LETTER

Formation of trapped-ion population in the process of charging of an absorbing sphere in a collisionless plasma

To cite this article: A. A. Kiselyov *et al* 2015 *EPL* **111** 15001

A Letters Journal Exploring the Frontiers of Physics

July 2015

EPL, **111** (2015) 15001
doi: 10.1209/0295-5075/111/15001

www.epljournal.org

Formation of trapped-ion population in the process of charging of an absorbing sphere in a collisionless plasma

A. A. Kiselyov[1,2(a)], M. S. Dolgonosov[1] and V. L. Krasovsky[1(b)]

[1] *Space Research Institute, Russian Academy of Sciences - Profsoyuznaya 84/32, Moscow 117997, Russia*
[2] *Moscow Institute of Physics and Technology - 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russia*

received 29 April 2015; accepted in final form 8 July 2015
published online 30 July 2015

PACS 52.25.Dg – Plasma kinetic equations
PACS 52.20.Dq – Particle orbits
PACS 52.65.Rr – Particle-in-cell method

Abstract – Dynamics of charging of an absorbing spherical body is studied by means of a numerical simulation. Upon saturation of the charge of the body and relaxation of transient oscillatory phenomena accompanying the charging, the disturbed plasma passes into a stable steady state. Along with the determination of space-time dependences of electrostatic quantities, the numerical experiment allows to observe the time evolution of electron and ion distributions in phase space. A dense cloud of trapped ions is formed near the spherical body, provided that the ion Debye length is of the order of the radius of the body, and the electron Debye length exceeds appreciably the radius. The trapped ions contribute substantially to the screening of the charged sphere, thereby affecting the structure of the disturbed plasma in the asymptotic steady state at long times.

Introduction. – The first theoretical studies of plasmas disturbed by an absorbing spherical body were paralleled by the development of the electric probe theory [1–3]. The investigation of the interaction of bodies with space plasmas calls for solving the same problem usually under the simplifying the assumption of the spherical shape of the body [4,5]. Such an assumption has also been in use for the computation of the charge of an individual grain of dust in dusty plasma physics [6]. Despite numerous applications of the problem of the charged sphere in plasmas [6–11], the long history of the studies and considerable progress in understanding of physics, there is a serious long-standing impediment to a rigorous consistent treatment of this classic problem. The fundamentally important question of the distribution function of trapped charged particles and their contribution to the screening of the absorbing sphere, as applied to collisionless plasmas, remains unresolved up to now.

In most cases, the problem is treated within the framework of the traditional formulation, *i.e.* possible steady states of the disturbed plasma are examined. However, if the radius of the sphere is small or of the order of the Debye length, a difficulty of a fundamental nature is typical of such an approach. The distribution function of trapped particles moving on finite orbits around the charged sphere remains undetermined. The reason is that, contrary to free particles moving infinitely, the trapped-particle distribution function does not obey the boundary condition at infinity, where the distribution function is assumed to be given. Although with allowance made for collisional phenomena the trapped-particle distribution can be established [4], in a fully collisionless plasma its uncertainty results in the absence of uniqueness of the solution [12]. In the case of sufficiently large radius of the sphere, the trapped particles cannot exist [3], so that the mentioned difficulty disappears. However, in order to solve the problem rigorously in the case of small radius of the sphere, it is necessary to investigate all possible steady states of the disturbed plasma with respect to stability, or to solve an initial value problem of transition to the stable state. Below, we will follow the second way.

Although the strongly nonlinear problem in a nonstationary formulation is extremely difficult for theoretical analysis, it can be solved by numerical methods. A numerical simulation of the charging may be found, for example, in the paper [13], where collisions have been taken into account and Cartesian coordinates have been used

(a) E-mail: alexander.kiselyov@stonehenge-3.net.ru
(b) E-mail: vkrasov@iki.rssi.ru

in the application to the spherically symmetric problem. Similar computations were later carried out with the help of the molecular-dynamics method [14,15]. In contrast to the mentioned work, we will treat the problem in the most refined formulation, *i.e.* under the assumption of the absence of collisions, and exploiting the spherical symmetry of the problem in full measure by the use of canonical equations of charged-particle motion. The purpose of our numerical simulation is to observe the dynamics of charging and to find the asymptotic steady state of the plasma disturbed by the charged absorbing sphere, including the specific form of the trapped-particle distribution function at long times, the charge of the sphere and other electrostatic characteristics of the physical system.

Formulation of the problem and input equations. – A simple formulation of the initial value problem on the charging of a sphere absorbing electrons and ions presumes the appearance of the sphere of radius R in a collisionless isotropic plasma at some instant of time $t = 0$. Under typical conditions, the flux of more mobile electrons exceeds the ion flux, so that the sphere acquires a negative charge. As time is going, the electron flux is decreasing while the ion flux is increasing. At long times, the fluxes equalize, so that the plasma tends to a certain asymptotic steady state. This stable equilibrium is of particular interest. The purpose of the numerical simulation described below is to trace the transition of the plasma to this state.

The input equations are Poisson and Vlasov equations for electrons and ions. Due to spherical symmetry of the problem and conservation of the angular momenta of charged particles, the motion of particles can be treated as a motion in one-dimensional effective potential, formally as a motion with one degree of freedom [3,4]. As a result, the solution of the problem and the technique of the numerical simulation are significantly simplified owing to the important advantage of the equations in canonical form. If the following units of measurement are in use

$$[r] = R, \quad [v_e] = [v_i] = u_e, \quad [t] = R/u_e, \quad [n] = n_0,$$
$$[\phi] = m_e u_e^2/e, \quad [E] = m_e u_e^2/eR, \quad [W_e] = m_e u_e^2,$$
$$[W_i] = m_i u_e^2, \quad [M_e] = m_e u_e R, \quad [M_i] = m_i u_e R,$$

where $[M_{e,i}]$ are units of the angular momenta of electrons M_e and ions M_i, $[W_{e,i}]$ are the energy units, n_0 is the number density of undisturbed plasma and u_e is a typical electron velocity, the Vlasov and Poisson equations take the dimensionless form

$$\frac{\partial f_{e,i}}{\partial t} + v_r \frac{\partial f_{e,i}}{\partial r} - \frac{\partial U_{e,i}}{\partial r}\frac{\partial f_{e,i}}{\partial v_r} = 0, \qquad (1)$$

$$(D_e/r)^2(\partial/\partial r)r^2 E = n_i - n_e, \quad E = -\partial\phi/\partial r. \qquad (2)$$

Every charged particle is moving in the effective potential taking into account the action of the centrifugal force,

$$U_{e,i}(r,t;M_{e,i}) = M_{e,i}^2/2r^2 + c_{e,i}\phi(r,t), \qquad c_e = -1, \qquad c_i = \mu = m_e/m_i, \qquad (3)$$

where the absolute value of the angular momentum of an individual charged particle M is a constant parameter.

On the whole, the problem is characterized by the following three parameters:

$$u = u_i/u_e, \quad \mu = m_e/m_i, \quad D_e = d_e/R = u_e/\omega_{pe}R, \qquad (4)$$

where u_e and u_i are typical velocities of electrons and ions in the unperturbed plasma, and D_e is the dimensionless electron Debye length. The values $u_{e,i}$ as well as the effective Debye radii $d_{e,i} = u_{e,i}/\omega_{pe,i}$ can be defined in accordance with a specific form of the unperturbed distribution functions of the particles $F_{e,i}$ at $t = 0$. In particular, the dimensionless form of the monoenergetic distributions is given by

$$F_{e,i} = (1/4\pi V^2)\delta(V - V_{e,i}), \quad V_e = 1, \quad V_i = u, \qquad (5)$$

where V is the absolute value of the velocity of a particle, so that the kinetic energy of the particle equals $W \equiv V^2/2 = (1/2)(v_r^2 + M^2/r^2)$. The distribution functions $F_{e,i}$ are also boundary conditions for eqs. (1) at infinity, $r = \infty$. The distributions (5) projected on the plane (z, v_r), where $z = r^3$, are uniform. This simplifies the observation and physical interpretation of the deformation of the phase space density of the particles in the course of simulation. At the same time, it is not difficult to generalize the scheme of computations and the corresponding programming codes to the case of an arbitrary isotropic in velocity space $F_{e,i}$.

PIC method for solving spherically symmetric problems. – Equations (1) and (2) represent the basis for a collisionless PIC ("particle-in-cell") simulation. As is known, the PIC technique is well suited as applied to studies of trapping phenomena [16]. The trapping of charged particles in potential wells is characterized by the appearance of multistream motions of Vlasov's fluid accompanied by overturning the equal-level curves of the distribution function and by the formation of vortex structures in phase space. A bright example of the observation of the trapping phenomenon is the numerical simulation of nonlinear saturation of a beam-plasma (or two-stream) instability in plane one-dimensional (1D) geometry. The numerical experiments described in [16] refer to one of the first achievements of computational plasma physics, since the nonlinear stage of the instability cannot be described analytically. The PIC method turns out to be very effective and quite economical also in the application to three-dimensional (3D) spherically symmetric problems of plasma kinetics. However, the spherical geometry of the problem calls for a certain modification of the PIC scheme in comparison with the plane 1D case. In particular, instead of "cells" in the form of plane layers, the cells represent spherical layers in our simulation. The second important distinction from the plane case is that charged particles are moving in the effective potential (3) which depends on the parameter M. As a consequence, every value of M determines an "individual" phase plane (r, v_r) for the particles with that angular momentum, in contrast to the unique phase plane shown, for example, in the figures from [16]. To observe entirely the phase space dynamics of all particles, these "individual" phase planes $(r, v_r; M)$ may be projected on one plane as is done below. It should be noted, however, that in this "combined" phase plane, the phase trajectories of particles with different M can intersect with each other. Needless to say that in the six-dimensional phase space such intersections are impossible in accordance with well-known principles of mechanics.

Taking into account the mentioned peculiarities, we have adopted the collisionless PIC method in the application to the studies of spherically symmetric kinetic plasma phenomena. As an example, below we will discuss results of the numerical simulation of the charging of the sphere in fully collisionless plasma.

Discussion of numerical simulation results. – We describe now a numerical solution of the formulated problem by means of the technique outlined above. For definiteness, here we consider the solution at the following values: $u = 0.005$, $D_e = 6.0$ and $\mu = 1/1836$, $(D_i \simeq 1.29)$, assuming unperturbed distribution functions of electrons and ions to be monoenergetic (5). In this case, the dimensionless effective length of screening equals approximately $D = d/R = D_e D_i/\sqrt{D_e^2 + D_i^2} \simeq 1.26$. Since the cells in our PIC model represent spherical layers of equal volume, in addition to the radial distance r, it is convenient to use also the new independent variable $z = r^3$ proportional to the volume of a sphere of radius r. Initially, an approximately equal number of PIC particles are located in every cell of width Δz similarly to the plane PIC model. At $t = 0$, macroparticles representing small finite elements of the phase space fluid uniformly fill the region limited by the inequalities $1 \le z \le Z_{max} = R_{max}^3, |v_r| \le V_{e,i}$ in the plane (z, v_r), where R_{max} is some maximum value of the radial distance r. In the run discussed below, the width of each cell is equal to $\Delta z = 0.022$. The number of particles in every cell is approximately $N_c \simeq 130$. The motion equations of individual PIC particles in the effective potential (3)

$$\mathrm{d}r/\mathrm{d}t = v_r, \qquad \mathrm{d}v_r/\mathrm{d}t = M_{e,i}^2/r^3 + c_{e,i}E(r,t), \tag{6}$$

are integrated by using the Runge-Kutta method with the time step $\Delta t = 0.38$. The calculation region is bounded by the radius of the sphere $r = 1$ $(z = 1)$ and the maximum radius $1 \le r \le R_{max}$ $(1 \le z \le Z_{max})$. The value R_{max} has been chosen so that the electric-field strength would appreciably exceed the level of pseudothermal noise typical for the collisionless PIC simulation in the calculation region, except for short times when the intensity of the regular field is very low. In the particular run discussed below $R_{max} = 9$. The maximum angular momentum of particles corresponds to the impact parameter R_{max}, so that $M_e \le R_{max}$ and $M_i \le uR_{max}$. The number of points used for the discretization of the variable M is about 6600.

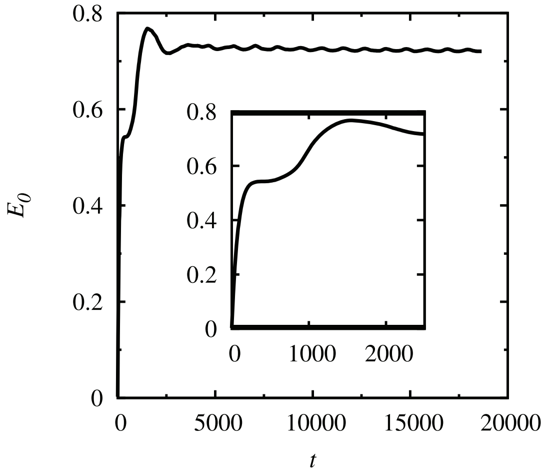

Fig. 1: Time dependence of the absolute value of the electric field on the surface of the sphere. The fast initial stage of charging is shown also in the insertion located in the centre of the figure. The dimensionless electron plasma period equals $T_{pe} \simeq 37.7$. The ion plasma period T_{pi} is about 1615. After saturation of the charge of the sphere the field E_0 oscillates slightly near the value of 0.72.

The time dependence of the electric field on the surface of the sphere $E_0(t) = |E(t, r = 1)|$ is shown in fig. 1. Note that the dimensionless charge of the sphere is simply equal to $Q(t) = -E_0(t)$ provided that the charge is normalized to $4\pi e n_0 R d_e^2$. The asymptotic value of the charge of the sphere normalized to the charge in the Debye's sphere is approximately $Q/(eN_d) \simeq 39$, where $N_d \equiv (4\pi/3)n_0 d^3$. At short times $(\omega_{pe}t \ll 1)$, the field grows almost linearly in accordance with the "neutral" approximation [4]. At long times, the intensity of field tends to the asymptotic value $E_0 \simeq 0.72$, and plasma passes into the steady state. In the process of saturation of the charge, transient oscillatory processes are observed. The intensity of the oscillations generally depends on the chosen set of the parameters (4). The oscillations may be caused by a combination of Langmuir and ion acoustic modes as well as by oscillations of trapped ions in troughs of the effective potential (bounce oscillations). The last ones are responsible for weak oscillations of $E_0(t)$ on the background of the asymptotic average value 0.72. It is noteworthy that the typical bounce frequency exceeds the ion plasma frequency $\omega_B \simeq 1.6\omega_{pi}$, as follows from fig. 1, since the typical bounce period $T_B = 2\pi/\omega_B \simeq 1000$, while the ion plasma period is about $T_{pi} = 2\pi/\omega_{pi} \simeq 1615$. As time progresses, the amplitude of the bounce oscillations decreases slowly due to the continuous phase mixing of the trapped ions. It should be emphasized that the frequencies of oscillations of individual trapped particles in the effective potential wells depend on their energies and angular momenta, so that the oscillations visible

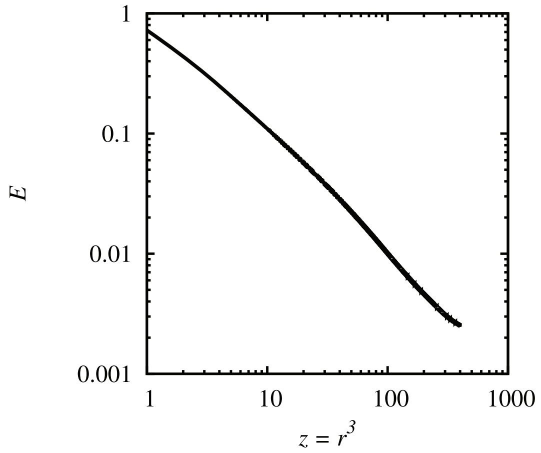

Fig. 2: Spatial profile of the electric-field absolute value at $t = 18661$ on a dual logarithmic scale. At the distances of order $r > 5$ ($z > 125$) the influence of thermal noise becomes noticeable.

in fig. 1 represent a summarized effect of many trapped particles oscillating, generally speaking, with different frequencies.

Figure 2 shows the spatial dependence of the electric field at $t = 18661$. In the vicinity of the sphere the profile $E(r)$ is very smooth, and the electric field is well approximated by the Coulomb law in the limit $r \to 1$. At large distances, of order R_{max}, the regular electric field is strongly attenuated owing to screening, so that the artificial thermal noise of the computer "plasma" characteristic of any PIC simulation manifests itself more appreciably. Due to the inevitable decrease in the electric-field strength at long distances and the presence of thermal noise, the location of the external boundary of the calculation region R_{max} has been chosen not too far from the sphere, although it must exceed, at least, several effective lengths of screening D (several ion Debye lengths for the set of parameters in hand). Here $R_{max}/D \simeq 7$.

Now let us consider the dynamics of the plasma. For the initial unperturbed distribution functions of the form (5), the initial distribution of PIC particles is uniform in the plane (z, v_r), reflecting the absence of plasma density perturbation. The evolution of electron and ion distributions has been observed with the help of an animation. The electron dynamics is physically quite simple since the electron effective potential is a monotonically decreasing function of r at any M. Because of this, it has been omitted in this letter. Of greatest interest is the evolution of ion distribution. A fraction of the ions moving toward the sphere ($v_r < 0$) with small angular momenta M is attracted, accelerated and absorbed by the sphere. The trajectories of particles with very large M (large impact parameters) do not suffer a noticeable effect from the action of the electric field. At last, a fraction of the ions with moderate values of M may be trapped in the wells of the effective potential U_i. Figure 3 shows the ion distribution in the phase plane at different times, including the initial distribution at $t = 0$ and asymptotic one. The trapping is accompanied by overturning of the equal-level lines of the distribution function and consequent phase mixing process. Depending on the parameters (4), the trapped ions are grouped into rather complex vortex-like structures, sometimes quite extensive in radial direction. However, for the set of parameters in hand, the trapped-ion bunch is formed in the immediate vicinity of the sphere. This can be shown by means of a qualitative consideration of the initial stage of charging and taking into account the smallness $\mu \ll 1$. According to such a scaling analysis the trapped-ion bunch must be concentrated near the sphere if the ion Debye length is of the order of $d_i \simeq \sqrt{d_e R}$, or in the accepted designations $D_i \simeq \sqrt{D_e}$, ($u^2 D_e/\mu = d_i^2/R d_e \equiv C \simeq 1$). For the parameters under consideration $C = 0.2754$. As another result of the scaling, the typical value of trapped-ion angular momentum can be evaluated by $M \simeq \mu^{1/2}$, that is approximately 4.7 times greater than the angular momentum of an ion moving with impact parameter equal to the radius of the sphere $M_R = u$. The typical values of kinetic and potential energies of the trapped ions are of the same order $W_T \simeq \mu/2$, *i.e.* they are about 20 times greater than the initial ion energy $W_{i0} = u^2/2$. These estimates are in good agreement with the results of our simulation as well as with theoretical studies on the structure of the trapping region [12].

The ratio of total trapped-ion charge to the charge of the sphere is about $Q_T/|Q| \simeq 0.18$, *i.e.* the charge of the trapped-particle cloud Q_T is commensurable with $|Q|$. Characterizing the trapped-ion contribution to screening of the charged sphere, the value of this simple parameter is one of the new findings of our study and gives the answer to the long-standing question of the trapped-particle effect. Since polarization charge of a screening plasma sheath is equal to the charge of the sphere with reversed sign, summarized charge caused by electron and ion density perturbation, in addition to the trapped-ion charge, equals obviously $Q_P = 0.82|Q|$.

Finally, we have compared, as far as possible, an outgrowth of our numerical experiments with a result of the study [9], wherein the trapped-ion effect was neglected. The case No. 2 in [9] corresponds to the set of parameters (4): $u = 3.31 \times 10^{-3}$, $\mu = 1.37 \times 10^{-4}$, $D_e = 1.68$. We have revealed that the asymptotic value $E_0 \simeq 0.98$ (also charge of the sphere) is about four times lower than the counterpart from [9]. However, it should be noted that there is no one-to-one correspondence between our study and the work [9] due to the difference in the form of $F_e(v)$. The stationary surface potential $|\phi_0|$ is about 2.2 times lower than the one from [9]. The potential is given by the simple equation $|\phi_0| = 0.5(1-u)/(1+\mu/u)$ following from the current balance $|j_e| = |j_i|$. This exact value has been used for testing the accuracy of the simulation. The typical fractional error of the calculated $|\phi_0|$ is about 0.01.

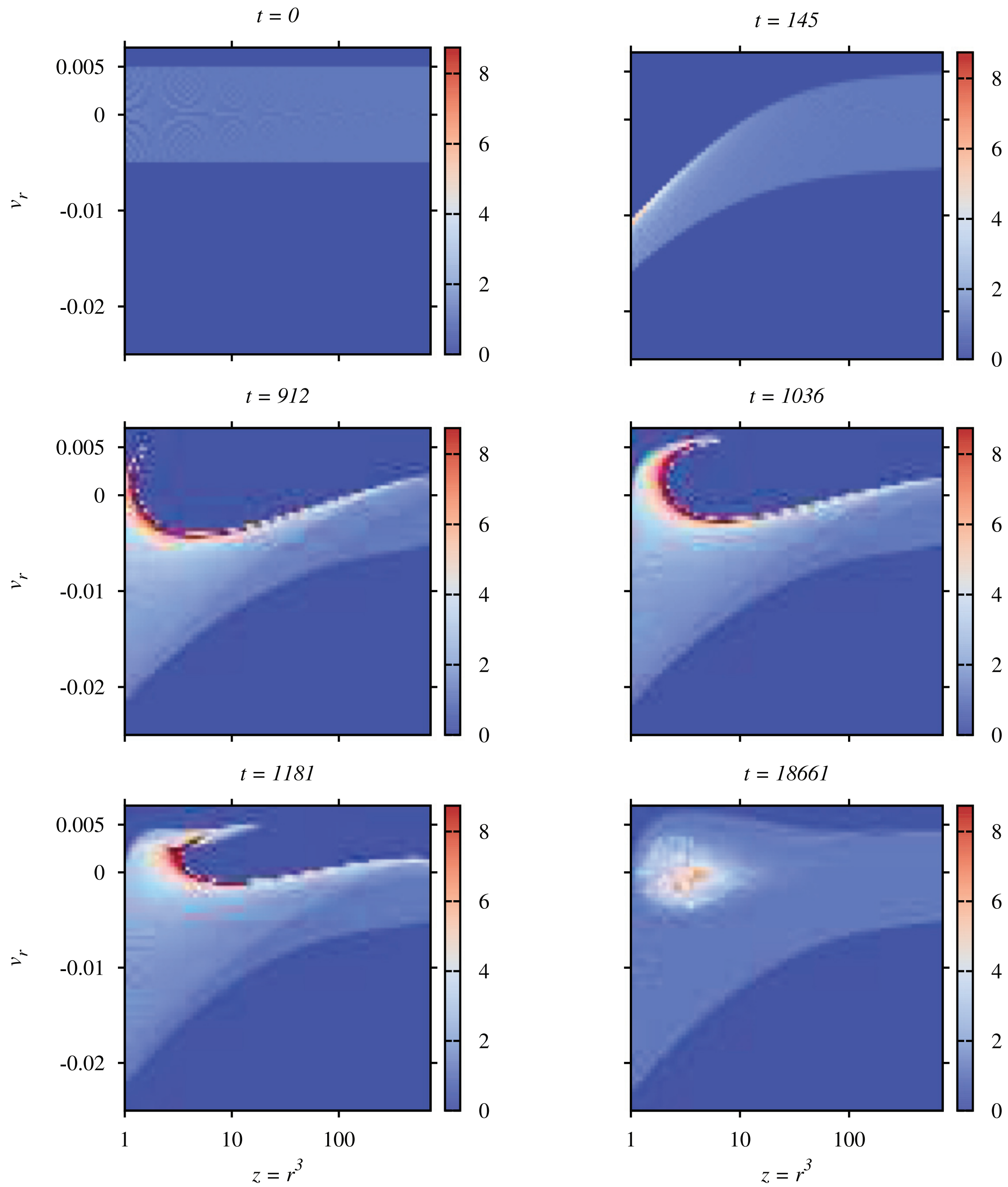

Fig. 3: Evolution of ion phase space distribution. The intensity of the colour corresponds to phase space density, with the scale unit corresponding to the initial density at $t = 0$. In the early stage of charging, electrons play a dominant role. The repulsion of the electrons by the sphere results in the saturation of the charging at the times of the order electron plasma period as seen in fig. 1. At times of the order of the ion plasma period the accelerated ions start to contribute to screening. As time passes, their contribution to screening becomes dominant. Concurrently, ion trapping starts at this stage ($t = 912, 1036, 1181$). Subsequently, the fine structure of the trapped-ion bunch is changing due to the oscillations of the trapped ions in the effective potential and phase mixing of the particles, while the coarse structure of the bunch, as a whole, is quite stable at long times. The trapped-ion population in the sheath screening the sphere is seen as a bright cloud at $t = 18661$.

Summary. – The study of charging and screening of absorbing bodies immersed in plasmas has attracted unremitting attention for many years due to a wide variety of applications of this classic problem of plasma physics [1–11]. As applied to the plasma free of collisions, straightforward analysis of steady states of the disturbed plasma is impeded by uncertainty of the trapped-particle distribution [3,4,12]. In the case of small radius of the absorbing sphere, this difficulty represents a major obstacle to a consistent solution of the problem in the standard, stationary, formulation. Above, as a way out, we have examined the problem in a nonstationary setting and carried out a numerical simulation of the charging under the assumption of the instantaneous appearance of the sphere in the plasma. As a result, the stable trapped-ion distribution has been determined in a collisionless regime for the first time.

The developed computational algorithm and computer code permit the direct measurement of all electrostatic quantities and observation of electron and ion motion in phase space. The main conclusion following from the performed simulation is that the plasma passes into a stable steady state at long times. In contrast to collision-dominating plasmas, the asymptotic equilibrium of the collisionless plasma is established under the influence of rapid dynamical processes. The trapping looks as the formation of a trapped-ion bunch in phase space. It is accompanied by the appearance of vortex structures in the phase space typical of the trapping phenomena with consequent grouping of the trapped ions into the dense bunch. At long times, phase mixing of the trapped particles oscillating in effective potential wells leads to gradual smoothing-out of their distribution function, resulting in the formation of the stable asymptotic trapped-ion population. The trapped ions contribute substantially to screening of the negatively charged sphere, especially near the sphere, where their density is high. Therefore, bunching of the trapped particles during the charging plays an important role and affects the final state of the disturbed plasma and the electrostatic properties of the physical system.

According to the numerical experiment, the total charge of the trapped ions is commensurable with the charge of the absorbing sphere. This indicates, in particular, that a conclusion of the work [17] about the insignificance of the ion trapping in rarefied space plasmas is invalid. Calculations of the trapped-particle distribution function in a collision-dominating plasma [14,15,18,19] cannot be applied to collisionless plasmas, since the passage to the limit of infinitesimal collision frequency, completely ignoring very important questions of plasma stability, is physically unjustified. In other words, highly probable development of instabilities typical of collisionless plasmas removes the "paradox" pointed out in [18] in view of the infinitely long time necessary for the relaxation of the trapped-ion distribution to the state discussed by the authors. Calculations performed in [20] on the basis of a trapped-ion number density postulated without sufficient physical grounds suffer from a similar drawback (at least as applied to collisionless plasmas described by the Vlasov equation neglecting dissipation), inasmuch as, in the strict sense, the concept of "thermodynamic equilibrium" is inapplicable to Vlasov's plasmas.

Finally, regarding the technique of modelling, the performed numerical experiments have demonstrated the high efficiency and reliability of the developed code, so that it may be applied to various problems of the collisionless plasma kinetics under the condition of spherical symmetry. In addition to the specific problem considered above, the numerical method described above may be easily modified for studies of the interaction of space bodies with plasma environment, spherical wave phenomena, including strongly nonlinear wave processes, expansion of plasma into vacuum, etc.

* * *

The work was partly supported within the framework of the program OFN-15 of the Russian Academy of Sciences.

REFERENCES

[1] Mott-Smith H. and Langmuir I., *Phys. Rev.*, **28** (1926) 727.

[2] Allen J. E., Boyd R. I. F. and Reynolds P., *Proc. Phys. Soc. B*, **70** (1957) 297.

[3] Bernstein I. N. and Rabinowitz I. N., *Phys. Fluids*, **2** (1959) 112.

[4] Al'pert Y. L., Gurevich A. V. and Pitaevskii L. P., *Space Physics with Artificial Satellites* (Consultant Bureau, New York) 1965.

[5] Whipple E. C., *Rep. Prog. Phys.*, **44** (1981) 1199.

[6] Fortov V. E. and Morfill G. E., *Complex and Dusty Plasmas: From Laboratory to Space* (CRC Press, London, New York) 2010.

[7] Morfill G. E., Grün E. and Johnson T. V., *Planet. Space Sci.*, **28** (1980) 1087.

[8] Goertz C. K., *Rev. Geophys.*, **27** (1989) 271.

[9] Daugherty J. E. *et al.*, *J. Appl. Phys.*, **72** (1992) 3934.

[10] Du C.-R. *et al.*, *EPL*, **99** (2012) 45001.

[11] Ussenov Y. A. *et al.*, *EPL*, **105** (2014) 15002.

[12] Krasovsky V. L., *Plasma Phys. Rep.*, **39** (2013) 572.

[13] Choi S. J. and Kushner M. J., *IEEE Trans. Plasma Sci.*, **22** (1994) 138.

[14] Zobnin A. V. *et al.*, *JETP*, **91** (2000) 483.

[15] Maiorov S. A., *Plasma Phys. Rep.*, **31** (2005) 690.

[16] Morse R., *Methods in Computational Physics,* Vol. **9**, *Plasma Physics* (Academic Press, New York, London) 1970.

[17] Goree J., *Phys. Rev. Lett.*, **69** (1992) 277.

[18] Lampe M. *et al.*, *Phys. Plasmas*, **10** (2003) 1500.

[19] Sukhinin G. I. *et al.*, *Phys. Rev. E*, **79** (2009).

[20] Bystrenko T. and Zagorodny A., *Phys. Lett. A*, **299** (2002) 383.