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Delay-induced Turing-like waves for one-species reaction-diffusion model on a network

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Abstract – A one-species time-delay reaction-diffusion system defined on a complex network is studied. Traveling waves are predicted to occur following a symmetry-breaking instability of a homogeneous stationary stable solution, subject to an external nonhomogeneous perturbation. These are generalized Turing-like waves that materialize in a single-species populations dynamics model, as the unexpected byproduct of the imposed delay in the diffusion part. Sufficient conditions for the onset of the instability are mathematically provided by performing a linear stability analysis adapted to time-delayed differential equations. The method here developed exploits the properties of the Lambert W-function. The prediction of the theory are confirmed by direct numerical simulation carried out for a modified version of the classical Fisher model, defined on a Watts-Strogatz network and with the inclusion of the delay.

Collective dynamics spontaneously emerge in a vast plethora of physical systems and often play a role of paramount importance for the efficient implementation of dedicated functions. Traveling waves are among the most studied phenomena for their ubiquitous and cross-disciplinary interest. Periodic traveling waves are for example encountered when describing self-oscillatory and excitable systems in different realms, from chemistry to biology, passing through physics. The Fisher [1,2] equation, introduced to characterize the spatial spread of an advantageous allele, defines the paradigmatic arena for addressing the peculiarities of traveling wave solutions in a reaction-diffusion system in a spatial continuous domain. This is a one-species population dynamic model, which assumes a logistic rule of replication and growth. The microscopic entities belonging to the scrutinized population can also delocalize in space, following a standard diffusion mechanism. The interplay between the aforementioned processes yields stable traveling wave solutions, which are selectively generated starting from a special class of initial conditions [3]. More generally, it is however interesting to speculate on the possibility for a system to yield self-organized collective patterns of the traveling-wave type, following a symmetry-breaking instability, seeded by diffusion. Starting from a homogeneous solution subject to a tiny, nonhomogeneous, initial perturbation, a reaction-diffusion system can destabilize via a dynamical instability, identified by Alan Turing in a seminal work [4]. The Turing instability, as the process is nowadays called, can drive the emergence of nonlinear stationary stable patterns, if at least two species, the activator and inhibitors, are diffusing and mutually interacting in the embedding environment. Alternatively, the Turing mechanism can instigate traveling wave solutions, provided that at least three species, one of which mobile, are assumed to interact via apt nonlinear couplings [5]. For a reaction-diffusion system hosted on a discrete heterogeneous spatial support, namely a network, the instability can eventually set in for a two-species model yielding stationary stable patterns [6]. Also in the network framework, three coupled species are the minimal request for a traveling wave to rise from a stochastic perturbation of an initial homogeneous stationary stable state. Observe, however, that taking into account finite-size corrections, stochastic Turing patterns can develop, even if just the inhibitor is allowed to crawl from one node to its adjacent neighbors [7].

Starting from these premises, the aim of this letter is to tackle the above problem under a radically different angle and thus introduce the simplest mathematical setting for which traveling wave solutions are generated, on a network, as a symmetry-breaking instability of the Turing type. To anticipate our finding, we will prove that a one-species model endowed with a delay in the diffusion can produce the sought instability. Note that this phenomenon would not be possible on continuous domains and thus it completely relies on the discrete nature of the support [8]. The analysis holds in general, but to demonstrate our conclusion we shall refer to a Fisher equation defined on a complex network and modified with the inclusion of a constant delay term.

The usage of time-delay differential equations (DDEs) defined on complex networks [9–11] is nowadays very popular, from pure to applied sciences [12,13]. Delays are for instance introduced to model finite communication or displacement time of quantities across network links. The imposed time-delay can nontrivially interfere with the reactive dynamics, taking place on each node of the graph, thus resulting in unexpected emergent properties. For example, the classical paper [14] deals with time-delayed systems made of two coupled phase oscillators and demonstrates the existence of multistability of synchronized solutions: an invariant manifold exists which attracts all the solutions of the system, yielding global oscillatory phenomena. Since this pioneering contribution, the subject has gained a lot of attention and several results have been reported, assuming linear systems [15], coupled oscillators [16–18], oscillations death [19–21] and oscillations control [22]. Theory has been fruitfully applied to tackle real problems, see for instance [23–29].

The work of this letter moves from this reference context to build an ideal bridge with the Turing-like theory of pattern formation on complex networks. The discrete Laplacian operator, that encodes for the diffusion of the mobile population, incorporates a delay term. Our analysis of the DDEs exploits the properties of the Lambert W-function [30,31] thus the solution of the linearized equation can be given in closed analytical form. Furthermore, we have full access to the associated eigenvalues and their dependence on the involved parameters is made explicit. This allows us to go beyond the technique based on the computation of the Hopf bifurcation, namely to determine the parameters for which the eigenvalue with the largest real part passes through a pure imaginary value.

We consider an undirected connected network composed by n nodes and assume one species to diffuse, from node to node, via the available links. Reactions also take place on each node, as dictated by a specific nonlinear function f of the local species concentration. Let us denote by $x_i(t)$ the species concentration on node i, at time t. Then its time evolution is governed by

$$\dot{x}_i(t) = f(x_i(t-\tau_r)) + D\sum_j L_{ij}x_j(t-\tau_d), \qquad (1)$$

where D stands for the diffusion coefficient, $L_{ij} = G_{ij} - k_i\delta_{ij}$ is the Laplacian matrix of the network whose adjacency matrix is given by G and $k_i = \sum_j G_{ij}$ identifies the degree of the i-th node. τ_r is the delay involved in the reaction occurring at each node, while τ_d is the delay due to the displacement across nodes. For the sake of simplicity we hereby assume the delay to be independent from the link indices[1]. As mentioned earlier, it is well known that the above one-species reaction-diffusion system cannot exhibit Turing-like instability, in the limiting case for $\tau_r = \tau_d = 0$. As we shall argue, the introduction of a finite delay $\tau_r = \tau_d = \tau > 0$, will significantly alter this conclusion.

To proceed in the analysis we assume a stable homogeneous equilibrium, $x_i(t) = \hat{x}$ for all $i = 1, \dots, n$ and $t \geq 0$ and look for sufficient conditions to destabilize such equilibrium, following the introduction of a nonhomogeneous perturbation, which in turn activates the diffusion part. To determine the preliminary conditions that have to be met for the homogeneous equilibrium to be stable, we linearize system (1) with $D = 0$, around $\hat{x}$, and recall that $\tau_r = \tau > 0$. Let $A = f'(\hat{x})$, then the characteristic equation reads $\lambda = Ae^{-\lambda\tau}$ whose solutions are

$$\lambda_k = \frac{1}{\tau}W_k(\tau A), \qquad k \in \mathbf{Z}, \qquad (2)$$

W_k being the k-th branch of the Lambert W-function [30]. To guarantee the needed stability of the homogeneous equilibrium $\hat{x}$, one has to require that (A, τ) and an integer k exist for which $\Re\lambda_k(A, \tau) < 0$.

The Lambert W-function is the complex multivalued function of the complex variable $z \in \mathbf{C}$ defined to be solution of the equation $z = W(z)e^{W(z)}$. It has infinitely many branches [30] denoted by $W_k(z)$, for $k \in \mathbf{Z}$; among them $W_0(x)$ —the principal branch— is obtained by restricting z to lie on the real axis, more precisely on $\Re z \in (-e^{-1}, +\infty)$, and with the constraint $W_0(z) \geq -1$. Hence the branch cut of W_0 is defined by $\{z : -\infty < \Re z \leq -e^{-1}, \Im z = 0\}$. Let us observe that $k = 0$ and $k = -1$ are the only branches for which the Lambert W-function can assume real values. For all remaining k, $\Im W_k(z) \neq 0$ for all z (see fig. 1).

Roughly speaking, W_0 bends the z plane (cut along $\Re z < -e^{-1}$) into a parabolic-like domain in the w plane, whose boundary curves $\Im w \mapsto \Re w = -\Im w \operatorname{cotan} \Im w$ (blue solid and dotted curves in fig. 1) are bounded by π and $-\pi$. Moreover, W_0 satisfies the following relevant condition (Lemma 3 of [32]):

$$\forall z \in \mathbf{C} : \max_{k\in\mathbf{Z}} \Re W_k(z) = \Re W_0(z). \qquad (3)$$

The previous eq. (3) allows to restate the stability condition of the homogeneous equilibrium as follows:

$$\exists (A, \tau) \quad \text{such that} \quad \Re W_0(\tau A) < 0. \qquad (4)$$

[1]A more general description can be provided by taking into account the possibility that each link introduces a different delay [10].

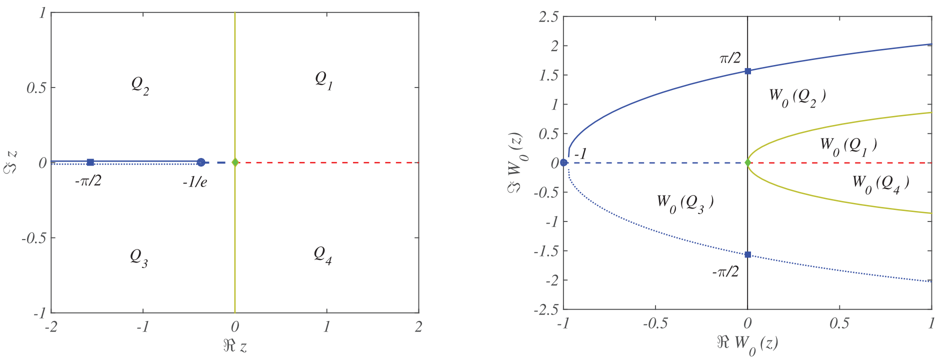

Fig. 1: (Colour on-line) The Lambert W-function: principal branch $W_0(z)$. Left panel: in the complex plane $z \in \mathbf{C}$, we represent the upper part of the branch cut $\{z : -\infty < \Re z \le -e^{-1}, \Im z = 0^+\}$ by a solid line and the lower part of the branch cut $\{z : -\infty < \Re z \le -e^{-1}, \Im z = 0^-\}$ by a dashed line; the circle denotes the point $(-1/e, 0)$ while the square refers to $(-\pi/2, 0)$. Right panel: the complex plane $w \in \mathbf{C}$, where $w = W_0(z)$. The solid blue line is the image of the upper part of the branch cut $\Im w \mapsto \Re w = -\Im w \operatorname{cotan} \Im w$ for $0 < \Im w < \pi$, while the dashed blue line is the image of the lower branch cut via W_0, $\Im w \mapsto \Re w = -\Im w \operatorname{cotan} \Im w$ for $-\pi < \Im w < 0$. The circle at coordinates $(-1, 0)$ is the image of the point $(-1/e, 0)$ and the squares positioned at $(0, \pi/2)$, respectively at $(0, -\pi/2)$, are the image of the point of coordinates $(-\pi/2, 0^+)$, respectively $(-\pi/2, 0^-)$. The red dashed line is the image of the positive real axis while the green curved line is the image of the imaginary axis $\Re z = 0$. Observe that if $\Im z = 0$ and $-\pi/2 < \Re z < -1/e$, then $-1 \le \Re W_0(z) < 0$, $|\Im W_0(z)| < \pi/2$ and $\Im W_0(z) \ne 0$ (blue solid and dotted lines), if $\Im z = 0$ and $-1/e \le \Re z \le 0$, then $-1 \le \Re W_0(z) \le 0$ and $\Im W_0(z) = 0$ (blue dashed line), if $\Im z = 0$ and $\Re z < -\pi/2$, then $\Re W_0(z) > 0$ and $|\Im W_0(z)| > \pi/2$ (blue solid and dotted lines) and if $\Im z = 0$ and $\Re z > 0$, then $\Re W_0(z) > 0$ and $\Im W_0(z) = 0$ (red dashed line).

Hence, from the properties of the Lambert W-function represented in fig. 1, this amounts to require

$$-\frac{\pi}{2} < \tau A < 0. \tag{5}$$

We now turn to considering the effect of a nonhomogeneous perturbation, superposed to the postulated homogeneous equilibrium. This implies studying the full system (1), with $\tau_r = \tau_d = \tau > 0$ and $D > 0$. Linearizing as before, we get

$$\dot{\delta x}_i(t) = A\delta x_i(t-\tau) + D\sum_j L_{ij}\delta x_j(t-\tau). \tag{6}$$

We then introduce as in [6] the Laplacian matrix and the corresponding complete set of orthonormal eigenvectors ϕ^α, associated to the topological eigenvalues $0 = \Lambda^1 > \Lambda^2 \ge \ldots \ge \Lambda^n$, $\sum_j L_{ij}\phi_j^\alpha = \Lambda^\alpha \phi_i^\alpha$, for $\alpha = 1, \ldots, n$. This approach has already been shown effective to study pattern formation for reaction-diffusion system defined on networks [6,33] but also on multiplex [34,35] allowing to use this basis to decompose $\delta x_i(t)$ instead of the Fourier basis used in the continuous setting. Employing once again the ansatz of exponential growth

$$\delta x_i(t) = \sum_\alpha c_\alpha e^{\lambda_\alpha t}\phi_i^\alpha, \quad \text{for all} \quad i = 1, \ldots, n, \tag{7}$$

we eventually get from (6) the characteristic equation

$$\lambda_\alpha - (A + D\Lambda^\alpha)e^{-\lambda_\alpha \tau} = 0, \tag{8}$$

whose solutions are

$$\lambda_{\alpha,k} = \frac{1}{\tau}W_k\left(\tau(A + D\Lambda^\alpha)\right), k \in \mathbf{Z} \text{ and } \alpha = 1, \ldots, n. \tag{9}$$

Symmetry-breaking instabilities seeded by diffusion can set in if at least a pair k and α exists such that $\Re\lambda_{\alpha,k} > 0$, provided $-\frac{\pi}{2} < \tau A < 0$ (the homogeneous equilibrium must be stable as a prerequisite of the analysis)[2]. When a bounded family of k exists for which $\Re\lambda_{\alpha,k} > 0$, the one with the largest real part dominates the instability and shapes the emerging pattern. Recalling again eq. (3), we finally get the following sufficient condition for the onset of Turing-like instability in a one-species DDE of the general type (1):

$$\exists \alpha \in [2, \ldots, n] \text{ such that } \frac{1}{\tau}\Re W_0\left(\tau(A + D\Lambda^\alpha)\right) > 0. \tag{10}$$

Exploiting the properties of the Lambert W-function represented in fig. 1, eq. (10) is satisfied whenever[3] there exists $\bar\alpha > 1$ such that $\tau D\Lambda^{\bar\alpha} < -\frac{\pi}{2} - \tau A$.

Observe also that for such $\bar\alpha$, $\Im\lambda_{0,\bar\alpha} \ne 0$. Hence, the instability materialize in the appearance of traveling waves. Since $\Re W_0(x)$ is increasing, for decreasing real $x < -\pi/2$, the dispersion relation attains its maximum at $\alpha = n$, the Laplacian eigenvalues being ordered for decreasing real parts. Hence, the Turing-like instabilities cannot develop if $\tau D\Lambda^n > -\frac{\pi}{2} - \tau A$. Summing up, traveling waves are expected to develop, for a fixed network topology, and sufficently large values of τ and D. Conversely, for a fixed choice of the parameters (τ, D and A), one should make

[2]Remark that $\alpha > 2$: in fact, for $\alpha = 1$ we have $\Lambda^1 = 0$ and thus, by assumption, $\Re\lambda_{1,k} = \Re W_k(\tau A) < 0$.

[3]It is worth emphasizing that, in principle, one could also satisfy eq. (10) with $\tau(A + D\Lambda^\alpha) > 0$. This alternative condition is however not compatible with the stability request (5) and the negativeness of Λ^α for all α.

the network big and so force Λ^n to be large enough, in absolute value.

As our goal is to determine the minimal model for which the aforementioned instability sets in, we now consider the reduced case for $\tau_r = 0$ and $\tau_d = \tau > 0$, that is the delay is only associated to the diffusion part. For $D = 0$ the homogeneous equilibrium $\hat{x}$, is stable if and only if $A < 0$. By repeating the procedure highlighted above we end up with the following characteristic equation: $\lambda_\alpha - A - D\Lambda^\alpha e^{-\lambda_\alpha \tau} = 0$, whose solution reads

$$\lambda_{\alpha,k} = \frac{1}{\tau} W_k(\tau D \Lambda^\alpha e^{-\tau A}) + A, \quad k \in \mathbf{Z}. \tag{11}$$

The equilibrium $\hat{x}$ is hence destabilized by diffusion if there exist $\alpha > 1$ for which

$$\Re W_0(\tau D \Lambda^\alpha e^{-\tau A}) > -A\tau. \tag{12}$$

Let us observe that $-A\tau$ is positive and $\tau D \Lambda^\alpha e^{-\tau A}$ is negative. Moreover, we have already remarked that $\Re W_0(x)$ is positive and increasing for decreasing negative x. Hence, a critical $x_c < 0$ exists[4] for which $\Re W_0(x_c) = -A\tau$ and eq. (12) is satisfied for all $\tau D \Lambda^\alpha e^{-\tau A} < x_c$. Traveling waves are hence predicted to manifest, for sufficiently large $\Lambda^{\bar{\alpha}}$, in absolute value. We again stress that $\Im \lambda_{0,\bar{\alpha}} \neq 0$.

To complete the general discussion we consider the dual problem, where $\tau_r = \tau > 0$ and $\tau_d = 0$. As already observed, the homogeneous equilibrium $\hat{x}$ is stable if $-\pi/2 < \tau A < 0$. The characteristic equation associated to this problem can be cast in the form $\lambda_\alpha - A e^{-\lambda_\alpha \tau} - D\Lambda^\alpha = 0$, whose solution is

$$\lambda_{\alpha,k} = D\Lambda^\alpha + \frac{1}{\tau} W_k(\tau A e^{-\tau D \Lambda^\alpha}), \quad k \in \mathbf{Z}. \tag{13}$$

One can prove[5] that $D\Lambda^\alpha + \frac{1}{\tau} W_0(\tau A e^{-\tau D \Lambda^\alpha}) < 0$ for all Λ^α, $\alpha > 1$. We are consequently led to conclude that the stable homogeneous equilibrium cannot undergo a diffusion driven Turing-like instability, if the delay term is solely confined in the reaction part.

As an application of the previous theory, we take f in eq. (1) to be the logistic function $f(x) = ax(1-x)$, as in the spirit of the Fisher model. At variance with the Fisher equation [1,2], we now imagine the species to be hosted on a discrete support, rather than on a continuum segment. In the original Fisher scheme the emerging wave relates to the heteroclinic orbit of the system and requires a specific, step-like, initial profile. In our case the traveling wave will originate following a symmetry-breaking instability of an initial random perturbation. To carry out the analysis, we select the homogeneous equilibrium solution $\hat{x} = 1$ and after look at its associated stability properties, as a function of a, τ, D and the network topology. Silencing the diffusion, $D = 0$, the stability condition in eq. (5) rewrites $0 < a\tau < \frac{\pi}{2}$. In demonstrating our findings, we consider a Watts-Strogatz network [36] made of 100 nodes, with average degree $\langle k \rangle = 6$ and probability to rewire a link $p = 0.03$. Other network topologies can be in principle assumed, returning similar qualitative conclusions. The chosen network is large enough so to have one eigenvalue for which the dispersion relation has positive real part (see left panel of fig. 3): thus Turing-like waves do exist. In fig. 2 we report the result of a numerical solution of the system under scrutiny obtained with a RK4 method adapted to deal with constant delay differential equation. The parameters are set to the values $a = 1.4$, $D = 0.05$ and $\tau = 1$. The DDE should be complemented with the value of the function on the delay interval $[-\tau, 0)$. In the spirit of a perturbation of the stable equilibrium, we decided to set $x_i(t) = 1 + \delta_i$ for all $t \in [-\tau, 0)$ where δ_i are random Gaussian numbers drawn from $N(0, 0.01)$. Observe that $a\tau = 1.4 < \pi/2$ and thus the equilibrium $\hat{x} = 1$ is stable in the absence of diffusion. On the other hand, one can clearly appreciate that after a transient period, patterns do manifest as stable oscillations around the solution $\hat{x} = 1$.

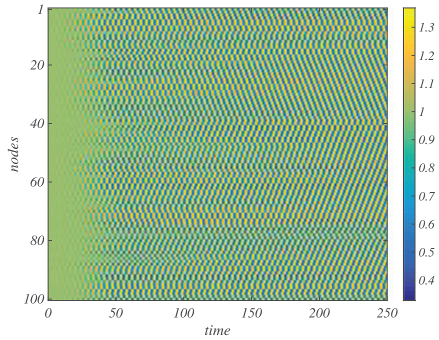

Fig. 2: (Colour on-line) Numerical solution of the system under scrutiny with parameters $a = 1.4$, $D = 0.05$ and $\tau = 1$. Initial conditions are set equal to $x_i(t) = 1 + \delta_i$ for all $t \in [-\tau, 0)$ where δ_i are random Gaussian numbers drawn from $N(0, 0.01)$. The underlying network is a Watts-Strogatz network [36] made of 100 nodes, with average degree $\langle k \rangle = 6$ and probability to rewire a link $p = 0.03$.

In fig. 3 the dispersion relation is displayed. In the left panel the quantity

$$\max_k \Re \lambda_{\alpha,k} = \Re \lambda_{\alpha,0} = \frac{1}{\tau} \Re W_0\left(-\tau a + \tau D \Lambda^\alpha\right), \tag{14}$$

is plotted as a function of Λ^α, for fixed a, D and τ (as specified in fig. 2), and for the same Watts-Strogatz network. One can clearly identify several eigenvalues for which $\Re \lambda_{\alpha,0} > 0$. The right panel reports the maximum of the dispersion relation as a function of the eigenvalues, *i.e.* $\max_\alpha \Re \lambda_{\alpha,k}$, *vs.* (a, τ) for fixed $D = 0.05$ and for the same Wattz-Strogatz network. The stability domain of the equilibrium without diffusion is bounded by $a > 0$ ($\tau > 0$, for physical reasons) and $a\tau < \pi/2$ (dash-dotted black curve). The Turing-like waves can emerge for all pairs (a, τ), inside such limited domain, for which $\max_\alpha \Re \lambda_{\alpha,0} > 0$.

[4] See appendix A for the explicit computation of x_c.
[5] See appendix B for a rigorous proof of the claim.

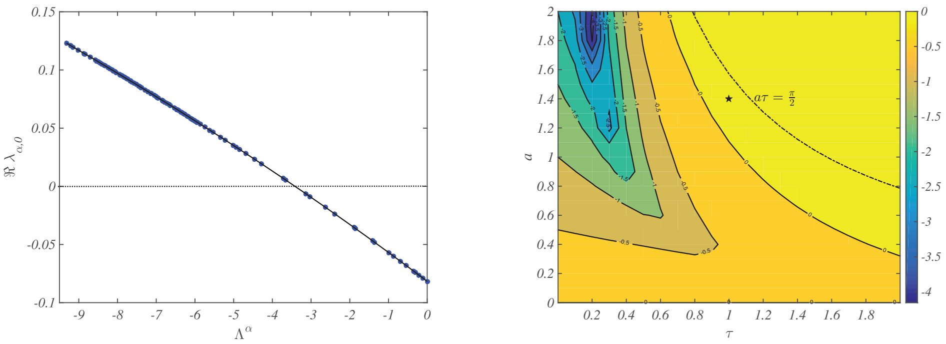

Fig. 3: (Colour on-line) Dispersion relation. Left panel: $\Re\lambda_{\alpha,0}$ as a function of Λ^α, for fixed a, D and τ, as specified in fig. 2 and for the same Wattz-Strogatz network. Blue dots represent the values of $\lambda_{\alpha,0}$ computed for a given Λ^α, while the solid black line is the continuous approximation. The horizontal dotted black line stands for the 0-th level. Right panel: $\max_\alpha \Re\lambda_{\alpha,0}$, as a function of (a,τ), for fixed $D = 0.05$ using the same Wattz-Strogatz network as employed in fig. 2. The dash-dotted curve $a\tau = \pi/2$ delineates the boundary (together with $a > 0$ and $\tau > 0$) of the stability region of the homogeneous solution $\hat{x} = 1$. Hence, all pairs (a,τ) for which $0 < a\tau < \pi/2$ correspond to stable solution of the homogeneous equilibrium. Turing-like instabilities are thus allowed to develop for all pairs (a,τ) inside such domain, for which $\max_\alpha \Re\lambda_{\alpha,0} > 0$. The star refers to the setting of the simulation reported in fig. 2.

In conclusion, we have hereby shown that a one-species time-delay reaction-diffusion system defined on a complex network can exhibit traveling waves, following a symmetry-breaking instability of a homogeneous stationary stable solution, subject to an external nonhomogeneous perturbation. These are Turing-like waves which emerge in a minimal model of single-species population dynamics, as the unintuitive byproduct of the imposed delay. Based on a linear-stability analysis adapted to time-delayed differential equations, we provided sufficient conditions for the onset of the instability, as a function of key quantities, such as the reaction parameters, the delay, the diffusion coefficient and the network topology. The wave possesses multiple fronts and persists in time, without fading away as it occurs for the customary Fisher equation. The observation that Turing-like instability can originate for a one-species model evolving on a heterogeneous graph, provided a delay is included in the transport term, enables us to significantly relax the classical constraints for the patterns to emerge and opens up the perspective for intriguing developments in a direction so far unexplored.

* * *

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Appendix A: Turing-like instability when the delay is confined in the diffusion term. – To study the stability of the equilibrium $\hat{x}$ in the case the delay is present only in the diffusion part, we have to look for $\alpha > 1$ for which

$$\Re W_0(\tau D \Lambda^\alpha e^{-\tau A}) > -A\tau. \tag{A.1}$$

Let $z = x+iy$ and $w = \xi + i\eta$, then the following relation is recovered once we impose $z = we^w$, namely $w = W_0(z)$:

$$x = e^\xi(\xi\cos\eta - \eta\sin\eta), \tag{A.2}$$
$$y = e^\xi(\eta\cos\eta + \xi\sin\eta). \tag{A.3}$$

Let us introduce $x_c = \tau D \Lambda^{\bar{\alpha}}$ and $\xi_c = -\tau A$, values for which the equality holds in (A.1). Observe that $x_c < 0$ and $\xi_c > 0$. Rewriting eq. (A.2) for $z_c = x_c$ (namely $y_c = 0$) and $w_c = \xi_c + i\eta_c$ we get

$$x_c = e^{\xi_c}(\xi_c\cos\eta_c - \eta_c\sin\eta_c), \tag{A.4}$$
$$0 = e^{\xi_c}(\eta_c\cos\eta_c + \xi_c\sin\eta_c). \tag{A.5}$$

From the second equation one can obtain (implicitly) η_c as a function of ξ_c: $\eta_c = -\xi_c \tan\eta_c$. Because $\xi_c < 0$ one can find a unique solution $\eta_c(\xi_c) \in (\pi/2, \pi)$ (and the opposite one). Inserting this result into the first equation we get

$$x_c(\xi_c) = -e^{-\eta_c(\xi_c)\,\mathrm{cotan}\,\eta_c(\xi_c)} \frac{\eta_c(\xi_c)}{\sin\eta_c(\xi_c)}. \tag{A.6}$$

So in conclusion given $-\tau A$ one can obtain the critical $x_c(-\tau A)$ for which we have equality in eq. (A.1), and thus conclude, by invoking the scaling properties of W_0, that for all $\Lambda^\alpha < x_c(-\tau A)/(\tau D)$ the strict inequality sign holds in eq. (A.1).

Appendix B: Turing-like instability are impeded when the delay only appears in the reaction term. – Let us consider system (1) once the delay dependence is given only through the reaction term. The condition for the stability of the homogeneous equilibrium without diffusion is given by eq. (5). Following the same procedure outlined in the main body of the paper —linearizing, then expanding the perturbation on the basis of the eigenvectors of the Laplacian— one obtains the following characteristic equation: $\lambda_\alpha - Ae^{-\lambda_\alpha \tau} - D\Lambda^\alpha = 0$, that can be solved using the W-Lambert function to give

$$\lambda_{\alpha,k} = D\Lambda^\alpha + \frac{1}{\tau} W_k(\tau A e^{-\tau D\Lambda^\alpha}), \qquad k \in \mathbf{Z}. \tag{B.1}$$

Turing-like instability can emerge if the homogeneous equilibrium becomes unstable in the presence of the diffusion. We then look for α and k such that $\Re\lambda_{\alpha,k} > 0$. Using Lemma 3 of [32] this is equivalent to

$$\Re W_0(\tau A e^{-\tau D\Lambda^\alpha}) > -\tau D\Lambda^\alpha. \tag{B.2}$$

Let us observe that $-\tau D\Lambda^\alpha \geq 0$. Because $\tau A < 0$, one cannot solve the previous equation with $\Im W_0(\tau A e^{-\tau D\Lambda^\alpha}) = 0$ (in this case one should have the argument of W_0 to be positive). Hence we look for $\tau A e^{-\tau D\Lambda^\alpha} < -\pi/2$.

Let us introduce $s = -D\Lambda^\alpha > 0$, $u = \tau A \in (-\pi/2, 0)$ and the function

$$g(s) = -s + \Re W_0(ue^s), \tag{B.3}$$

our goal is to prove that $g(s) < 0$ for all $s > 0$ which is in turn equivalent to stating that Turing-like instability cannot develop.

Let us rewrite eq. (A.2) for $x = ue^s$ and $y = 0$:

$$\begin{aligned} ue^s &= e^\xi(\xi\cos\eta - \eta\sin\eta), &\text{(B.4)}\\ 0 &= e^\xi(\eta\cos\eta + \xi\sin\eta). &\text{(B.5)}\end{aligned}$$

Isolating ξ in the second equation and inserting it in the first one, we get $e^s = -\frac{\eta}{u\sin\eta}e^\xi$. One can thus rewrite $g(s)$ as follows:

$$\begin{aligned} g(s) &= -s + \Re W_0(ue^s) = -s + \xi &\text{(B.6)}\\ &= -\xi - \log\left(-\frac{\eta}{u\sin\eta}\right) + \xi &\text{(B.7)}\\ &= -\log\left(-\frac{\eta}{u\sin\eta}\right). &\text{(B.8)}\end{aligned}$$

Because $u > -\pi/2$ and $\pi/2 < \eta < \pi$ we obtain $-\eta/(u\sin\eta) > 1$ and thus $\log(-\frac{\eta}{u\sin\eta}) > 0$.

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