LETTER

Synchronization in delayed multiplex networks

View the article online for updates and enhancements.

You may also like

Synchronization in delayed multiplex networks

ARADHANA SINGH[1], SAPTARSHI GHOSH[1], SARIKA JALAN[1,2(a)] and JÜRGEN KURTHS[3,4]

[1] *Complex Systems Lab, Discipline of Physics, Indian Institute of Technology Indore - Khandwa Road,
Indore-452017, India*
[2] *Centre for Biosciences and Biomedical Engineering, Indian Institute of Technology Indore - Khandwa Road,
Indore-452017, India*
[3] *Potsdam Institute for Climate Impact Research - P.O. Box 601203, D-14412 Potsdam*
[4] *Institute for Complex Systems and Mathematical Biology, University of Aberdeen - Aberdeen AB24 3FX, UK*

Abstract – We study the impact of multiplexing on the global phase synchronizability of different layers in the delayed coupled multiplex networks. We find that at strong couplings, the multiplexing induces the global synchronization in sparse networks. The introduction of global synchrony depends on the connection density of the layers being multiplexed, which further depends on the underlying network architecture. Moreover, multiplexing may lead to a transition from a quasi-periodic or chaotic evolution to a periodic evolution. For the periodic case, the multiplexing may lead to a change in the period of the dynamical evolution. Additionally, delay in the couplings may bring upon synchrony to those multiplex networks which do not exhibit synchronization for the undelayed evolution. Using a simple example of two globally connected layers forming a multiplex network, we show how delay brings upon a possibility for the inter-layer global synchrony, that is not possible for the undelayed evolution.

Introduction. – The realization that many real-world systems such as transport, banks, stock market [1–5], etc. can be represented by multiple levels of interactions, has led to a spurt in the activities of understanding and characterizing various properties of multiplex networks. The prime motivation of the multiplex framework is that the function of individuals in one level gets affected by the interactions and functions in the other levels. A multiplex network consists of layered networks with one-to-one correlation between the replica nodes in different layers [6–11]. Each layer in the multiplex network constitutes different types of relations between the same units, presenting a more realistic framework of modeling real-world interactions [1]. Further, one of the most fascinating emergent behaviors of the interacting nonlinear dynamical units is the observation of synchronization [12–14], which is defined as the appearance of a relation between two processes due to the interactions between them [15,16]. Synchronization has been investigated a lot due to its wide range of applicability [13,16]. For example, synchronization plays a crucial role in proper functioning of systems

as diverse as motor functions of a neural network [17], efficiency in a business or academic system [2], signal processing in a communication network to proper flow of traffic in transport networks [18]. A recent work has revealed that there exists an onset of explosive synchronization in multilayer networks [19,20]. Other recent works have investigated changes in the static and dynamic behavior of multiplex networks with the interlink strength variation [21] as well as they have revealed the intra-layer synchronization without the inter-layer synchronization [22].

Furthermore, delays naturally arise in real-world systems due to the finite speed of information propagation [23]. The delays are shown to lead to many emerging phenomena in coupled dynamical units such as oscillation death, stabilizing periodic orbits, enhancement or suppression of synchronization, chimera state, etc. [24–34]. In this letter, we investigate the impact of multiplexing as well as the delay on the global synchronization of various networks. Instead of exact synchronization we analyze the phase synchronization as sparse networks exhibit a negligible number of nodes manifesting the exact synchronization. Furthermore, in many realistic situations the connection density as well as the degree distribution

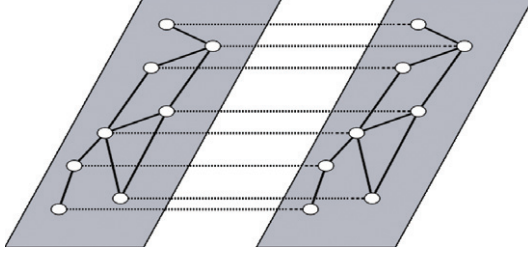(a)E-mail: sarikajalan9@gmail.com (corresponding author)

Fig. 1: Schematic diagram depicting a two-layer multiplex network. The solid lines represent the intra-layer connections and the dashed lines represent the inter-layer connections.

of various layers can be different. For instance, a collaboration and a friendship networks formed by the people employed in the same institution can have different architecture as well as connection density, and hence here we consider different layers being represented by different network architectures.

Theoretical framework. – We consider a network of N nodes and N_c connections. The dynamical evolution of each node at time t in the network is represented by a variable $x^i(t), i = 1, 2, \ldots, N$. This evolution of the dynamic variable with the delay (τ) can be described by a delayed coupled map model as [35]

$$x_i(t+1) = (1-\epsilon)f(x_i(t)) + \frac{\epsilon}{k_i} \sum_{j=1}^{mN} A_{ij} f(x_i(t-\tau)), \quad (1)$$

where ε is the overall coupling strength ($0 \le \epsilon \le 1$), τ represents the communication delay between the nodes, and m is the number of layers in the multiplex network. A is the adjacency matrix with elements A_{ij} taking values 1 and 0 depending upon whether there exists a connection between nodes i and j or not. We consider an undirected network with no self-loop and hence A is a symmetric matrix with diagonal elements being zero. Without loss of generality, we consider a simple multiplex network with two layers and N nodes in each layer (see fig. 1), and the adjacency matrix A for a two-layer multiplex network can be given as

$$A = \begin{pmatrix} A^1 & I \\ I & A^2 \end{pmatrix}, \quad (2)$$

where A^1 and A^2 are the adjacency matrices corresponding to the layer 1 and layer 2. $k_i = (\sum_{j=1}^{N} A_{ij}^l) + 1$ is the degree of the i-th node in the lth layer of the multiplex network. The function $f(x)$ defines the local nonlinear map and the coupling between the nodes. In the present framework, we consider local dynamics given by the logistic map, $f(x) = 4x(1-x)$ and the circle map, for which $f(x)$ shows chaotic evolution [36].

We quantify the global phase synchronization defined as follows [15]. Let n_i and n_j denote the number of times when the variables $x_i(t)$ and $x_j(t)$, $t = 1, 2, \ldots, T$ for the nodes i and j exhibit local minima during the time interval T. Let n_{ij} denote the number of times these local
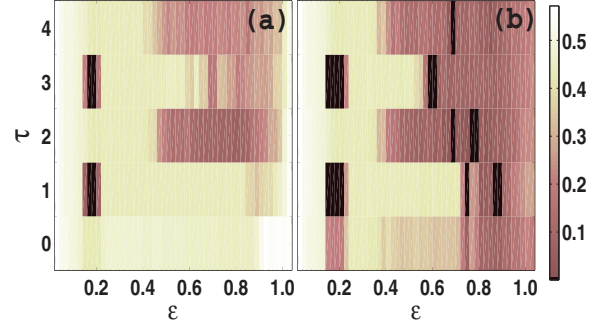

Fig. 2: (Colour on-line) Phase diagram depicting the variation of global phase synchronization measured by D for the SF network with $\langle k \rangle = 4$ in the ε-τ plane, (a) when it is isolated and (b) after multiplexing with another SF network of $\langle k \rangle = 10$, with respect to ε. The number of nodes in each layer is $N = 250$. All results are averaged over 20 different realization.

minima match each other. The phase distance between two nodes i and j is then given as

$$d_{ij} = 1 - \frac{2n_{ij}}{(n_i + n_j)}.$$

The nodes i and j are phase synchronized if $d_{ij} = d_{ji} = 0$. We used as global phase synchronization measure $D = \sum_{j=1}^{nN} d_{ij}$ for the whole multiplex network, $D_1 = \sum_{j=1}^{N} d_{ij}$ and $D_2 = \sum_{j=N}^{2N} d_{ij}$ for the first and the second layer, respectively. The global phase synchronized state for the whole multiplex network exists for $D = 0$, whereas one of the layers being global synchronized is indicated by $D_1 = 0$ or $D_2 = 0$ and $D \neq 0$. We study global phase synchronizability of a 1-d lattice, scale-free (SF), random and the globally connected networks upon multiplexing with another layer of 1-d lattice, SF, random and globally connected network architectures. The 1-d lattices used in the simulation have circular boundary conditions with each node having k nearest neighbors. SF and random networks are obtained by using BA and ER models, respectively [37]. The multiplex network is constructed by making one-to-one connections between the replica nodes in two layers.

Results. – We evolve eq. (1) starting from a set of random initial conditions and study global phase synchronization after an initial transient for various combinations such as regular-regular, regular-random, random-global, SF-global, random-random and SF-SF multiplex networks. First, we discuss changes in global phase synchronization of sparse SF networks upon multiplexing with a SF network. We take a SF network ($\langle k_1 \rangle = 4$) and multiplex it with another SF network ($\langle k_2 \rangle = 10$). For the undelayed evolution, an isolated SF network with $\langle k_1 \rangle = 4$ does not exhibit global phase synchronization for any of the coupling values as depicted in fig. 2(a). The multiplexing in this case does not bring upon any change in global synchronizability.
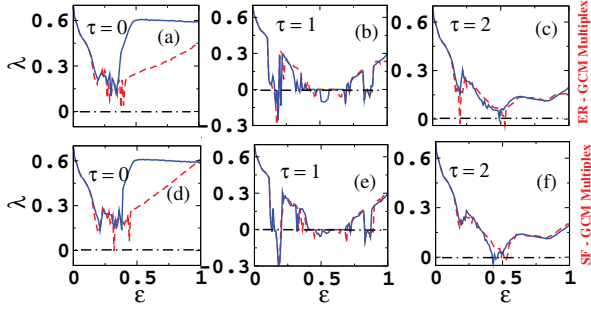
Fig. 3: (Colour on-line) Variation of the largest Lyapunov exponent with ε for the isolated SF network with $\langle k_1 \rangle = 10$ (dotted line) and after multiplexing with the globally connected network (solid line) for (a) undelayed, (b) delayed ($\tau = 1$), (c) delayed ($\tau = 2$) evolution. Similarly, for isolated random networks with $\langle k \rangle = 10$ (dotted lines) and after multiplexing with the globally connected networks (solid line) for (d) undelayed, (e) delayed ($\tau = 1$), (f) delayed ($\tau = 2$) evolution. Here the number of nodes N is taken to be 50 in each layer. All plots are for the average over 20 different realizations of the initial conditions.

An interesting phenomenon is displayed for the delayed evolution, while at the weak coupling multiplexing does not influence the synchronizability, at strong couplings multiplexing brings upon the global synchronization in the layer having sparse connections. The isolated SF networks, irrespective of the average degree of the network, exhibit global phase synchronization at weak coupling for the odd parity of delays in the coupling range $0.16 \lesssim \varepsilon \lesssim 0.18$ (fig. 2). The dynamical evolution in this coupling range is periodic with periodicity depending on the delay value (fig. 3(b)). While the multiplexing does not introduce any significant change in the synchronizability of the network for this coupling range, it leads to an enhancement in the global phase synchronizability at strong couplings. Thus, the delayed coupled isolated network ($\langle k_1 \rangle = 4$) does not exhibit a global synchronization, and multiplexing with different denser networks induces global synchrony. The coupling range for which the global synchronization is induced may change with the delay value even when the network architecture remains the same. For example at $\tau = 1$ the global synchronization is induced for the coupling range $0.71 \lesssim \varepsilon \lesssim 0.72$ and $0.84 \lesssim \varepsilon \lesssim 0.86$, and for $\tau = 3$ synchronization is observed for the coupling range $0.56 \lesssim \varepsilon \lesssim 0.58$. The same phenomenon is also observed for the other network architectures which, in the isolated state, do not exhibit the synchrony for the delayed evolution but upon multiplexing with the another denser network they exhibit the global synchronization at strong-coupling values. One such example is the sparse 1-d lattice, which, in the isolated state, does not exhibit the global synchronization at strong couplings, but upon multiplexing with the globally connected network it exhibits the global synchronization in the coupling range $0.56 \lesssim \varepsilon \lesssim 0.76$ (fig. 4(a)).
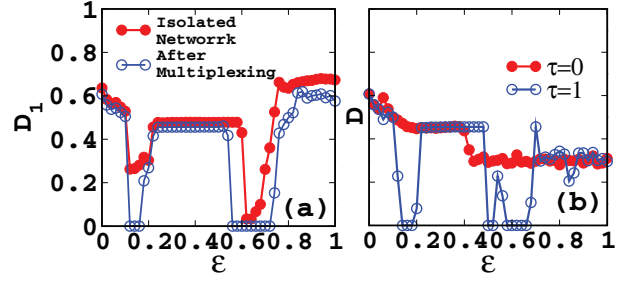

Fig. 4: (Colour on-line) (a) Variation of D_1 for the 1-d lattice ($\langle k_1 \rangle = 10$) before (closed circles) and after multiplexing (open circles) with the globally connected network for $\tau = 1$. (b) Variation of D for a multiplex network consisting of two globally connected layers for the undelayed (closed circles) and for the delayed evolution (open circles). The number of nodes in each layer is $N = 250$.

Further, two layers do not get synchronized with each other for the undelayed evolution, whereas the introduction of the delay induces the global synchronization in the multiplex network. An introduction of delay is already known to enhance the synchronizability of a network [29]; however, finding that the multiplex network exhibits global synchronization for the delayed evolution is more interesting as for the undelayed evolution the multiplex network does not exhibit global synchronization even if the globally connected network forms the individual layer (fig. 4(b)). Note that the isolated networks for sufficiently high connection density are known to exhibit global synchronization for the undelayed evolution. For example, the multiplex network consisting of the SF layer with $\langle k_1 \rangle = 10$ and the globally connected layer does not exhibit global synchronization at $\varepsilon = 0.69$ for the undelayed evolution (fig. 5(a)), whereas the introduction of delay results in the global synchronization of the multiplex network (fig. 5(b)). For the case of a multiplex network consisting of two globally connected layers, the undelayed evolution does not bring upon the global synchronization to the whole network, while the individual layer keeps on showing the global synchronization as observed for the isolated globally connected network (fig. 6(a)). An introduction of the delay leads to the global synchronization of the multiplex network (fig. 6(b)). Similarly, an introduction of the delay causes global synchronization in the multiplex network consisting of the random and the globally connected layers, which was not observed for the undelayed evolution as discussed above.

The introduction of global synchrony for the delayed evolution can be explained by considering the case of exact synchronization in a simple network architecture as follows. Let $x_i^1(t) = y^1(t)$, $\forall i$ and $\forall t > t_0$ be the global synchronized state of a globally connected isolated network and $z_i^1(t) = y^2(t)$ be the global synchronized state of another isolated network. Upon multiplexing, the difference variable between the two nodes in the same layer 2
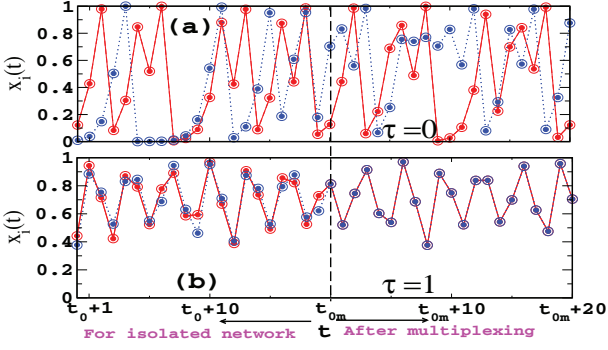
Fig. 5: (Colour on-line) Time series of two nodes (open and closed circles with solid lines) from the SF network (layer 1) and two nodes (open and closed circles with dotted lines) from the globally connected network (layer 2) before and after multiplexing for (a) undelayed and (b) delayed ($\tau = 1$) evolution. $N = 250$ in each layer and $\varepsilon = 0.69$. For the SF network, $\langle k_1 \rangle = 10$. The time series is potted after the initial transient of 10000 time steps before (t_0) and after the multiplexing (t_{0m}).
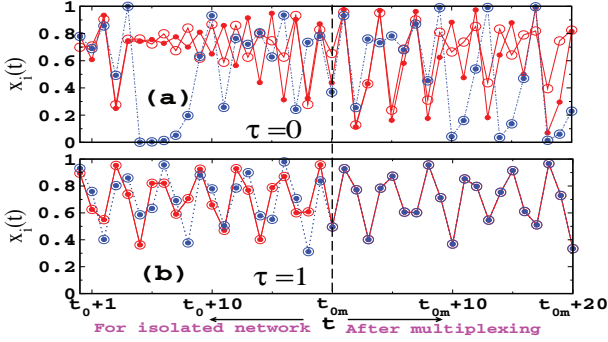

Fig. 6: (Colour on-line) Time series of four nodes (open and closed circles) from the globally connected layers (two from each) before and after multiplexing for (a) undelayed and (b) delayed ($\tau = 1$) evolution. $N = 250$ in each layer and $\varepsilon = 0.65$. The time series is potted after the initial transient of 10000 time steps before (t_0) and after multiplexing (t_{0m}).

at $\varepsilon = 1$ will be given as

$$dx^2_{ij}(t+1) = \frac{(k^1_i - k^1_j)}{(k^1_i + 1)(k^1_j + 1)} \times (f(y^1(t-\tau)) - f(y^2(t-\tau))). \quad (3)$$

Thus, for $k^1_i = k^1_j$, which is the case of the globally connected network, the intra-layer synchronization manifested by the isolated networks remains unaffected after multiplexing, whereas the difference variable between the mirror nodes i and j from two different layers will be

$$x^1_i(t+1) - x^2_i(t+1) = f(y^1(t)) - f(y^2(t)).$$

The above difference variable will not vanish for the nodes having the chaotic dynamics and therefore restricting the synchronization between them. However, for the delayed
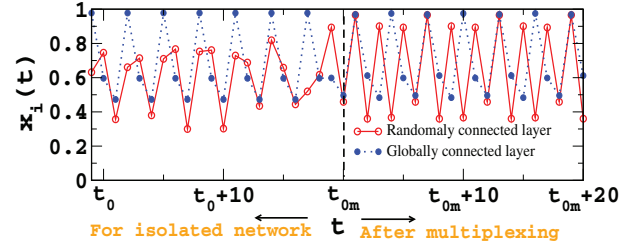

Fig. 7: (Colour on-line) Time series of two nodes (open and closed circles) from the random network (layer 1) and two nodes (open and closed circles) from the globally connected network (layer 2) before and after multiplexing. $N = 50$ in each layer, $\varepsilon = 0.58$ and for the random network the average degree is $\langle k_1 \rangle = 10$. The time series is potted after the initial transient of 10000 time steps before (t_0) and after multiplexing (t_{0m}).

evolution the difference variable will be

$$x^1_i(t+1) - x^2_i(t+1) = (1-\varepsilon)f(y^1(t)) - f(y^2(t) + (\varepsilon)(f(y^1(t-\tau)) - f(y^2(t-\tau))); \quad (4)$$

and depending on the value of ε and τ, nodes from the different layers may get synchronized even when both nodes have chaotic dynamics.

Further, in order to investigate the changes in the dynamical evolution upon multiplexing, we calculate the largest Lyapunov exponent as a function of ε (fig. 3). For the undelayed evolution, the multiplexing does not bring upon any significant change in the dynamical evolution and the dynamics remains chaotic for all the coupling values as observed for the isolated random network (fig. 3(a)), whereas, for the delayed evolution, multiplexing may lead to a transition from the quasi-periodic to periodic, or from the chaotic to a periodic evolution. For example, in the coupling range $0.54 \lesssim \varepsilon \lesssim 0.58$ and for $\tau = 1$, the multiplexing leads to a transition from the quasi-periodic to a periodic evolution (fig. 3(b)). In the same coupling range an isolated network exhibits the global phase synchronization, while multiplexing destroys the global phase synchronization as discussed above. In the coupling range $(0.83 \lesssim \varepsilon \lesssim 0.87)$, where the isolated network leads to a periodic evolution, multiplexing retains the periodic evolution with the same period (fig. 3(b)). But for the same coupling range the multiplexing also retains the global phase synchronization manifested by the isolated random network. In the coupling range $0.88 \lesssim \varepsilon \lesssim 0.89$, the multiplexing leads to a transition from a periodic to the chaotic evolution (fig. 3(b)).

Furthermore, as discussed above, the globally coupled network exhibits a transition from a periodic state to another periodic state with a different period upon multiplexing. For example, in the coupling range $0.54 \lesssim \varepsilon \lesssim 0.58$, the isolated random network for $\tau = 1$ exhibits the chaotic dynamics, and the isolated globally connected network shows a periodic evolution with periodicity three.
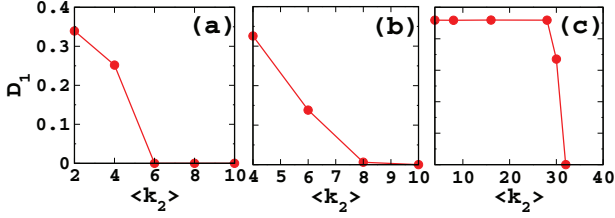
Fig. 8: (Colour on-line) Variation of (D_1) for a SF network with $\langle k_1 \rangle = 4$ multiplexed with (a) SF network at $\varepsilon = 0.86$, (b) random network at $\varepsilon = 0.86$ and (c) 1-d lattice at $\varepsilon = 0.76$, for various average degrees. Here the number of nodes N is taken to be 250 in each layer. All results are for the average over 20 different realizations of the initial condition.

The multiplexing leads to a periodic state with periodicity six for both the networks (fig. 7). Note that in the same coupling range there is no synchronization between nodes of the different layers (fig. 7).

Furthermore, in order to see an impact of the network architecture on the enhancement in the global synchronizability of a network upon multiplexing, we present results for the multiplexing of the SF network with various different network architectures viz. SF, random network and 1-d lattice of various average degrees. We find that the network architecture plays an important role in deciding the denseness of connection in a layer which leads to the global synchronization. For instance, SF networks with $\langle k_1 \rangle = 4$ exhibit the global synchronization upon multiplexing with a SF network having $\langle k_2 \rangle = 6$ (fig. 8(b), (c)), while the same phenomenon is observed for multiplexing with the random and the 1-d lattice of $\langle k_2 \rangle = 10$ and $\langle k_2 \rangle = 30$, respectively (fig. 8(b), (c)).

In order to demonstrate the robustness of the phenomenon for which the multiplexing introduces the global synchronization in the sparse networks at strong couplings, we present results for the coupled circle maps as well. The local dynamics then can be given by

$$f(x) = x + \omega + (p/2\pi)\sin(2\pi x) \quad (\text{mod } 1). \quad (5)$$

Here we discuss results with the parameters of a circle map corresponding to the chaotic evolution ($\omega = 0.44$ and $p = 6$).

For the local evolution being governed by the circle map and the coupled dynamics given by eq. (1), the sparse networks —those which do not exhibit the global synchrony— upon multiplexing with the denser networks exhibit the global phase synchronization. Figure 9 presents an example where the isolated SF network ($\langle k_1 \rangle = 2$) leads to the cluster formation (fig. 9(a)), while multiplexing with the SF network ($\langle k_2 \rangle = 10$) leads to a transition to the globally synchronized state at $\varepsilon = 0.96$ (fig. 9(b)). Furthermore, at the same coupling value there is a transition from the chaotic to the periodic evolution with periodicity four (fig. 9(b)).
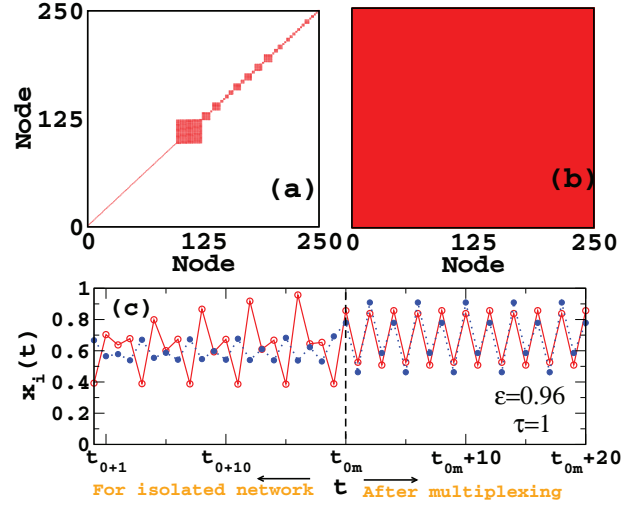

Fig. 9: (Colour on-line) (a) and (b): node-vs.-node diagram; (c) time series for SF networks with $\langle k_1 \rangle = 2$. (a) Cluster state for the undelayed evolution, (b) globally synchronized state indicated by the formation of a single cluster after multiplexing with the another SF network having $\langle k_2 \rangle = 10$ and (c) time series of two nodes from the SF network with $\langle k_1 \rangle = 2$ before and after the multiplexing. Here the number of nodes (N) is taken to be 250 in each layer, $\varepsilon = 0.96$ and $\tau = 1$.

Conclusion. – We have investigated the impact of multiplexing on the global phase synchronization and the dynamical evolution of the nodes in the individual layer in delayed multiplex networks. We mainly find that the impact of multiplexing depends on the network architecture of different layers as well as on the overall coupling strength. For the undelayed evolution and at the weak couplings for delayed evolution, multiplexing does not lead to any significant impact on the synchronizability of a network, yielding a similar dynamical evolution irrespective of the network architecture. For these cases, the same parity of delay values brings upon a similar impact as that observed for the isolated networks [29,30]. Upon multiplexing, the odd parity of delay values exhibits the global phase synchronization with a periodic evolution, whereas in the same coupling range the even parity of delays may exhibit the global phase synchronization with the dynamical evolution being chaotic. At strong couplings, the delayed sparse networks exhibit the global synchrony upon multiplexing with the denser networks. The connection density of the layer being multiplexed plays an important role in deciding if there will be an introduction of the global synchronization in the sparse networks which further depends on the network architecture. Multiplexing with a SF network with the connection density lower than the corresponding ER and regular networks may lead to the global synchrony in a sparse SF network at the strong couplings. Furthermore, for the undelayed evolution, the multiplex network does not display the global synchrony even though the nodes in each layer are globally connected while incorporation of delay leads to the global synchronization of the entire network. By using

a simple network architecture consisting of two globally connected layers, we demonstrate that for the chaotic intrinsic dynamical evolution, the global synchronization in a multiplex network is not possible. The introduction of delays in the coupling provides a possibility for the same. Additionally, the multiplexing leads to a transition from a quasi-periodic or the chaotic evolution to a periodic evolution as well as it may lead to a change in the periodicity.

This analysis can be further extended to get a better understanding of various dynamical processes in real-world systems, such as controlling congestion in the multiplex transport networks [18] as well as the occurrence of different diseases such as epileptic seizure by representing brain as delayed multiplex networks [38,39].

* * *

REFERENCES

[1] GARDENES J. G. *et al.*, *Nat. Sci. Rep.*, **2** (2012) 620.
[2] GRANELL C., GOMEZ S. and ARENAS A., *Phys. Rev. Lett.*, **111** (2013) 128701.
[3] AGUIREE J. *et al.*, *Phys. Rev. Lett.*, **112** (2014) 248701.
[4] HOLME P. and SARAMÄKI J., *Phys. Rep.*, **519** (2012) 97.
[5] WASSERMAN S. and FAUST K., *Structural Analysis in the Social Sciences*, Vol. **8** (Cambridge University Press, Cambridge, England) 1994.
[6] NEKORKIN V. I., KAZANTSEV V. B. and VELARDE M. G., *Physica D*, **151** (2001) 1.
[7] NEKORKIN V. I. *et al.*, *Phys. Rev. E*, **64** (2001) 036602.
[8] KAZANTSEV V. B. *et al.*, *Phys. Rev. E*, **63** (2001) 016212.
[9] BOCCALETTI S. *et al.*, *Phys. Rep.*, **544** (2014) 1.
[10] LEE K.-M., KIM J. Y., CHO W.-K., GOH K.-I. and KIM I.-M., *New J. Phys.*, **14** (2012) 033027.
[11] BIANCONI G., *Nat. Phys.*, **10** (2014) 712.
[12] NEWMANN E., SUSHKO I., MAISTRENKO Y. and FEUDEL U., *Phys. Rev. E*, **67** (2003) 026202.
[13] PIKOVSKY A., ROSENBLUM M. and KURTHS J., *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge) 2003; ARENAS A. *et al.*, *Phys. Rep.*, **469** (2008) 93.
[14] PECORA L. M., SORRENTINO F., HAGERSTROM A. M., MURPHY T. E. and ROY R., *Nat. Commun.*, **5** (2013) 1038.
[15] JALAN S. and AMRITKAR R. E., *Phys. Rev. Lett.*, **90** (2003) 0141011; AMRITKAR R. E. and JALAN S., *Physica A*, **321** (2003) 220; JALAN S., AMRITKAR R. E. and HU C.-K., *Phys. Rev. E*, **72** (2005) 016211.
[16] KURAMOTO Y., *Chemical Oscillations, Waves and Turbulence* (Springer-Verlag, Berlin) 1984.
[17] LAIRD A. R., ROGERS B. P. and MEYERAND M. E., *Proc. Int. Soc. Magn. Reson. Med.*, **10** (2012); VARELA F. *et al.*, *Nat. Rev. Neurosci.*, **2** (2001) 229.
[18] BARTHÉLEMY M., *Phys. Rep.*, **499** (2011) 1.
[19] ZHANG X., BOCCALETTI S., GUAN S. and LIU Z., *Phys. Rev. Lett.*, **114** (2015) 038701.
[20] LEYVA I. *et al.*, *Sci. Rep.*, **3** (2013) 1281.
[21] BASTAS N., LAZARIDIS F., ARGYRAKIS P. and MARAGAKIS M., arXiv:1501.06700v1 (2015).
[22] GAMBUZZA L. V., FRASCA M. and GÓMEZ-GARDEÑES J., *EPL*, **110** (2015) 20010.
[23] LAKSHMANAN M. and SENTHILKUMAR D. V., *Dynamics of Nonlinear Time-Delay Systems* (Springer-Verlag, Berlin) 2010.
[24] REDDY D. *et al.*, *Phys. Rev. Lett.*, **80** (1998) 5109; PUNETHA N. *et al.*, *Phys. Rev. E*, **85** (2012) 046204.
[25] ATAY F. M. and KARABACK Ö., *SIAM J. Appl. Dyn. Syst.*, **5** (2006) 508.
[26] HEIL T. *et al.*, *Phys. Rev. Lett.*, **87** (2001) 87117; HÖFNER J. M. *et al.*, *EPL*, **95** (2011) 40002.
[27] SCHÖLL E. *et al.*, *Philos. Trans. R. Soc. A*, **367** (2009) 1079.
[28] HEIL T. *et al.*, *Phys. Rev. Lett.*, **86** (2001) 795.
[29] DHAMALA M. *et al.*, *Phys. Rev. Lett.*, **92** (2004) 0741404; ATAY F. *et al.*, *Phys. Rev. Lett.*, **92** (2004) 144101; SHRII M. *et al.*, *EPL*, **98** (2012) 10003.
[30] SINGH A., JALAN S. and KURTHS J., *Phys. Rev. E*, **87** (2013) 030902(R).
[31] SEN M. K. *et al.*, *J. Stat. Mech.* (2010) P08018.
[32] SETHIA G. *et al.*, *Phys. Rev. Lett.*, **100** (2008) 144102.
[33] WANG Q. *et al.*, *EPL*, **83** (2008) 50008.
[34] GOSAK M., MARKOVIČ R. and MARHL M., *Physica A*, **391** (2012) 2764.
[35] NEWMAN M. E., *Networks An Introduction* (Oxford University Press Inc., New York) 2010.
[36] STROGRATZ S. H., *Nature (London)*, **410** (2001) 268 and reference therein.
[37] ALBERT R. and BARABÁSI A.-L., *Rev. Mod. Phys.*, **74** (2002) 47.
[38] GRANELL C., GÓMEZ S. and ARENAS A., *Phys. Rev. Lett.*, **111** (2013) 128701.
[39] TIMME SHINYA N. *et al.*, *PLoS ONE*, **9** (12) (2014).