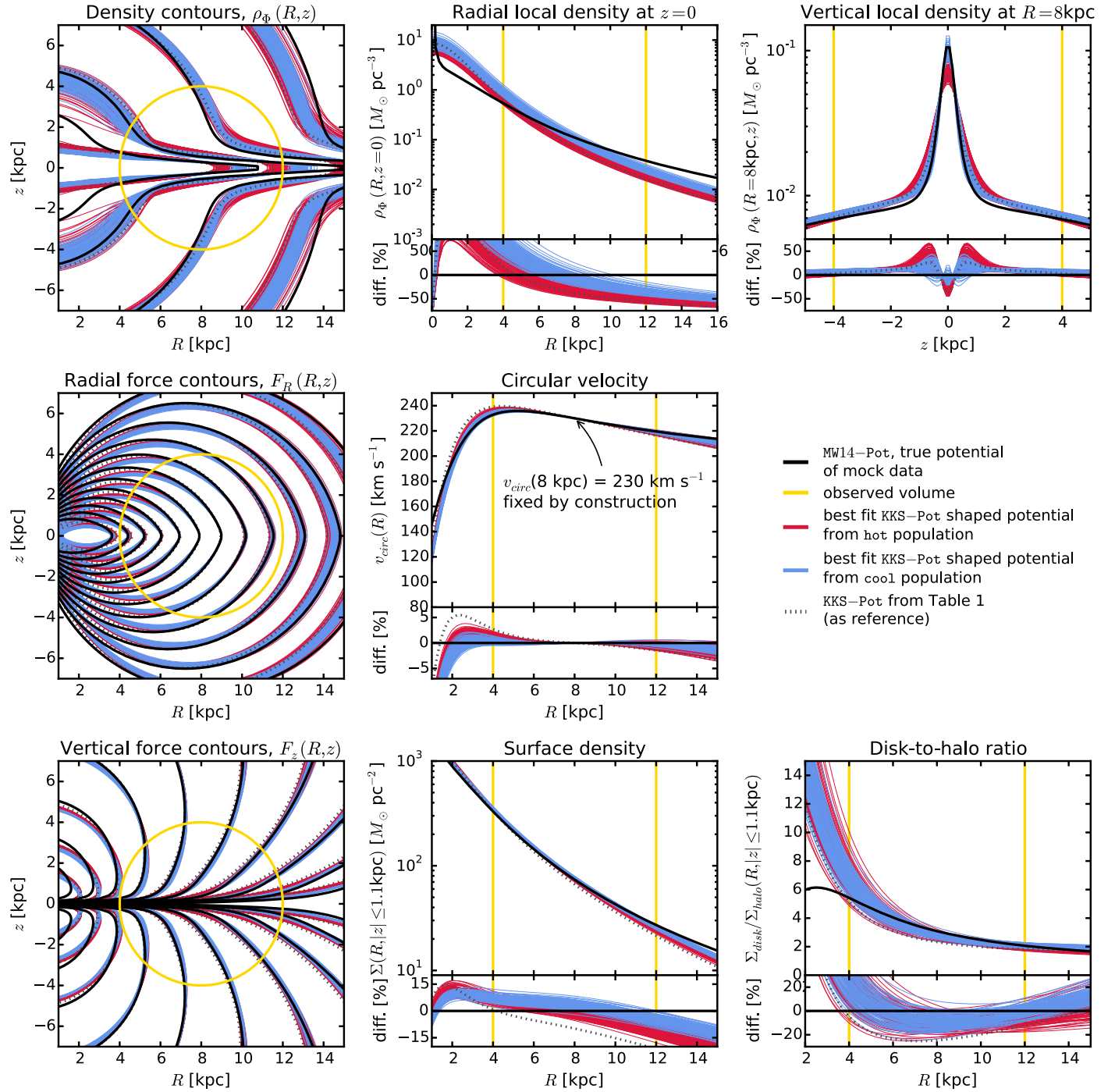
Figure 17 compares the true qDF parameters with the best fit qDF parameters belonging to the best fit potentials from Figure 16, and we also overplot the actual physical scale lengths and velocity dispersion as estimated directly from the mock data. While we recover  $h_R$ ,  $\sigma_{R,0}$ , and  $h_{\sigma,R}$  within the errors, we misjudge the parameters of the vertical velocity dispersion ( $\sigma_{0,z}$  and especially  $h_{\sigma,z}$ ), even though the actual mock data distribution is well reproduced. This discrepancy could be connected to the KKS-Pot not being able to reproduce the flatness of the disk. Also,  $\sigma_z$  and  $\sigma_R$  in Equations (6)–(7) are scaling profiles for the qDF (cf. BR13), and how close they are to the actual velocity profile depends on the choice of potential; that is, the *physical* velocity dispersion is well recovered even if the qDF velocity dispersion parameters are not. Figure 17 stresses once more that the actual parameter values of action-based DFs have always to be considered together with the potential in which they were derived. This is of importance in studies that use a fiducial potential to fit action-based DFs to stellar data, like, e.g., Sanders & Binney (2015) and Das & Binney (2016).

### 3.7. The Influence of the Stellar Population’s Kinematic Temperature

Overall, we found that it does not make a big difference if we use hot or cool stellar populations in our modeling.

How precise and reliable model parameters can be recovered does, to a certain extent, depend on the kinematic temperature of the data, as well as on the model parameter in question and

<sup>10</sup> We made sure that  $v_{\text{circ}}(R_\odot)$  can be very well recovered when included in the fit of a `cool` population. The model assumption that  $v_{\text{circ}}(R_\odot)$  is known does therefore not affect the discussion qualitatively.

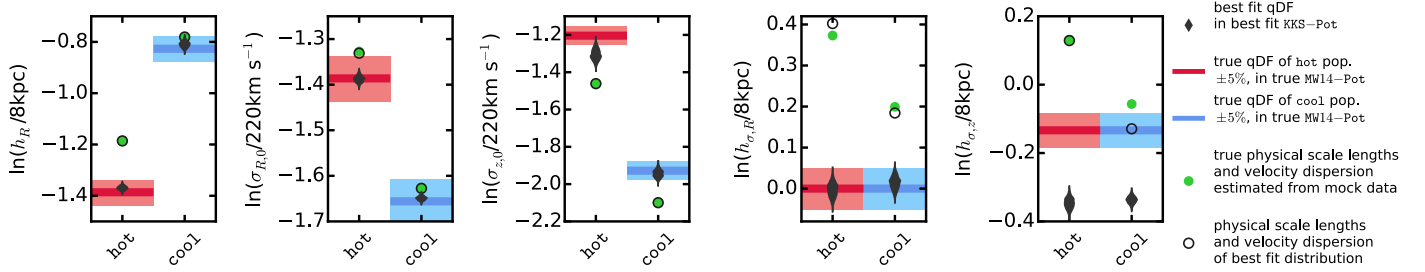


**Figure 16.** Recovery of the gravitational potential if the assumed potential model family (KKS-Pot with fixed  $v_{\text{circ}}(R_{\odot})$ ) and the true potential of the (mock data) stars (MW14-Pot in Table 1) have slightly different parametric forms. In addition to contours of equal density,  $\rho_{\Phi}$ , radial and vertical force,  $F_R$  and  $F_z$ , in the  $(R, z)$  plane (left column), we show local density profiles,  $\rho_{\Phi}(R, z=0)$  and  $\rho_{\Phi}(R=8 \text{ kpc}, z)$ , as well as the circular velocity curve,  $v_{\text{circ}}(R)$ , the total surface density profile within  $|z| \leq 1.1 \text{ kpc}$ ,  $\Sigma(R) \equiv \int_{-1.1 \text{ kpc}}^{1.1 \text{ kpc}} \rho_{\Phi}(R, z) dz$ , and the ratio of the disk and halo contributions to the total surface density,  $\Sigma_{\text{disk}}(R)/\Sigma_{\text{halo}}(R)$ . We compare the true potential (black lines) with 100 sample potentials (red and blue lines) drawn from the *pdf* found with MCMC for a *hot* (red) and a *cool* (blue) stellar population, and also display the relative difference as a percentage of the true value. (All mock data model parameters are given as Test 8 in Table C1.) Overall, the true potential is well recovered—especially in regions where most of the observed stars are located.

on the observation volume. But there is no easy rule of thumb as to what combination would give the best results (see Figure 5). There are two exceptions.

First, the circular velocity at the Sun,  $v_{\text{circ}}(R_{\odot})$ , is always best recovered with cooler populations (see Figures 8, 10, 13, and 14, and for the recovery of  $v_{\text{circ}}(R)$  at  $R \neq R_{\odot}$  see

Figure 16) because more stars are on near-circular orbits (see Figure 18). Cooler populations are also less sensitive to misjudgements of (spatial) selection functions at large  $|z|$  (see Figure 7). There is, however, the caveat that cool populations are more susceptible to non-axisymmetric streaming motions in the disk.



**Figure 17.** Recovery of the qDF parameters for the case where the true and assumed potential deviate from each other (see Test 8 in Table C1). The thick red (blue) lines represent the true qDF parameters of the `hot` (`cool`) qDF in Table 2 used to create the mock data, surrounded by a 5% error region. The gray violins are the marginalized *pdfs* for the qDF parameters found simultaneously with the potential constraints shown in Figure 16. We compare the qDF parameters with the actual physical scale lengths and velocity dispersion at the Sun estimated from the mock data and the best fit distribution by fitting exponential functions to the data. First, this shows that—apart from some small deviations in the velocity dispersion scale lengths—the velocity distribution of the mock data is very well reproduced by the best fit. Second, this demonstrates how the qDF parameters in different potentials do not necessarily agree with each other or with the actual physical velocity distribution.

Second, hotter populations seem to be less sensitive to misjudgements of proper motion measurement uncertainties (see Figure 10) and pollution with stars from a cooler population (see Figures 13 and 14) because of their higher intrinsic velocity dispersion (see Figure 19).

In addition, we find indications in Figure 16 that different regions within the Galaxy are best probed by populations of different kinematic temperature; the `hot` population gives the best constraints on the radial local and surface density profiles at a smaller radius than the `cool` population because of its smaller tracer scale length. The `cool` population with most stars close to the mid-plane recovers the flatness of the disk more reliably.

#### 4. SUMMARY AND DISCUSSION

Recently implementations of action DF-based modeling of 6D data in the Galactic disk have been put forth, in part to lay the ground-work for *Gaia* (BR13; McMillan & Binney 2013; Piffl et al. 2014; Sanders & Binney 2015).

We present *RoadMapping*, an improved implementation of the dynamical modeling machinery of BR13, to recover the MW’s gravitational potential by fitting an orbit DF to stellar populations within the Galactic disk. In this work we investigated the capabilities, strengths, and weaknesses of *RoadMapping* by testing its robustness against the breakdown of some of its assumptions—for well-defined, isolated test cases using mock data. Overall, the method works very well and is robust, even when there are small deviations of the model assumptions from the “real” Galaxy.

*RoadMapping* applies a full likelihood analysis and is statistically well-behaved. It goes beyond BR13 by allowing for a straightforward and flexible implementation of different model families for potential and DF. It also accounts for selection effects by using full 3D selection functions (given some symmetries).

##### 4.1. Computational Speed

Large data sets in the age of *Gaia* require increasingly accurate likelihood evaluations and flexible models. To be able to deal with these computational demands we sped up the *RoadMapping* code by combining a nested-grid approach with MCMC and by faster action calculation using the Stäckel (Binney 2012a) interpolation grid by Bovy (2015). Our

approach therefore allows us to explore the full *pdf*, while similar studies (e.g., Piffl et al. 2014; Sanders & Binney 2015; Das & Binney 2016) focus more on the model with maximum likelihood only. This also makes *RoadMapping* slower; fitting three DHB-Pot and five qDF parameters in each of the analyses in Tests 5 and 7 (see Table C1) takes, for example, ~25–30 hr on 25 CPUs. This is still a feasible computational effort as long as we restrict ourselves to potentials with a closed-form expression for  $\Phi(R, z)$  (as done in this work). An equivalent analysis using, e.g., a double exponential disk (requiring integrals over Bessel functions) would take several days to weeks for  $N_* = 20,000$ . In any case, the application of *RoadMapping* to millions of stars will be a task for supercomputers, and calls for even more improvements and speed-up in the fitting machinery.

##### 4.2. Properties of the Data Set

We could show that *RoadMapping* can provide potential and DF parameter estimates that are very accurate (i.e., unbiased) and precise in the limit of large data sets as long as the modeling assumptions are fulfilled.

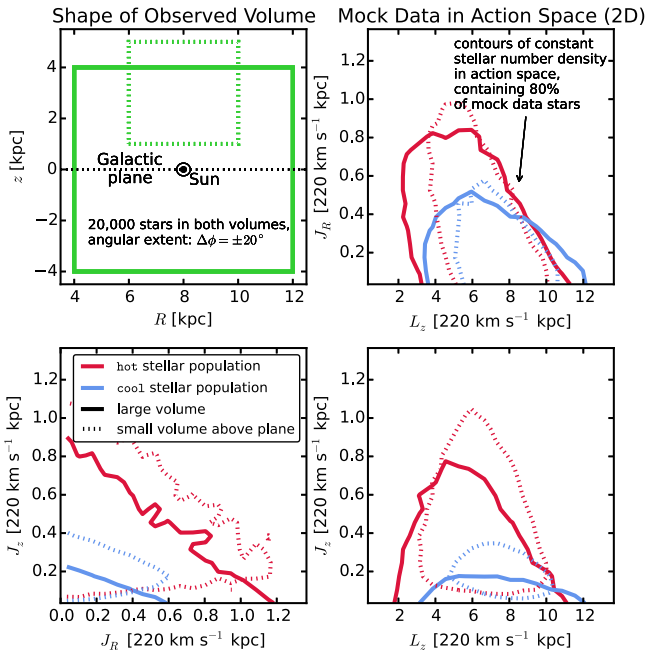
In case the data set is affected by substantive measurement uncertainties, the potential can still be recovered to high precision, as long as these uncertainties are perfectly known and distance uncertainties are negligible. For large proper motion uncertainties, e.g.,  $\delta\mu \sim 5 \text{ mas yr}^{-1}$ , the formal errors on the parameters are only twice as large as in the case of no measurement uncertainties. However, properly accounting for measurement uncertainties is computationally expensive.

For the results to be accurate within  $2\sigma$  (for 10,000 stars), we need to know to within 10% both the true stellar distances (at  $r_{\text{max}} \leq 3 \text{ kpc}$  and  $\delta\mu \lesssim 2 \text{ mas yr}^{-1}$ ) and the true proper motion uncertainties (with  $\delta\mu \lesssim 3 \text{ mas yr}^{-1}$ ).

The distance condition is an artifact of the likelihood approximation (Equation (16)) that *RoadMapping* uses to save computation time, and the reason why we will have to restrict the *RoadMapping* modeling to stars with small distance uncertainties.

Fortunately, the measurement uncertainties of the final *Gaia* data release with  $\delta\mu \lesssim 0.3 \text{ mas yr}^{-1}$  at  $G \lesssim 20 \text{ mag}$ , and  $\delta r/r \lesssim 5\%$  at  $r \sim 3 \text{ kpc}$  for  $G < 15 \text{ mag}$  (see Section 3.4 and de Bruijne et al. 2014) will be well below these limits and





**Figure 18.** Distribution of mock data in action space (2D iso-density contours, enclosing 80% of the stars) depending on shape and position of a wedge-like survey observation volume (upper left panel, see also Appendix B), and temperature of the stellar population (indicated in the legend). The four mock data sets are generated in the KKS-Pot from Table 1 from either the hot or cool DF in Table 2. The distribution in action space visualizes how orbits with different actions reach into different regions within the Galaxy. The corresponding mock data in configuration space is shown in Figure 19.

promise accurate potential constraints. Before the final *Gaia* data release, however, we might have to restrict the modeling to suitable giant tracers with small uncertainties.

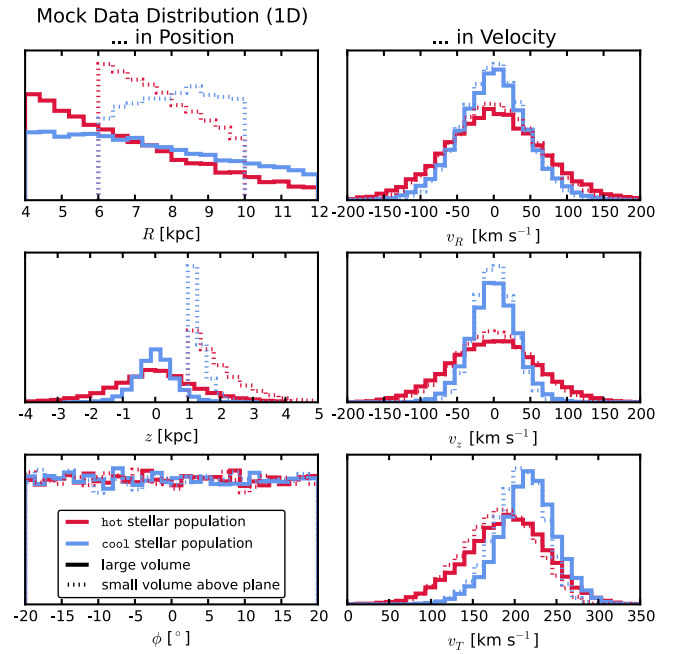
The main caveat of Tests 2 and 6.1–6.3 in Section 3.4 (see Table C1) concerning measurement uncertainties is the use of the Iso-Pot, which we chose for computational speed reasons. However, Tests 5 and 7, which we run for both DHB-Pot and Iso-Pot, gave qualitatively and quantitatively very similar results for both potentials. This makes us confident that also our results about measurement uncertainties are independent of the actual choice of potential.

We also found that the location of the survey volume within the Galaxy matters little. At a given sample size a larger survey volume with large coverage in both radial and vertical direction will give the tightest constraints on the model parameters.

The potential recovery with *RoadMapping* seems to be robust against minor misjudgements of the spatial data SF, in particular to a completeness overestimation of  $\lesssim 15\%$ – $20\%$  at the edge of a survey volume with  $r_{\max} = 3$  kpc.

We found indications that populations of different scale lengths and temperatures probe different regions of the Galaxy because the best potential constraints are achieved where most of the stellar tracers are located. This supports the approach by BR13, who measured for each MAP the surface mass density only at one single best radius to account for missing flexibility in their potential model.

While cooler populations probe the Galaxy rotation curve better and hotter populations are less sensitive to pollution, overall stellar populations of different kinematic temperature seem to be equally well suited for dynamical modeling.



**Figure 19.** Distribution of the mock data from Figure 18 in configuration space. The corresponding observation volumes (as indicated in the legend) are shown in Figure 18, upper left panel. The 1D histograms illustrate that qDFs generate realistic stellar distributions in Galactocentric coordinates ( $R, z, \phi, v_R, v_z, v_T$ ). More stars are found at smaller  $R$  and  $|z|$ , and are distributed uniformly in  $\phi$  according to our assumption of axisymmetry. The distribution in radial and vertical velocities,  $v_R$  and  $v_z$ , is approximately Gaussian with the (total projected) velocity dispersion being of the order of  $\sim \sigma_{R,0}$  and  $\sim \sigma_{z,0}$  (see Table 2). The distribution of tangential velocities,  $v_T$ , is skewed because of asymmetric drift.

#### 4.3. Deviations from the DF Assumption

*RoadMapping* assumes that stellar sub-populations can be described by simple DFs. We investigated how much the modeling would be affected if the assumed family of DFs would differ from the stars' true DF.

In Example 1 in Section 3.5 we considered true stellar DFs being (i) hot with more stars with low velocities and less stars at small radii than assumed (reddish data sets in Figures 11 and 13), or (ii) cool with broader velocity dispersion wings and less stars at large radii than assumed (bluish data sets). We find that case (i) would give more reliable results for the potential parameter recovery.

Binning of stars into MAPs in  $[\alpha/\text{Fe}]$  and  $[\text{Fe}/\text{H}]$ , as done by BR13, could introduce systematic errors due to abundance uncertainties or too large bin sizes—always assuming MAPs follow simple DF families (e.g., the qDF). In Example 2 in Section 3.5 we found that, in the case of 20,000 stars per bin, differences of  $\lesssim 20\%$  in the qDF parameters of two neighboring bins can still give quite good constraints on the potential parameters.

The relative differences in the qDF parameters  $\sigma_{R,0}$  and  $\sigma_{z,0}$  of neighboring MAPs in Figure 6 of BR13 (which have bin sizes of  $[\text{Fe}/\text{H}] = 0.1$  dex and  $\Delta[\alpha/\text{Fe}] = 0.05$  dex) are indeed smaller than 20%. For the  $h_R$  parameter, however, the bin sizes in Figure 6 of BR13 might not yet be small enough to ensure no more than 20% of difference in neighboring bins.

The qDF is a specific example for a simple DF for stellar sub-populations that we used in this paper, but it is not essential

for the *RoadMapping* approach. Future studies might apply slight alternatives or completely different DFs to the data.

#### 4.4. Gravitational Potential Beyond the Parameterized Functions Considered

In addition to the DF, *RoadMapping* also assumes a parametric model for the gravitational potential. We test how using a potential of Stäckel form (KKS-Pot, Batsleer & Dejonghe 1994) affects the *RoadMapping* analysis of mock data from a different potential family with halo, bulge, and exponential disk (MW14-Pot, Bovy 2015). The potential recovery is quite successful; we properly reproduce the mock data distribution in configuration space, and the best fit potential is—within the limits of the model—as close as it gets to the true potential, even outside of the observation volume of the stellar tracers.

For as many as 20,000 stars, constraints already become so tight that it should presumably be possible to distinguish between different parametric MW potential models (e.g., the DHB-Pot and the KKS-Pot).

Fitting parameterized potentials of Stäckel form to MW data (see, e.g., Batsleer & Dejonghe 1994; Famaey & Dejonghe 2003) has the advantage of allowing action calculations that are accurate and fast. It does, however, limit the space of potentials that can be investigated, as different potential components are all required to have the same focal distance. Using the *Stäckel fudge* (Binney 2012a) together with parameterized potentials made up from physically motivated building blocks (exponential disks, power-law dark matter halo etc.), as was done by BR13, seems to be the most promising approach—even though there still remain several challenges concerning computational speed to be solved.

#### 4.5. Different Modeling Approaches Using Action-Based DFs

BR13 focused on MAPs for a number of reasons. First, they seem to permit simple DFs (Bovy et al. 2012a, 2012b, 2012c), i.e., approximately qDFs (Ting et al. 2013). Second, all stars must orbit in the same potential. While each MAP can yield different DF parameters, it will also provide a (statistically) independent estimate of the potential. This allows for a valuable cross-checking reference. In some sense, the *RoadMapping* approach focuses on constraining the potential, treating the DF parameters as nuisance parameters. That we were able to show in this work that *RoadMapping* results are quite robust to the form of the DF not being entirely correct motivates this approach further.

Magorrian (2014) introduced a framework that avoids specific parameterizations of action-based DFs and marginalizes over all possible DFs to constrain the potential. While this is the proper way to treat a nuisance DF, it appears to be computationally very challenging.

For reasons of galaxy and chemical evolution, the DF properties are astrophysically linked between different MAPs (Sanders & Binney 2015). In its current implementation, *RoadMapping* treats all MAPs as independent and does not exploit such correlations. Ultimately, the goal is to do a consistent chemodynamical model that simultaneously fits the potential and DF( $\mathbf{J}$ ,  $[\text{X}/\text{H}]$ ) (where  $[\text{X}/\text{H}]$  is  $[\text{Fe}/\text{H}]$  and other elements either referenced to H or Fe, i.e.,  $[\text{X}/\text{H}]$  denotes the whole abundance space) with a full likelihood analysis. This

has not yet been attempted with *RoadMapping* because the behavior is quite complex.

Since the first application of *RoadMapping* by BR13 there have been two similar efforts by Piffl et al. (2014) and Sanders & Binney (2015) to constrain the Galactic potential and/or orbit DF for the disk.

Piffl et al. (2014) fitted both potential and a  $f(\mathbf{J})$  to giant stars from the RAVE survey (Steinmetz et al. 2006) and the vertical stellar number density profiles in the disk by Jurić et al. (2008). They did not include any chemical abundances in the modeling. Instead, they used a superposition of action-based DFs to describe the overall stellar distribution at once: a superposition of qDFs for cohorts in the thin disk, a single qDF for the thick disk stars, and an additional DF for the halo stars. Taking proper care of the selection function requires a full likelihood analysis, which is computationally expensive. Piffl et al. (2014) choose to circumvent this difficulty by directly fitting (i) histograms of the three velocity components in eight spatial bins to the velocity distribution predicted by the DF and (ii) the vertical density profile predicted by the DF to the profiles by Jurić et al. (2008). The vertical force profile of their best fit mass model nicely agrees with the results from BR13 for  $R > 6.6$  kpc. The disadvantage of their approach is that by binning the stars spatially, a lot of information is not used.

Sanders & Binney (2015) focused on understanding the abundance-dependence of the DF, relying on a fiducial potential. They developed extended distribution functions (eDF), i.e., functions of both actions and metallicity for a superposition of thin and thick disks, each consisting of several cohorts described by qDFs, a DF for the halo, a functional form of the metallicity of the interstellar medium at the time of birth of the stars, and a simple prescription for radial migration. They applied a full likelihood analysis accounting for selection effects and found a best fit for the eDF in the fixed fiducial potential by Dehnen & Binney (1998) to the stellar phase-space data of the Geneva-Copenhagen Survey (Nordström et al. 2004; Holmberg et al. 2009), metallicity determinations by Casagrande et al. (2011), and the stellar density curves by Gilmore & Reid (1983). Their best fit predicted the velocity distribution of SEGUE G-dwarfs (Ahn et al. 2014) quite well, but had biases in the metallicity distribution, which they attributed to being a problem with the SEGUE metallicities.

Das & Binney (2016) proceeded recently in a similar fashion to constrain an eDF for halo stars.

#### 4.6. Future Work

We know that real galaxies, including the MW, are not axisymmetric. Using  $N$ -body models, we will explore in a subsequent paper how the recovery of the gravitational potential with *RoadMapping* will be affected when data from a non-axisymmetric disk galaxy system with spiral arms gets interpreted through axisymmetric models. There are several interesting scientific questions for which a *RoadMapping* investigation of galaxy simulations could be a pragmatic approach to address them: (i) What is the influence of spiral arms and resonances on the modeling outcome? (ii) Can we recover the potential well enough to calculate actions so accurate that clumps in orbit space can be identified? This is important to be able to compare clumps in action space to clustering of stars in abundance space. (iii) How do results

from *RoadMapping*, i.e., the potential and DF, compare with results from Jeans models?

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## APPENDIX A MOCK DATA

The mock data in this work is generated according to the following procedure. We assume that the positions and velocities of our stellar mock sample are indeed drawn from our assumed family of potentials (Section 2.3) and DFs (Section 2.4), with given parameters  $p_\Phi$  and  $p_{\text{DF}}$ . The DF is in terms of actions, while the transformation  $(\mathbf{x}_i, \mathbf{v}_i) \xrightarrow{\Phi} \mathbf{J}_i$  is computationally much less expensive than its inversion. We therefore employ the following efficient two-step method for creating mock data, which also accounts for a spatial survey selection function  $\text{SF}(\mathbf{x})$  (see Appendix B).

In the first step we draw stellar positions  $\mathbf{x}_i$ . We start by setting up the interpolation grid for the tracer density  $\rho(R, |z| | p_\Phi, p_{\text{DF}})$  generated according to Section 2.4.<sup>11</sup> Next, we sample random positions  $(R_i, z_i, \phi_i)$  uniformly within the observable volume. Using an MC rejection method we then shape the samples distribution to follow  $\rho(R, |z| | p_\Phi, p_{\text{DF}})$ . To apply a non-uniform completeness function, we use the rejection method a second time. The resulting set of positions,  $\mathbf{x}_i$ , follows the distribution  $p(\mathbf{x}) \propto \rho_{\text{DF}}(R, |z| | p_\Phi, p_{\text{DF}}) \times \text{SF}(\mathbf{x})$ .

In the second step we draw velocities,  $\mathbf{v}_i$ . For each of the positions  $(R_i, z_i)$  we first sample velocities from a Gaussian envelope function in velocity space which is then shaped toward DF ( $\mathbf{J}[R_i, z_i, \mathbf{v} | p_\Phi] | p_{\text{DF}}$ ) using a rejection method. We now have a mock data set satisfying  $(\mathbf{x}_i, \mathbf{v}_i) \rightarrow p(\mathbf{x}, \mathbf{v}) \propto \text{DF}(\mathbf{J}[\mathbf{x}, \mathbf{v} | p_\Phi] | p_{\text{DF}}) \times \text{SF}(\mathbf{x})$ .

Measurement uncertainties can be added to the mock data by applying the following modifications to the above procedure. We assume Gaussian uncertainties in the heliocentric phase-space coordinates  $\tilde{\mathbf{x}} = (\text{R.A.}, \text{decl.}, (m - M))$ ,  $\tilde{\mathbf{v}} = (\mu_{\text{R.A.}} \cdot \cos(\text{decl.}), \mu_{\text{decl.}}, v_{\text{los}})$  (see Section 2.1). In the case of distance and position uncertainties, stars virtually scatter in and out of the observed volume. To account for this, we draw the *true*  $\mathbf{x}_i$  from a volume that is larger than the actual observation volume, perturb the  $\mathbf{x}_i$  according to the position uncertainties and then reject all stars that now lie outside of the observed volume. This mirrors the random scatter around the detection threshold for stars whose distances are determined from the apparent brightness and the distance modulus. We then sample *true*  $\mathbf{v}_i$  (given the *true*  $\mathbf{x}_i$ ) as described above and perturb them according to the velocity uncertainties.

We show examples of mock data sets (without measurement uncertainties) in action space (Figure 18) and configuration

space  $(\mathbf{x}, \mathbf{v})$  (Figure 19). The mock data generated from the qDF follow the expected distributions in configuration space. The distribution in action space illustrates the intuitive physical meaning of actions. The stars of the `c001` population have, in general, lower radial and vertical actions, as they are on more circular orbits. Circular orbits with  $J_R = 0$  and  $J_z = 0$  can only be observed in the Galactic mid-plane. The different ranges of angular momentum,  $L_z$ , in the two example observation volumes reflect  $L_z \sim R \times v_{\text{circ}}$  and the volumes’ different radial extent. The volume at larger  $z$  contains stars with higher  $J_z$ . An orbit with  $L_z \ll$  or  $\gg L_z(R_\odot)$  can only reach into a volume at  $\sim R_\odot$ , if it is more eccentric and has therefore larger  $J_R$ . This, together with the effect of asymmetric drift, explains the asymmetric distribution of  $J_R$  versus  $L_z$  in Figure 18.

## APPENDIX B SELECTION FUNCTIONS

Any survey’s selection function (SF) can be understood as defining an effective sample sub-volume in the space of observables, e.g., position on the sky (limited by the pointing of the survey), distance from the Sun (limited by brightness and detector sensitivity), colors, and metallicity of the stars (limited by survey mode and targeting). The SF can therefore be thought of as having both a spatial small scale structure (due to pencil beam pointing, dust obscuration, etc.) and some overall spatial characteristics (e.g., mean height above the plane and mean Galactocentric radius of the stars). The treatment of realistic and complex SFs was already demonstrated in BR13 (who used the pencil beam SF of the SEGUE survey; Bovy et al. 2012c) and Bovy et al. (2016) (who investigated the effect of dust extinction). In this work we aim to make a generic and basic exploration of search volume shapes and, as shown by Bovy et al. (2016), this should be possible without explicitly considering the spatial SF substructure. Inspired by the contiguous nature of the *Gaia* SF, which is basically only limited by a magnitude cut, and the fact that this magnitude cut would—in the absence of small scale structure—translate to a sharp distance cut for standard candle tracer populations like red clump stars, we therefore use in our modeling a simple spatial SF of spherical shape with radius  $r_{\text{max}}$  around the Sun,

$$\text{SF}(\mathbf{x}) \equiv \begin{cases} \text{completeness}(\mathbf{x}) & \text{if } |\mathbf{x} - \mathbf{x}_\odot| \leq r_{\text{max}}, \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

We set  $0 \leq \text{completeness}(\mathbf{x}) \leq 1$  everywhere inside the observed volume, so it can be understood as a position-dependent detection probability for a star at  $\mathbf{x}$ . Unless explicitly stated otherwise, we simplify to  $\text{completeness}(\mathbf{x}) = 1$ . Additionally, we use in Figure 6 (Test 4 in Table C1) and Figures 18–19 for illustrative purposes some rather unrealistic survey volumes which are angular segments of a cylindrical annulus (wedge), i.e., the volume with  $R \in [R_{\text{min}}, R_{\text{max}}]$ ,  $\phi \in [\phi_{\text{min}}, \phi_{\text{max}}]$ ,  $z \in [z_{\text{min}}, z_{\text{max}}]$  within the model Galaxy.

## APPENDIX C SUMMARY OF TEST SUITES

Table C1 lists all the different tests in this work and summarizes the model parameters, figures and results.

<sup>11</sup> For the creation of the mock data we use  $N_x = 20$ ,  $N_v = 40$  and  $n_\sigma = 5$  in Equation (8).

**Table C1**  
Summary of Test Suites in this Work.

| Test  |  | Model for Mock Data   | Model in Analysis  | Figures and Results   |
|---|--|---|--|---|
| Test 1 :<br>Numerical accuracy<br>in calculating<br>the likelihood normalization                | <i>Potential:</i><br><i>DF:</i><br><i>Survey volume:</i><br><i>Numerical accuracy:</i>   | DHB-Pot<br>hot or cool qDF<br>sphere around Sun, $r_{\max} = 0.2, 1, 2, 3$ or 4 kpc<br>$N_x \in [5, 32]$ , $N_y \in [4, 48]$ , $n_\sigma \in [3, 7]$  |  | Figure 2<br>Suitable accuracy for our tests:<br>$N_x = 16$ , $N_y = 24$ , $n_\sigma = 5$ .<br>Higher spatial resolution is<br>required for cooler populations.  |
| Test 2 :<br>Numerical convergence<br>of convolution<br>with measurement<br>uncertainties        | <i>Potential:</i><br><i>DF:</i><br><i>Survey Volume:</i><br><i>Uncertainties:</i><br><br><i>Numerical accuracy:</i><br>$N_*$ : | Iso-Pot<br>hot qDF<br>sphere around Sun, $r_{\max} = 3$ kpc<br>$\delta R. A. = \delta \text{decl.} = \delta(m - M) = 0$<br>$\delta v_{\text{los}} = 2 \text{ km s}^{-1}$<br>$\delta \mu = 2, 3, 4$ or $5 \text{ mas yr}^{-1}$<br><br>10,000 | Iso-Pot, all parameters free<br>qDF, all parameters free<br>(fixed and known)<br>(fixed and known)<br><br>$N_{\text{samples}} \in [25, 1200]$  | Figure 3<br>The number of required MC samples<br>scales as $N_{\text{samples}} \propto (\delta v_{\max})^2$ ,<br>with $\delta v_{\max}$ being the largest $\delta v$<br>in the data set. If $N_*$ is smaller,<br>less accuracy, i.e., less $N$ ,<br>is needed to reach a given<br>accuracy. |
| Test 3.1:<br>Shape of the model<br>parameters' <i>pdf</i><br>for large data sets                | <i>Potential:</i><br><i>DF:</i><br><i>Survey Volume:</i><br>$N_*$ :  | Iso-Pot<br>hot qDF<br>sphere around Sun, $r_{\max} = 2$ kpc<br>20,000   | Iso-Pot, all parameters free<br>qDF, all parameters free<br>(fixed and known)  | Figure 4<br>The <i>pdf</i> becomes a multivariate<br>Gaussian in the limit of<br>large data. The width of the<br><i>pdf</i> scales as $1/\sqrt{N_*}$ .  |
| Test 3.2 :<br>Parameter estimates<br>are unbiased;<br>Influence of survey<br>volume size        | <i>Potential:</i><br><i>DF:</i><br><br><i>Survey volume:</i><br>$N_*$ :  | Iso-Pot<br>hot or cool qDF<br><br>sphere around Sun, $r_{\max} = 0.2, 1, 2, 3$ or 4 kpc<br>20,000   | Iso-Pot, free parameter: $b$<br>qDF, free parameters:<br>$\ln h_R$ , $\ln \sigma_{z,0}$ , $\ln h_{\sigma,z}$<br>(fixed and known)  | Figure 5<br><i>RoadMapping</i> behaves like an<br>unbiased maximum likelihood<br>estimator. Larger survey volumes<br>lead to tighter constraints,<br>even at the same $N_*$ .   |
| Test 4 :<br>Influence of<br>position and shape<br>of survey volume<br>on parameter recovery     | <i>Potential:</i><br><br><i>DF:</i><br><i>Survey volume:</i><br>$N_*$ :  | (i) Iso-Pot or (ii) DHB-Pot<br><br>hot qDF<br>four different wedges, see Figure 6, upper panel<br>20,000  | (i) Iso-Pot, all parameters free<br>(ii) DHB-Pot, free parameters:<br>$v_{\text{circ}}(R_\odot)$ , $a_{\text{disk}}$ , $f_{\text{halo}}$<br>qDF, all parameters free<br>(fixed and known)              | Figure 6<br>The exact position and shape of the<br>survey volume plays only a<br>minor role. Having both large<br>radial and vertical extent should<br>give the tightest constraints.   |
| Test 5 :<br>Influence of<br>wrong assumptions<br>about the spatial SF<br>on parameter recovery  | <i>Potential:</i><br><br><i>DF:</i><br><i>Survey volume:</i><br><i>Completeness:</i><br><br>$N_*$ :                            | DHB-Pot<br><br>hot or cool qDF<br>sphere around Sun, $r_{\max} = 3$ kpc<br>Equation (17)<br>with $\epsilon_r \in [0, 0.7]$<br>20,000  | DHB-Pot, free parameters:<br>$v_{\text{circ}}(R_\odot)$ , $a_{\text{disk}}$ , $f_{\text{halo}}$<br>qDF, all parameters free<br>(fixed and known)<br>completeness( $x$ ) = 1,<br>i.e., $\epsilon_r = 0$ | Figures 7<br>For minor misjudgements<br>of a radially symmetric SF<br>(i.e., $\epsilon_r \lesssim 0.15$ for hot and<br>$\epsilon_r \lesssim 0.2$ for cool populations)<br>the potential recovery<br>is still robust.  |
| Test 6.1 :<br>Effect of proper motion<br>uncertainties on<br>precision of<br>potential recovery | <i>Potential:</i><br><i>DF:</i><br><i>Survey volume:</i><br><i>Uncertainties:</i>  | Iso-Pot<br>hot or cool qDF<br>sphere around Sun, $r_{\max} = 3$ kpc<br>(i) $\delta R. A. = \delta \text{decl.} = \delta(m - M) = 0$ ,<br>$\delta v_{\text{los}} = 2 \text{ km s}^{-1}$ ,  | Iso-Pot, all parameters free<br>qDF, all parameters free<br>(fixed and known)<br>(fixed and known)   | Figure 8<br>If $\delta \mu$ is perfectly known, the<br>precision of the potential para-<br>meter recovery is only a<br>factor $\sim 1.15$ –2 worse  |



**Table C1**  
(Continued)

| Test  |   | Model for Mock Data  | Model in Analysis   | Figures and Results  |
|---|---|--|---|--|
|   | $N_*$ :   | $\delta\mu = 1, 2, 3, 4$ or $5 \text{ mas yr}^{-1}$<br>(ii) no measurement uncertainties<br>10,000   |   | for $\delta\mu \sim 1\text{--}5 \text{ mas yr}^{-1}$<br>than for a data set without<br>proper motion uncertainties.  |
| Test 6.2 :<br>Testing the convolution<br>with measurement<br>uncertainties in Equation (16)<br>with and without<br>distance uncertainties | <i>Potential:</i><br><i>DF:</i><br><i>Survey Volume:</i><br><i>Uncertainties:</i><br><br>$N_*$ :      | Iso-Pot<br>hot qDF<br>sphere around Sun, $r_{\text{max}} = 3 \text{ kpc}$<br>$\delta R$ . A. $= \delta \text{decl.} = 0$ ,<br>$\delta v_{\text{los}} = 2 \text{ km s}^{-1}$ ,<br>$\delta\mu = 1, 2, 3, 4$ , or $5 \text{ mas yr}^{-1}$ ,<br>(i) $\delta(m - M) = 0$ or<br>(ii) $\delta(m - M) \neq 0$ (see Figure 9)<br>10,000   | Iso-Pot, all parameters free<br>qDF, all parameters free<br>(fixed and known)<br>(fixed and known)  | Figure 9<br>The approximate likelihood<br>in Equation (16) is the true<br>likelihood in the absence of<br>position errors. For distance<br>uncertainties $\lesssim 10\%$ at<br>$r_{\text{max}} \sim 3 \text{ kpc}$ , which is the case<br>for <i>Gaia</i> and $G \lesssim 15$ , the intro-<br>duced bias is still less than $2\sigma$ .  |
| Test 6.3:<br>Underestimation<br>of proper motion<br>uncertainties   | <i>Potential:</i><br><i>DF:</i><br><i>Survey volume:</i><br><i>Uncertainties:</i><br><br>$N_*$ :      | Iso-Pot<br>hot or cool qDF<br>sphere around Sun, $r_{\text{max}} = 3 \text{ kpc}$<br>only proper motion uncertainties<br>1, 2, or $3 \text{ mas yr}^{-1}$<br>10,000  | Iso-Pot, all parameters free<br>qDF, all parameters free<br>(fixed and known)<br>proper motion uncertainties<br>10% or 50% underestimated   | Figure 10<br>The bias introduced by under-<br>estimating $\delta\mu$ grows with $\delta\mu$ .<br>If $\delta\mu \lesssim 3 \text{ mas yr}^{-1}$ and $\delta\mu$<br>is underestimated by 10%<br>the bias is only $\sim 2\sigma$ or less.   |
| Test 7 :<br>Deviations of the<br>assumed DF<br>from the<br>stars' true DF   | <i>Potential:</i><br><br><i>DF:</i><br><br><br><br><br><br><br><br><i>Survey volume:</i><br>$N_*$ :   | DHB-Pot<br><br>mix of two qDFs...<br>(i) Example 1:<br>... with different mixing rates<br>and fixed qDF parameters<br>(hot and cooler qDF from Table 2)<br>(ii) Example 2:<br>... with 50/50 mixing rate<br>and varying qDF parameters (by $X\%$ ):<br>a) hot and colder qDF or<br>b) cool and warmer qDF (see Table 2)<br>sphere around Sun, $r_{\text{max}} = 2 \text{ kpc}$<br>20,000 | DHB-Pot, free parameters:<br>$v_{\text{circ}}(R_{\odot})$ , $a_{\text{disk}}$ , $f_{\text{halo}}$<br>single qDF, all parameters free<br><br>(fixed and known)   | Figures 11–14<br>Example 1:<br>Hot qDF-like populations polluted<br>by up to $\sim 30\%$ of stars from a much<br>cooler population give still reliable<br>potential constraints. This is not<br>true for the opposite case.<br>Example 2:<br>Differences of $\lesssim 20\%$ in qDF<br>parameters do not matter<br>when mixing sub-populations,<br>e.g., due to finite binning in<br>abundance space and abundance<br>Errors. |
| Test 8 :<br>Deviations of the<br>assumed potential model<br>from the stars'<br>true potential   | <i>Potential:</i><br><br><i>DF:</i><br><i>Survey volume:</i><br>$N_*$ :<br><i>Numerical accuracy:</i> | MW14-Pot<br><br>hot or cool qDF<br>sphere around Sun, $r_{\text{max}} = 4 \text{ kpc}$<br>20,000   | KKS-Pot, all parameters free,<br>only $v_{\text{circ}}(R_{\odot}) = 230 \text{ km s}^{-1}$ fixed<br>qDF, all parameters free<br>(fixed and known)<br><br>$N_k = 20$ , $N_v = 28$ , $n_{\sigma} = 5.5$ | Figures 15–17<br>We find a good approximation<br>for the true potential, especially<br>where most stars are located,<br>given the limitations of the wrong<br>assumed potential model family.  |

**Note.** Reference potentials and qDFs are introduced in Tables 1 and 2, respectively. Parameters that are not left free in the analysis are always fixed to their true value. Unless stated otherwise, all mock data sets have SFs with completeness  $(x) = 1$  and no measurement uncertainties, and we use  $N_k = 16$ ,  $N_v = 24$ ,  $n_{\sigma} = 5$  as the numerical accuracy for calculating the likelihood normalization (see Section 2.6). The first column indicates the test suite; the second column the potential, DF, and SF model, etc., used for the mock data creation; the third column the corresponding model assumed in the *RoadMapping* analysis; and the last column lists the figures belonging to the test suite and summarizes the results

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