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Numerical Test of Analytical Theories for Perpendicular Diffusion in Small Kubo Number Turbulence

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Abstract

In the literature, one can find various analytical theories for perpendicular diffusion of energetic particles interacting with magnetic turbulence. Besides quasi-linear theory, there are different versions of the nonlinear guiding center (NLGC) theory and the unified nonlinear transport (UNLT) theory. For turbulence with high Kubo numbers, such as two-dimensional turbulence or noisy reduced magnetohydrodynamic turbulence, the aforementioned nonlinear theories provide similar results. For slab and small Kubo number turbulence, however, this is not the case. In the current paper, we compare different linear and nonlinear theories with each other and test-particle simulations for a noisy slab model corresponding to small Kubo number turbulence. We show that UNLT theory agrees very well with all performed test-particle simulations. In the limit of long parallel mean free paths, the perpendicular mean free path approaches asymptotically the quasi-linear limit as predicted by the UNLT theory. For short parallel mean free paths we find a Rechester & Rosenbluth type of scaling as predicted by UNLT theory as well. The original NLGC theory disagrees with all performed simulations regardless what the parallel mean free path is. The random ballistic interpretation of the NLGC theory agrees much better with the simulations, but compared to UNLT theory the agreement is inferior. We conclude that for this type of small Kubo number turbulence, only the latter theory allows for an accurate description of perpendicular diffusion.

Key words: cosmic rays – diffusion – magnetic fields – turbulence

1. Introduction

The problem of particle diffusion across a large-scale magnetic field is well known in plasma physics and astrophysics. Particles diffuse in the perpendicular direction because of complicated interactions between the charged energetic particle and turbulent magnetic fields. Apart from the problem that the properties of those magnetic fields are not fully known, the interaction process between particles and fields itself is very complicated. The first attempt to resolve this matter was the development of quasi-linear theory (see Jokipii 1966). This theory describes perpendicular diffusion as a process where the particles follow stochastic magnetic field lines while they move with constant velocity in the parallel direction.

Quasi-linear theory, however, does usually not agree with test-particle simulations (see, e.g., Qin et al. 2002a, 2002b and Shalchi 2009 for a review). Therefore, different nonlinear theories have been developed in the past (see, e.g., Owens 1974). A breakthrough was achieved in Matthaeus et al. (2003) where the nonlinear guiding center (NLGC) theory was developed. The latter theory agrees well with simulations as long as a slab/2D composite turbulence model with a dominant two-dimensional component is considered. For slab turbulence, however, one expects to find subdiffusive behavior of perpendicular transport (see, e.g., Jokipii et al. 1993; Jones et al. 1998 and Qin et al. 2002a), whereas NLGC theory provides a finite diffusion coefficient corresponding to normal Markovian transport (see Shalchi 2009; Tautz & Shalchi 2011).

Because of the problems of previous linear and nonlinear theories, the unified nonlinear transport (UNLT) theory was presented in Shalchi (2010). This theory is based on some ideas used before in quasi-linear and nonlinear treatments, but it uses the propagator of the pitch-angle-dependent cosmic-ray Fokker–Planck equation, albeit in an indirect way. UNLT theory provides a nonlinear integral equation for the perpendicular diffusion coefficient, which, in turn, contains different asymptotic limits (see Shalchi 2015). The theory contains quasi-linear theory as well as the nonlinear field line diffusion theory of Matthaeus et al. (1995) as special limits. For slab turbulence, the theory provides a vanishing diffusion parameter as expected.

According to Shalchi (2015), the perpendicular diffusion coefficient based on UNLT theory depends on two parameters. The first one is the ratio of the parallel mean free path of the particle $\lambda_\parallel$ and the parallel correlation scale of the turbulence $l_\parallel$. The second parameter is the *Kubo number* defined as

$$K := \frac{l_\parallel}{l_\perp}\frac{\delta B_x}{B_0}. \tag{1}$$

Here, we have used the parallel and perpendicular correlation scales of the turbulence[1], the x-component of the turbulent magnetic field, and the mean field. According to UNLT theory, the quasi-linear limit can be found for long parallel mean free paths and small Kubo numbers. For other extreme values of $\lambda_\parallel/l_\parallel$ and K, one finds different limits (see below and Shalchi 2015 for more details).

In more recent years, alternative theories for perpendicular transport have been developed. A *Random Ballistic Interpretation of the Nonlinear Guiding Center Theory* was proposed in Ruffolo et al. (2012). In Shalchi (2016), a theory for perpendicular diffusion in two-component turbulence was

[1] In the current paper, we use the term *correlation scales* for characteristic scales in the turbulence spectrum. More specifically, the parameters $l_\parallel$ and $l_\perp$ are the bendover and cutoff scales in the used model spectra. Therefore, these two parameters are not equal to the integral scales (see, e.g., Matthaeus et al. 2007 for more details) often used in the definition of the Kubo number.

presented that takes into account the implicit contribution of the slab modes.

It is the purpose of the current paper to perform test-particle simulations for small Kubo number turbulence. We compare our simulations with the following analytical theories

1. Quasi-linear theory of Jokipii (1966);
2. Original NLGC theory of Matthaeus et al. (2003);
3. UNLT theory of Shalchi (2010); and
4. The Random Ballistic Interpretation of the Nonlinear Guiding Center Theory developed in Ruffolo et al. (2012).

The remainder of this paper is organized as follows. In Section 2, we discuss the noisy slab model that is an extension of the usual slab model. This model corresponds to turbulence with a small Kubo number. In Section 3, we briefly discuss the four analytical theories used in the current paper. In Section 4, we perform test-particle simulations and compare them with the aforementioned analytical theories. In Section 5, we summarize and conclude.

2. The Noisy Slab Turbulence Model

A simple model for magnetic turbulence is the *slab model* where all wave vectors are assumed to be parallel with respect to the mean field. It is well known that perpendicular transport is subdiffusive in this case (see, e.g., Jokipii et al. 1993; Jones et al. 1998; Qin et al. 2002a; Shalchi 2005). Of course, this subdiffusive behavior only occurs if the parallel mean free path is finite. In the asymptotic limit $\lambda_\parallel \to \infty$, the usual quasi-linear scaling would be obtained. For a finite parallel mean free path, we find a transport process which is usually called *compound diffusion* (see, e.g., Kóta & Jokipii 2000; Webb et al. 2006; Shalchi & Kourakis 2007 for details).

In the current paper, we use a broadened slab model so that diffusion is restored. The broadening of a turbulence model with reduced dimensionality was discussed before in Weinhorst & Shalchi (2010), as well as in Ruffolo & Matthaeus (2013). Physically, this corresponds to the case where wave vectors are mainly oriented parallel with respect to the mean field, but weak fluctuations are taken into account. We refer to this model as the *noisy slab model*, and we define it via the magnetic correlation tensor

$$P_{mn}(\boldsymbol{k}) = \frac{2 l_\perp}{k_\perp} g(k_\parallel) \Theta(1 - k_\perp l_\perp) \left(\delta_{mn} - \frac{k_m k_n}{k_\perp^2} \right), \tag{2}$$

where we have used the *Kronecker delta*, the *Heaviside step function* $\Theta(x)$, and we have used cylindrical coordinates for the wave vector. They are related to Cartesian coordinates via $k_x = k_\perp \cos(\Psi)$, $k_y = k_\perp \sin(\Psi)$, and $k_z = k_\parallel$. The model described here was originally introduced in particle diffusion theory in Shalchi (2015). For the spectrum we employ

$$g(k_\parallel) = \frac{1}{2\pi} C(s) \delta B^2 l_\parallel [1 + (k_\parallel l_\parallel)^2]^{-s/2}, \tag{3}$$

as suggested by Bieber et al. (1994). The normalization function

$$C(s) = \frac{\Gamma(s/2)}{2\sqrt{\pi}\,\Gamma((s-1)/2)} \tag{4}$$

depends on the inertial range spectral index s and Gamma functions. Spectrum (3) is flat in the energy range defined via $k_\parallel l_\parallel \ll 1$. For $k_\parallel l_\parallel \gg 1$ we find $\sim (k_\parallel l_\parallel)^{-s}$, where it is usually assumed that $s = 5/3$ (see Kolmogorov 1941). This part of the spectrum is known as the inertial range. The parameter δB is the total turbulent magnetic field satisfying $\delta B^2 = \delta B_x^2 + \delta B_y^2$. The noisy slab model discussed here contains two length scales. The parameter $l_\parallel$ denotes the usual bendover scale in the parallel direction as used in the standard slab model. The parameter $l_\perp$ corresponds to the correlation scale across the mean magnetic field. The usual slab model used before in the literature can be recovered by the limiting process $l_\perp \to \infty$.

3. Different Analytical Theories for Perpendicular Transport

In this section, we discuss four analytical theories for perpendicular diffusion. We do not present the mathematical details of those theories, but rather briefly discuss the assumptions and approximations used in such theories, and then we show the results in the following sections.

3.1. Quasi-linear Theory

Quasi-linear theory can be understood as a first-order perturbation theory. If a diffusion coefficient is calculated based on that theory, one uses unperturbed orbits in fundamental formulas. For magnetostatic turbulence the quasi-linear perpendicular diffusion coefficient is given by (see Jokipii 1966 for the original derivation or Shalchi 2009 for more details)

$$\kappa_\perp = \frac{\pi}{2} \frac{v}{B_0^2} \int d^3k \; P_{xx}(\boldsymbol{k}) \delta(k_\parallel), \tag{5}$$

where we have used again *Dirac's delta*. If this formula is combined with the noisy slab model represented by Equation (2) and with spectrum (3), this becomes

$$\kappa_\perp = \frac{\pi}{2} C(s) l_\parallel v \frac{\delta B^2}{B_0^2} \tag{6}$$

or, in terms of mean free paths[2],

$$\frac{\lambda_\perp}{l_\parallel} = \frac{3\pi}{2} C(s) \frac{\delta B^2}{B_0^2}. \tag{7}$$

The latter formula agrees perfectly with the quasi-linear perpendicular mean free path derived before for slab turbulence (see, e.g., Equation (3.48) of Shalchi 2009). Therefore, no new effect can be observed if the quasi-linear approach is applied. The mathematical reason for that is the *Dirac delta* in Equation (5). Very characteristic for the quasi-linear formula is that the perpendicular mean free path does not depend on the parallel mean free path or particle energy. Also characteristic of Equation (7) is the scaling $\lambda_\perp \propto \delta B^2 / B_0^2$.

3.2. The Nonlinear Guiding Center Theory

It was shown before that quasi-linear theory is incomplete if it comes to the calculation of the perpendicular diffusion coefficient (see, e.g., Shalchi 2009 for a review). Because of the problems associated with the application of quasi-linear theory,

[2] The parallel and perpendicular mean free paths are related to the corresponding spatial diffusion coefficients via $\lambda_\parallel = 3\kappa_\parallel / v$ and $\lambda_\perp = 3\kappa_\perp / v$.

nonlinear transport theories have been developed. Matthaeus et al. (2003), for instance, have proposed the Nonlinear Guiding Center (NLGC) theory. The latter theory is based on several ad hoc assumptions and approximations. One of those is to replace fourth-order correlations by a product of two second-order correlations:

$$\langle v_z(t) v_z(0) \delta B_x[\mathbf{x}(t)] \delta B_x^*[\mathbf{x}(0)] \rangle \approx \langle v_z(t) v_z(0) \rangle \langle \delta B_x[\mathbf{x}(t)] \delta B_x^*[\mathbf{x}(0)] \rangle. \quad (8)$$

It was argued analytically that the latter approximation is incorrect for slab and small Kubo number turbulence (see, e.g., Shalchi 2005; Shalchi 2010). Furthermore, it was shown numerically by employing test-particle simulations that approximation (8) is only accurate for two-dimensional dominated turbulence and breaks down for slab-dominated turbulence (see Qin & Shalchi 2016).

By employing Equation (8) together with other approximations and assumptions, Matthaeus et al. (2003) derived the following nonlinear integral equation for the perpendicular diffusion coefficient:

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \int d^3k \, \frac{P_{xx}(\mathbf{k})}{v/\lambda_\parallel + \kappa_\perp k_\perp^2 + \kappa_\parallel k_\parallel^2}. \quad (9)$$

Here, we have introduced the *correction factor* a^2 as was done in Matthaeus et al. (2003). In that paper, it was suggested that $a^2 = 1/3$.

If one combines Equation (9) with the noisy slab model represented by Equation (2), we find after straightforward algebra

$$\kappa_\perp = \frac{2\pi l_\perp a^2 v^2}{3B_0^2} \int_0^{l_\perp^{-1}} dk_\perp \times \int_{-\infty}^{+\infty} dk_\parallel \, \frac{g(k_\parallel)}{v/\lambda_\parallel + \kappa_\perp k_\perp^2 + \kappa_\parallel k_\parallel^2}. \quad (10)$$

The $k_\perp$-integral can be expressed by an arctan function, and we obtain

$$\kappa_\perp^2 = \frac{2\pi l_\perp^2 a^2 v^2}{3B_0^2} \int_{-\infty}^{+\infty} dk_\parallel \, g(k_\parallel) \frac{\arctan(1/\alpha)}{\alpha} \quad (11)$$

with

$$\alpha = \left[\frac{3 l_\perp^2}{\lambda_\parallel \lambda_\perp} + \frac{\lambda_\parallel l_\perp^2}{\lambda_\perp l_\parallel^2} (l_\parallel k_\parallel)^2 \right]^{1/2}. \quad (12)$$

Now we employ spectrum (3) and the integral transformation $x = l_\parallel k_\parallel$ to derive

$$\left(\frac{\lambda_\perp}{l_\parallel}\right)^2 = 6C(s) a^2 \frac{l_\perp^2}{l_\parallel^2} \frac{\delta B^2}{B_0^2} \times \int_0^\infty dx \, (1 + x^2)^{-s/2} \frac{1}{\alpha(x)} \arctan\left[\frac{1}{\alpha(x)}\right]. \quad (13)$$

We like to emphasize that this is an implicit expression for the perpendicular mean free path because α depends on $\lambda_\perp$. The remaining integral in Equation (13) can easily be evaluated numerically. This is done in Section 4 where we compare original NLGC results with test-particle simulations and other theories.

We like to point out that in the limit $\lambda_\parallel \to 0$, NLGC theory predicts

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{a^2}{2} \frac{\delta B^2}{B_0^2}, \quad (14)$$

corresponding to a constant ratio of the perpendicular and parallel mean free path.

3.3. The UNLT Theory

According to Shalchi (2010), the original NLGC theory fails in the general case because approximation (8) is not valid. This is particularly the case for slab and small Kubo number turbulence. The latter statement was confirmed numerically in Qin & Shalchi (2016). Based on the pitch-angle-dependent cosmic-ray Fokker–Planck equation, Shalchi (2010) developed a nonlinear theory that no longer requires approximation (8). The following nonlinear integral equation has been found after lengthy algebra:

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2} \int d^3k \, \frac{P_{xx}(\mathbf{k})}{v/\lambda_\parallel + (4/3)\kappa_\perp k_\perp^2 + F(k_\parallel, k_\perp)} \quad (15)$$

with $F(k_\parallel, k_\perp) = (v^2 k_\parallel^2)/(3\kappa_\perp k_\perp^2)$. One can easily see that for two-dimensional turbulence, Equation (15) agrees with Equation (9) apart from the factor $4/3$ in the denominator. From Equation (15) one can also derive the Matthaeus et al. (1995) theory for field line random walk by considering the limit $\lambda_\parallel \to \infty$ (see again Shalchi 2010, 2015 for details).

If we combine Equation (15) with the noisy slab model (2), we obtain

$$\kappa_\perp = \frac{2\pi l_\perp a^2 v^2}{3B_0^2} \int_0^{l_\perp^{-1}} dk_\perp \times \int_{-\infty}^{+\infty} dk_\parallel \, \frac{g(k_\parallel)}{v/\lambda_\parallel + (4/3)\kappa_\perp k_\perp^2 + (v^2 k_\parallel^2)/(3\kappa_\perp k_\perp^2)}. \quad (16)$$

By using spectrum (3) again and the integral transformations $x = l_\parallel k_\parallel$ as well as $y = l_\perp k_\perp$, we derive

$$\frac{\lambda_\perp}{\lambda_\parallel} = 2a^2 C(s) \frac{\delta B^2}{B_0^2} \int_0^1 dy \times \int_0^\infty dx \, \frac{(1 + x^2)^{-s/2}}{1 + (4\lambda_\parallel \lambda_\perp y^2)/(9 l_\perp^2) + (\lambda_\parallel l_\perp^2 x^2)/(\lambda_\perp l_\parallel^2 y^2)}. \quad (17)$$

Different asymptotic limits of Equation (15) have been derived and discussed in Shalchi (2015). It was shown in that paper, that there are two asymptotic limits for the case of small Kubo numbers. If the parallel mean free path is long, we obtain the quasi-linear limit (see Equations (5)–(7) of the current paper) from Equation (15). If the parallel mean free path is short, the following formula can be deduced from Equation (15):

$$\kappa_\perp = \frac{\pi^2 \kappa_\parallel a^4}{B_0^4} \left[\int d^3k \, P_{xx}(\mathbf{k}) k_\perp \delta(k_\parallel)\right]^2. \quad (18)$$

In Shalchi (2015), the latter formula was called the *collisionless Rechester & Rosenbluth (CLRR) scaling* as Equation (18) has a lot of similarity with the scaling originally derived in the famous paper of Rechester & Rosenbluth (1978). By combining the noisy slab model with Equation (18), one obtains

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{\kappa_\perp}{\kappa_\parallel} = \left[\frac{\pi}{2} C(s) a^2 \frac{l_\parallel}{l_\perp} \frac{\delta B^2}{B_0^2}\right]^2, \tag{19}$$

where we have also used spectrum (3). Very clearly this formula disagrees with Equation (14) derived from NLGC theory. In Appendix A, we present a more detailed analytical investigation of UNLT theory in the limit of small parallel mean free path in order to explore the validity of Equation (19). As demonstrated, the CLRR limit is accurate as long as the Kubo number is smaller than one.

We can recover the case of pure slab turbulence by considering the limit $l_\perp \to \infty$ in Equation (19), and thus we find $\kappa_\perp = 0$. A vanishing diffusion coefficient has to be interpreted as subdiffusive transport.

3.4. Random Ballistic Interpretation of Nonlinear Guiding Center Theory

Ruffolo et al. (2012) developed the *Random Ballistic Interpretation of the NLGC theory*. Instead of assuming a pure diffusive motion as in the original NLGC theory, the authors used a random ballistic distribution function. The following integral equation has been derived:

$$\kappa_\perp = \frac{a^2 v^2}{3B_0^2}\sqrt{\pi}\int d^3k\, \frac{P_{xx}(\boldsymbol{k})}{\sigma(\boldsymbol{k})}\mathrm{Erfc}\left[\frac{v^2}{3\kappa_\parallel \sigma(\boldsymbol{k})}\right], \tag{20}$$

where the *complementary error function* and

$$\sigma(\boldsymbol{k}) = \left[2\sum_{i=x,y,z} k_i^2 \langle \tilde{v}_i^2 \rangle\right]^{1/2} \tag{21}$$

have been used. Here, we have assumed already that the turbulence is magnetostatic, whereas the original work of Ruffolo et al. (2012) allowed the inclusion of dynamical turbulence effects. If Equation (20) is combined with the noisy slab model (2), we derive

$$\kappa_\perp = \frac{2\pi^{3/2} l_\perp a^2 v^2}{3B_0^2}\int_0^{l_\perp^{-1}} dk_\perp \times \int_{-\infty}^{+\infty} dk_\parallel\, \frac{g(k_\parallel)}{\sigma(\boldsymbol{k})}\mathrm{Erfc}\left[\frac{v^2}{3\kappa_\parallel \sigma(\boldsymbol{k})}\right]. \tag{22}$$

For axi-symmetric turbulence, Equation (21) reduces to

$$\sigma(k_\parallel, k_\perp) = [2k_\perp^2 \langle \tilde{v}_x^2 \rangle + 2k_\parallel^2 \langle \tilde{v}_z^2 \rangle]^{1/2}. \tag{23}$$

Ruffolo et al. (2012) proposed the following models for the guiding center velocities:

$$\langle \tilde{v}_x^2 \rangle = \frac{a^2 v^2}{6}\frac{\delta B^2}{B_0^2} \tag{24}$$

and

$$\langle \tilde{v}_z^2 \rangle = \frac{v^2}{3}\left(1 - a^2\frac{\delta B^2}{B_0^2}\right). \tag{25}$$

If we combine spectrum (3) with Equation (22) and employ the integral transformations $x = l_\parallel k_\parallel$ and $y = l_\perp k_\perp$, we find after lengthy straightforward algebra

$$\frac{\lambda_\perp}{\lambda_\parallel} = 2\sqrt{\pi} C(s) a^2 \frac{\delta B^2}{B_0^2}\int_0^1 dy \times \int_0^\infty dx\, (1 + x^2)^{-s/2}\rho(x, y)\mathrm{Erfc}[\rho(x, y)] \tag{26}$$

with

$$\rho(x, y) = \frac{l_\parallel}{\lambda_\parallel}\left(2\frac{l_\parallel^2}{l_\perp^2}\frac{\langle \tilde{v}_x^2 \rangle}{v^2} y^2 + 2\frac{\langle \tilde{v}_z^2 \rangle}{v^2} x^2\right)^{-1/2}, \tag{27}$$

where the guiding center velocities are still given by Equations (24) and (25). Again, Equation (26) can be evaluated numerically. This is done in the next section.

4. Test-particle Results and Test of the Different Theories

We have performed test-particle simulations for the noisy slab model. The simulation code used here is a modification of a code used before (see, e.g., Shalchi & Hussein 2014; Hussein et al. 2015 for more details). Similar simulations have been performed in the past (see, e.g., Giacalone & Jokipii 1994; Michałek & Ostrowski 1996; Qin et al. 2002a, 2002b; Zimbardo et al. 2006; Reville et al. 2008; Tautz 2010; Zimbardo et al. 2012).

The aforementioned simulations solve the Newton–Lorentz equation numerically for an ensemble of test-particles to obtain their trajectories. From the obtained orbits one can compute the diffusion coefficients in the different directions of space. In the Newton–Lorentz equation, one has to specify the magnetic field vector. This is done via a Fourier representation in which the integrals are replaced by sums. In order to generate random magnetic fluctuations, we superpose a large number of plane waves with different random polarizations and phases.

We obtained numerical results for the parallel mean free path, the perpendicular mean free path, and the ratio of these two transport parameters. Some of the simulations were already presented in Hussein et al. (2015). All diffusion parameters were computed as functions of the magnetic rigidity as it is usually done in this type of numerical work. In Tables 1–4 of Appendix B, we list all diffusion parameters and the corresponding rigidities. In analytical theories, however, the rigidity does not explicitly enter the nonlinear integral equations. Therefore, we have plotted perpendicular versus parallel diffusion coefficients instead of showing these parameters versus magnetic rigidity. By doing this, we included the simulations for all considered values of the rigidity.

For all runs we have set the inertial range spectral index to $s = 5/3$. The other parameters are changed in the different runs. For the calculations based on NLGC theory and its random ballistic interpretation we have set $a^2 = 1/3$ as originally suggested in Matthaeus et al. (2003). For the computations with UNLT theory, however, we have set $a^2 = 1$ as suggested in Qin & Shalchi (2016).

Table 1
Simulation Run 1: $\delta B/B_0 = 1$ and $l_\parallel/l_\perp = 0.5$; in This Case the Kubo Number is $K = 0.35$

Rigidity R:	0.001	0.01	0.1	0.316	1	3.16	10	31.6	100
$\lambda_\parallel/l_\parallel$	0.3	0.66	1.35	2.15	4.15	12.4	77	730	7700
$\lambda_\perp/l_\parallel$	0.002	0.005	0.0085	0.012	0.032	0.181	0.40	0.51	0.59
$\lambda_\perp/\lambda_\parallel$	0.0067	0.0076	0.0063	0.0056	0.0077	0.0146	0.00519	7×10^{-4}	7.7×10^{-5}

Table 2
Simulation Run 2: $\delta B/B_0 = 0.75$ and $l_\parallel/l_\perp = 0.5$ Corresponding to a Kubo Number of $K = 0.27$

Rigidity R:	0.001	0.01	0.1	0.316	1	3.16	10	31.6	100
$\lambda_\parallel/l_\parallel$	0.68	1.16	2.3	3.99	7.6	24.3	135	1285	13400
$\lambda_\perp/l_\parallel$	0.0019	0.0032	0.0061	0.0105	0.021	0.1	0.215	0.291	0.32
$\lambda_\perp/\lambda_\parallel$	0.0028	0.0028	0.00265	0.00263	0.00276	0.00412	0.00159	2.26×10^{-4}	2.39×10^{-5}

Table 3
Simulation Run 3: $\delta B/B_0 = 0.5$ and $l_\parallel/l_\perp = 0.5$ Corresponding to a Kubo Number of $K = 0.18$

Rigidity R:	0.001	0.01	0.1	0.316	1	3.16	10	31.6	100
$\lambda_\parallel/l_\parallel$	1.82	2.97	5.76	8.8	17.3	52.9	301	2860	29600
$\lambda_\perp/l_\parallel$	0.001	0.0016	0.0032	0.0048	0.0095	0.043	0.099	0.128	0.143
$\lambda_\perp/\lambda_\parallel$	5.6×10^{-4}	5.39×10^{-4}	5.55×10^{-4}	5.45×10^{-4}	5.5×10^{-4}	8.1×10^{-4}	3.3×10^{-4}	4.5×10^{-5}	4.8×10^{-6}

Table 4
Simulation Run 4: $\delta B/B_0 = 0.1$ and $l_\parallel/l_\perp = 0.5$ Corresponding to a Kubo Number of $K = 0.035$

Rigidity R:	0.316	1	3.16	10	31.6	100
$\lambda_\parallel/l_\parallel$	405	670	1980	14300	7.5×10^4	5×10^5
$\lambda_\perp/l_\parallel$	0.00033	0.0006	0.0039	0.0057	0.0061	0.0058
$\lambda_\perp/\lambda_\parallel$	8.15×10^{-7}	8.96×10^{-7}	1.97×10^{-6}	3.99×10^{-7}	8.13×10^{-8}	1.16×10^{-8}

In Figures 1 and 2, we show results for $\delta B/B_0 = 1$ and $l_\parallel/l_\perp = 0.5$. In this case, the Kubo number defined via Equation (1) is $K = 0.35$. Very clearly we can see that UNLT theory agrees well with the simulations. Quasi-linear theory, which is contained in UNLT theory in the limit of $\lambda_\parallel \to \infty$, agrees only for large parallel mean free paths. For small parallel mean free paths we obtain the CLRR scaling as predicted by UNLT theory (see Shalchi 2015). In general, original NLGC theory and its ballistic extension do not agree well with the simulations for the case discussed here.

In Figures 3 and 4, we show results for $\delta B/B_0 = 0.75$ and $l_\parallel/l_\perp = 0.5$ corresponding to a Kubo number of $K = 0.27$. Figures 5 and 6 show results for $\delta B/B_0 = 0.5$ and $l_\parallel/l_\perp = 0.5$. In this case, the Kubo number is $K = 0.18$. The last set of simulations has been performed for $\delta B/B_0 = 0.1$ and $l_\parallel/l_\perp = 0.5$ corresponding to a Kubo number of $K = 0.035$. The results are visualized in Figures 7 and 8, respectively. In all cases, we find that UNLT theory agrees well with the simulations. Quasi-linear theory, which is contained in UNLT theory in the limit of $\lambda_\parallel \to \infty$, agrees only for large parallel mean free paths. For small parallel mean free paths we find again the CLRR limit.

5. Summary and Conclusion

In the current paper, we have explored the transport of energetic particles in turbulence with small Kubo numbers. To do this, we have employed the noisy slab model that can be understood as a broadened slab model. Therefore, it is not as extreme as the usual slab model used before in diffusion theory where the Kubo number is exactly equal to zero.

We have employed four different analytical theories in order to compute the perpendicular diffusion coefficient, namely the quasi-linear theory, the NLGC theory, the UNLT theory, and the random ballistic interpretation of NLGC theory. Our aim was to test those theories by comparing the diffusion parameters they provide with test-particle simulations in the small Kubo number regime.

We found that UNLT theory agrees very well with the simulations for all considered cases and parameter values. In that theory, we have set $a^2 = 1$ as found in the simulations of Qin & Shalchi (2016). Quasi-linear theory works well as long as the Kubo number is small and very long parallel mean free paths are considered. The CLRR scaling is valid for small Kubo number turbulence and short parallel mean free paths. We can observe a smooth turnover from CLRR scaling to quasi-linear transport if the parallel mean free path is increased. This is exactly what is predicted by UNLT theory. The NLGC theory does not agree with the simulations. We have set $a^2 = 1/3$ as originally proposed in Matthaeus et al. (2003). If one would change the latter parameter to $a^2 = 1$, for instance, this would only have a minor effect on the perpendicular diffusion coefficient. It also seems that the random ballistic interpretation of NLGC theory agrees better with simulations,

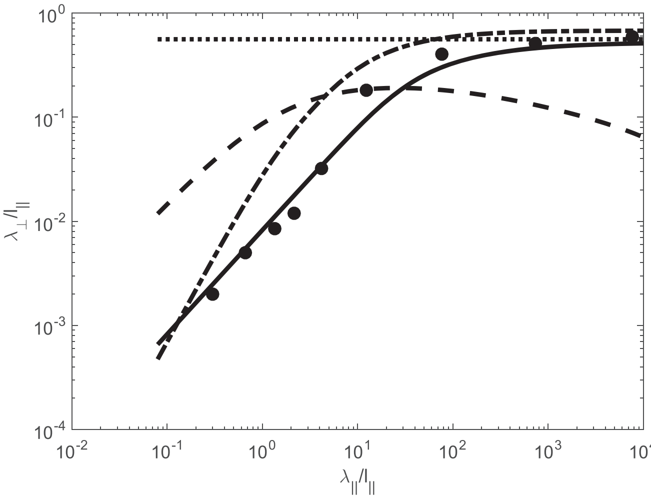

Figure 1. Perpendicular mean free path vs. the parallel mean free path computed by using different theories and test-particle simulations (dots). For the theoretical results we have shown NLGC theory (dashed line), Random Ballistic Interpretation of the NLGC theory (dashed–dotted line), UNLT theory (solid line), and quasi-linear theory (dotted line). Here, we have set $\delta B/B_0 = 1$ and $l_\parallel/l_\perp = 0.5$. The simulations were obtained for different values of the magnetic rigidity.

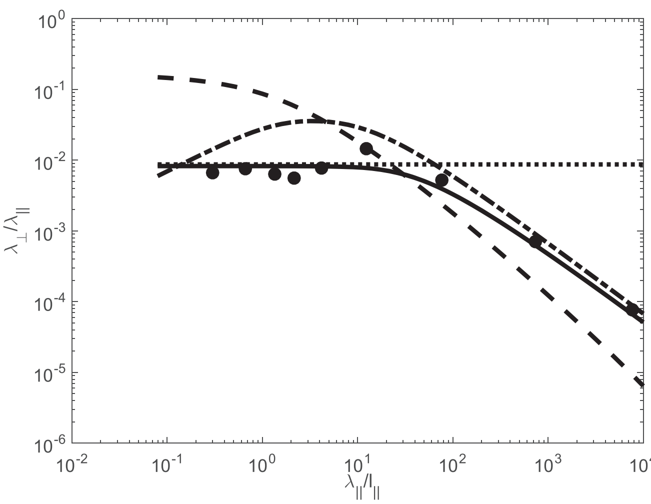

Figure 2. Ratio $\lambda_\perp/\lambda_\parallel$ vs. the parallel mean free path computed by using different theories and test-particle simulations (dots). For the theoretical results we have shown NLGC theory (dashed line), Random Ballistic Interpretation of the NLGC theory (dashed–dotted line), UNLT theory (solid line), and the CLRR limit (dotted line). Here, we have set $\delta B/B_0 = 1$ and $l_\parallel/l_\perp = 0.5$. The simulations were obtained for different values of the magnetic rigidity.

but in general the agreement is not as good as with UNLT theory.

We conclude that only UNLT theory can describe perpendicular diffusion for this type of small Kubo number turbulence accurately. We also like to emphasize that for the other extreme case, namely the case of large Kubo numbers, NLGC and UNLT theories provide very similar results, and thus both theories agree with simulations for this scenario.

A.S. acknowledges support by the Natural Sciences and Engineering Research Council (NSERC) of Canada. The simulations shown in this article were obtained by using the national computational facility provided by WestGrid.

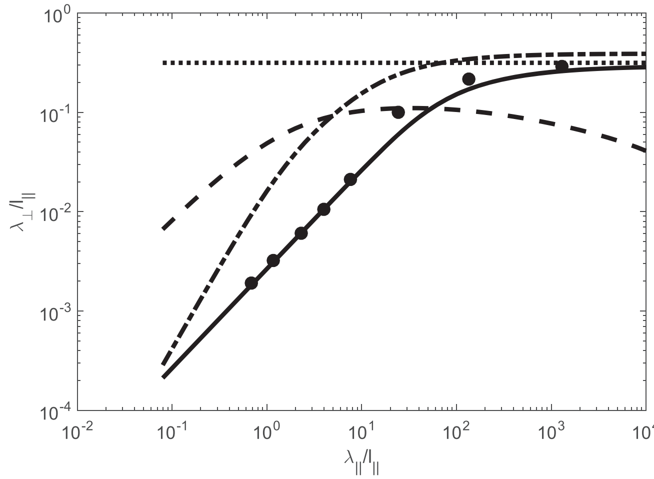

Figure 3. As in Figure 1, but here we have set $\delta B/B_0 = 0.75$ and $l_\parallel/l_\perp = 0.5$.

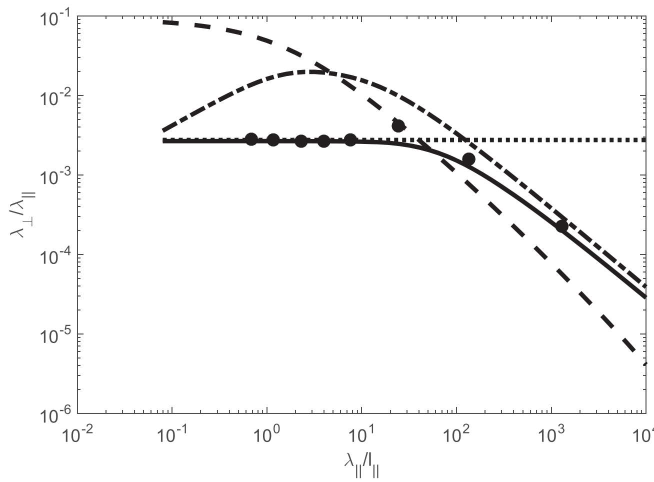

Figure 4. As in Figure 2, but here we have set $\delta B/B_0 = 0.75$ and $l_\parallel/l_\perp = 0.5$.

Appendix A
Analytical Results for Short Parallel Mean Free Paths

We consider the limit $\lambda_\parallel \to 0$ corresponding to short parallel mean free paths. Furthermore, in the theory of perpendicular diffusion, the inertial range spectral index s is not important. Therefore, we can set $s = 2$ and use $C(s = 2) = 1/(2\pi)$. Therefore, Equation (17) becomes

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{2}{\pi}\frac{\delta B_x^2}{B_0^2} \times \int_0^1 dy \int_0^\infty dx \frac{1}{(1 + x^2)[1 + (\lambda_\parallel l_\perp^2 x^2)/(\lambda_\perp l_\parallel^2 y^2)]} \tag{28}$$

where we have also set $a^2 = 1$. The x-integral can be solved by

$$\int_0^\infty dx \frac{1}{(1 + x^2)(1 + \alpha x^2)} = \frac{\pi}{2(1 + \sqrt{\alpha})}. \tag{29}$$

Thus, Equation (28) can be written as

$$\frac{\lambda_\perp}{\lambda_\parallel} = \frac{\delta B_x^2}{B_0^2}\int_0^1 dy \frac{1}{1 + \sqrt{(\lambda_\parallel l_\perp^2)/(\lambda_\perp l_\parallel^2 y^2)}}. \tag{30}$$

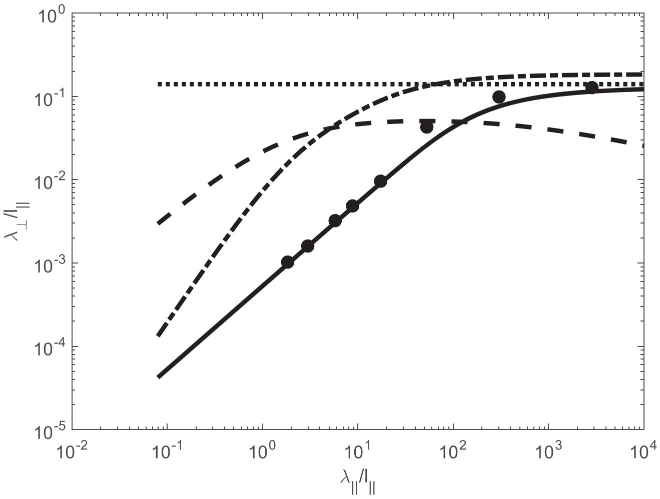

Figure 5. As in Figure 1, but here we have set $\delta B/B_0 = 0.5$ and $l_\|/l_\perp = 0.5$.

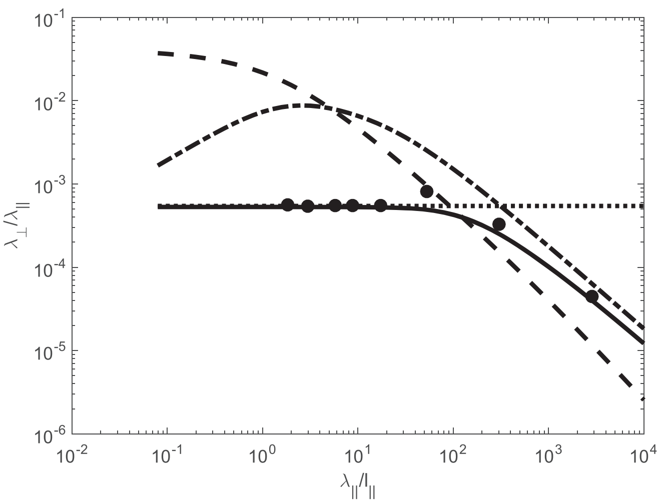

Figure 6. As in Figure 2, but here we have set $\delta B/B_0 = 0.5$ and $l_\|/l_\perp = 0.5$.

Now we use the dimensionless diffusion ratio

$$D := \frac{\lambda_\perp l_\|^2}{\lambda_\| l_\perp^2} \tag{31}$$

and the Kubo number defined in Equation (1) to find

$$D = K^2 \int_0^1 dy\, \frac{\sqrt{D}y}{1 + \sqrt{D}y}. \tag{32}$$

The remaining integral can be solved and we derive

$$D = K^2\left[1 - \frac{1}{\sqrt{D}}\ln(1 + \sqrt{D})\right]. \tag{33}$$

Obviously, D increases with increasing Kubo number. Therefore, for $K \to \infty$ we also expect $D \to \infty$, and Equation (33) can be approximated by

$$D = K^2. \tag{34}$$

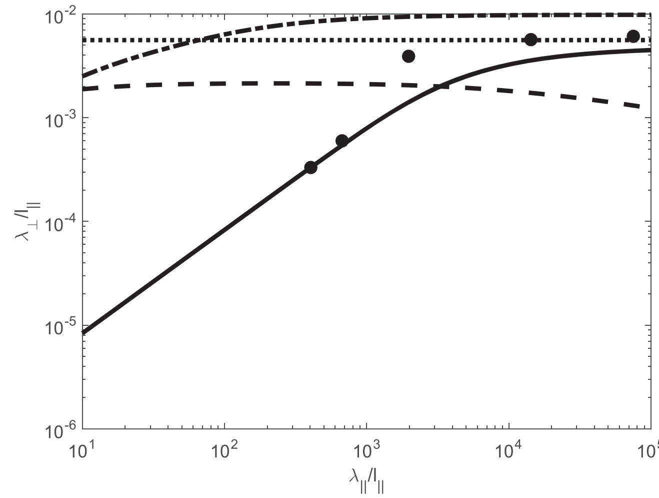

Figure 7. As in Figure 1, but here we have set $\delta B/B_0 = 0.1$ and $l_\|/l_\perp = 0.5$.

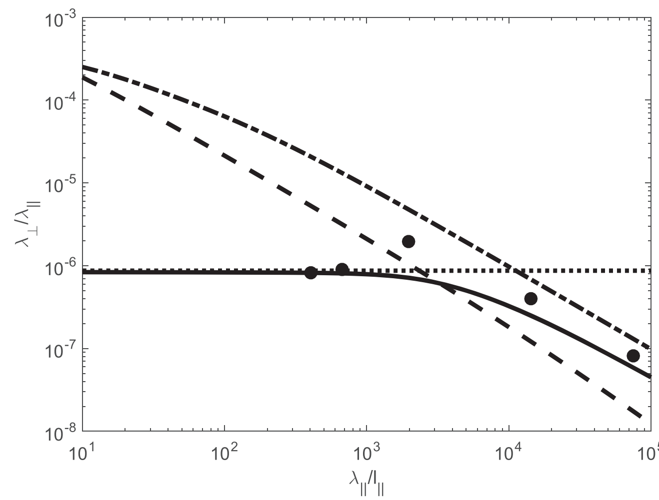

Figure 8. As in Figure 2, but here we have set $\delta B/B_0 = 0.1$ and $l_\|/l_\perp = 0.5$.

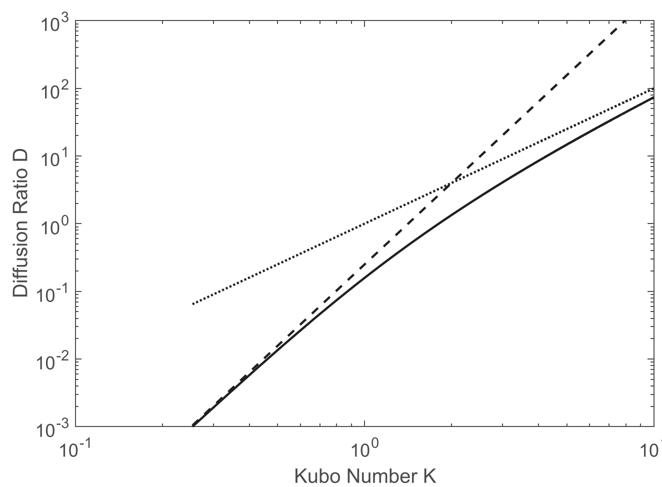

Figure 9. Diffusion ratio D vs. the Kubo number K in the strong pitch-angle scattering regime corresponding to short parallel mean free paths. Shown is the general formula (solid line), the small Kubo number limit (dashed line), and the large Kubo number limit (dotted line). The used formulas are given by Equations (33), (34), and (35), respectively.

For $K \to 0$, on the other hand, we expect $D \to 0$ and Equation (33) can be approximated by

$$D = \frac{1}{4}K^4. \quad (35)$$

The general formula (33) can easily be used in order to plot the diffusion ratio D versus the Kubo number K. In Figure 9, we compare the general formula with the asymptotic limits (34) and (35), respectively. According to our plot, the small Kubo number approximation is valid as long as the Kubo number is smaller than one as expected. The analytical considerations presented here also confirm the validity of Equation (18), which would lead to Equation (35) as well if the noisy slab model is employed.

Appendix B
Test-particle Simulations

In Tables 1–4, we summarize all performed test-particle simulations. We have calculated the parallel mean free path $\lambda_\parallel$, the perpendicular mean free path $\lambda_\perp$, and the ratio of the two diffusion parameters $\lambda_\perp/\lambda_\parallel$. All quantities were obtained for a certain value of the magnetic rigidity $R = R_L/l_\parallel$ where we have used the (unperturbed) Larmor-radius R_L and the parallel correlation scale of the turbulence $l_\parallel$. Some of these simulations were already published in Hussein et al. (2015).

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