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LETTER

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The value of conflict in stable social networks

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Abstract – A cooperative network model of sociological interest is examined to determine the sensitivity of the global dynamics to having a fraction of the members behaving uncooperatively, that is, being in conflict with the majority. We study a condition where in the absence of these uncooperative individuals, the contrarians, the control parameter exceeds a critical value and the network is frozen in a state of consensus. The network dynamics change with variations in the percentage of contrarians, resulting in a balance between the value of the control parameter and the percentage of those in conflict with the majority. We show that, as a finite-size effect, the transmission of information from a network B to a network A, with a small fraction of lookout members in A who adopt the behavior of B, becomes maximal when both networks are assigned the same critical percentage of contrarians.

Introduction. – The regulatory dynamics of the brain [1], the cardiovascular and other physiological systems [2], and indeed most biological/sociological networks appear to be poised at criticality [3]. The existence of phase transitions is so common, in part, because criticality is the most parsimonious way for a many-body system, with nonlinear interactions to exert self-control. Inhibitory links in neurophysiology and contrarians in sociology are the names given to interactions that evoke the disruption of organization and consensus, thereby suggesting that the well-being of either the brain or human society requires the containment of those negative agents. However, recent neurophysiological literature shows that this perspective may be overly restrictive, and that a sufficiently large concentration of inhibitory links may counter-intuitively have the beneficial effect of promoting a ceaseless activity [4], a characteristic that can provide evolutionary advantage.

One of the first explanations of abrupt social transitions in terms of criticality was made by Callen and Shapiro in 1974 [5]. They put together the concepts of social imitation and critical behavior a generation before Gladwell popularized the concept of the tipping point [6]. It is convenient to mention also ref. [7]. The authors of this approach to strike in big companies, made also a call to the creation of sociophysics. In the sociological phenomenon of interest to us here the role of inhibitory links is played by individuals called *contrarians* [8], yielding instabilities produced by frustration [9], an interesting phenomenon more recently discussed in refs. [10,11]. As the term frustration suggests, the action of contrarians is found to quench consensus or prevent its occurrence in accordance with the sociological conclusions of Crokidakis *et al.* [12]. However, we reach a different conclusion and find value in those individuals whose method of decision making are in conflict with the majority.

Herein the observation made in neurophysiology [4] is adapted to sociology using the decision making model (DMM). The complex network described by the DMM implements the echo response hypothesis, which assumes that the dynamic properties of a network of identical individuals is determined by individuals imperfectly copying the behavior of one another [13]. The effect of introducing contrarians into a cooperative social network is analyzed using a system of coupled two-state master equations. Using analytical calculations we show that in the presence of contrarians, increased cooperation effort, in the form of increased values of the DMM control parameter, is necessary to achieve consensus. At the same time, contrarians may promote a condition of ceaseless activity similar to that found in the cognitive context [4].

With the growing evidence of phase transitions in biological and sociological systems, recent studies have turned to information-theoretic analyses of canonical models for more insight. For instance, refs. [14–17] show that mutual information between elements peaks at the phase transition, while ref. [18] shows that in the Ising model information flow between elements peaks in the disordered regime. Following a different but related approach, we study the transmission of information from one DMM network to another through cross-correlation measurements. Our approach is related to the pioneer work of Galam and Moscovici [19], where the society individuals make a decision under the influence of both an internal and an external field, with the basic difference that, in this letter, the external influence is exerted by another network with the same complexity. We demonstrate that information transfer is maximally efficient when both systems are in a critical condition, as already found in earlier work [20,21]. The interesting novelty is that the control parameter necessary to yield criticality here is not the imitation strength, but the concentration of contrarians that for a specific value, hereby theoretically predicted, is proved to turn the supercritical condition into the critical condition.

Decision making model. – DMM network dynamics is a member of the Ising universality class and is found to be useful for describing phenomena related to social group behavior [13,22]. The network model is based on the dynamics of single individuals selecting one of two options. Denoting the two options as $+1$ and -1, the i-th individual generates the stochastic time series $s^{(i)}(t) = \pm 1$. Using the Gibbs perspective this time series is analyzed using the solution to the two-state master equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_1^{(i)}(t) &= -\frac{1}{2}g_{12}^{(i)}(t)p_1^{(i)}(t) + \frac{1}{2}g_{21}^{(i)}(t)p_2^{(i)}(t),\\ \frac{\mathrm{d}}{\mathrm{d}t}p_2^{(i)}(t) &= -\frac{1}{2}g_{21}^{(i)}(t)p_2^{(i)}(t) + \frac{1}{2}g_{12}^{(i)}(t)p_1^{(i)}(t),\end{aligned} \qquad (1)$$

where the $+1$ and -1 options have been labeled 1 and 2, respectively. The time-dependent transition rates $g_{12}^{(i)}(t)$ and $g_{21}^{(i)}(t)$ determine the production of decision events for the i-th individual. At the moment of making a decision the single individual tosses a coin to decide whether to keep the same opinion or to change her mind, hence the factors of 1/2 present in the master equation. Although this implies a random decision, the interaction with the other individuals may prolong or shorten the time necessary to make a decision, thereby generating a bias toward one of the two choices.

For notational simplicity let us describe the behavior of the i-th individual omitting the superscript i, while keeping in mind for now that this is a single individual and that there are $N-1$ other individuals in the network. The transition rate from state $|1\rangle$ to state $|2\rangle$ reads $g_{12} = g\exp\left[-K\left(M_1 - M_2\right)/M\right]$, where M is the number of individuals linked to the i-th individual, M_1 is the number of its neighbors in the state $|1\rangle$ and M_2 the number of its neighbors in the state $|2\rangle$. When the interaction coupling parameter K vanishes, the i-th individual generates a Poisson sequence $s(t)$ with decision events generated at the fixed rate g. When $K > 0$ the i-th individual cooperates with her neighbors. That is, when the i-th individual is in the state $|1\rangle$ and the majority of her neighbors share this state, the rate of her decision event productions decreases, thereby indicating that the i-th individual is likely to remain in the state $|1\rangle$ for a more extended time than in the absence of interaction. The same cooperative prescription holds true when the i-th individual is in the state $|2\rangle$, leading in this case to $g_{21} = g\exp\left[-K\left(M_2 - M_1\right)/M\right]$ and indicating that if the majority of her neighbors are in the state $|1\rangle$ the i-th individual makes decisions with a faster rate, thereby reducing her sojourn time in the state $|2\rangle$.

All-to-all coupling condition. – In this paper we adopt the all-to-all (ATA) coupling condition, which assigns to all the individuals the same number of neighbors, $M = N-1$, where N denotes the total number of network members. Since the total number of members usually satisfies $N \gg 1$, we set $M = N$. Under the ATA coupling condition all the individuals are described by only two transition rates.

The mean field of the network,

$$\xi\left(t\right) = \frac{1}{N}\sum_{i=1}^{N} s^{(i)}(t) = \frac{N_1\left(t\right) - N_2\left(t\right)}{N}, \qquad (2)$$

becomes identical to a probability difference in the limit $N \to \infty$ where $p_i = N_i/N$; $i = 1, 2$. Defining this probability difference as $x \equiv p_1 - p_2$, the master equation describing the mean-field behavior of an ATA DMM, consisting of an infinite number of cooperative individuals, becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{g}{2}(e^{Kx} - e^{-Kx}) - \frac{g}{2}(e^{-Kx} + e^{Kx})x. \qquad (3)$$

The equilibrium value of the mean field can be determined by setting the left-hand side of eq. (3) equal to zero, which yields the equation for the equilibrium value of the mean field: $x_{eq} = \tanh\left(Kx_{eq}\right)$. A second-order phase transition occurring in the cooperative system at $K = 1$ can be predicted as follows. If we make the assumption that at the phase transition the equilibrium value of x is very close to zero, then using the Taylor series expansion of the hyperbolic tangent gives us $x_{eq} = Kx_{eq}$, which is compatible with a small but non-vanishing solution only for $K = 1$.

An individual is a contrarian if she is inclined to make a decision that is the opposite of the one made by her neighbors [8]. Thus, for instance, the g_{12} transition rate would become $g_{12} = g\exp\left[K\left(M_1 - M_2\right)/M\right]$, and the g_{21} transition rate would become $g_{21} = g\exp\left[-K\left(M_1 - M_2\right)/M\right]$.

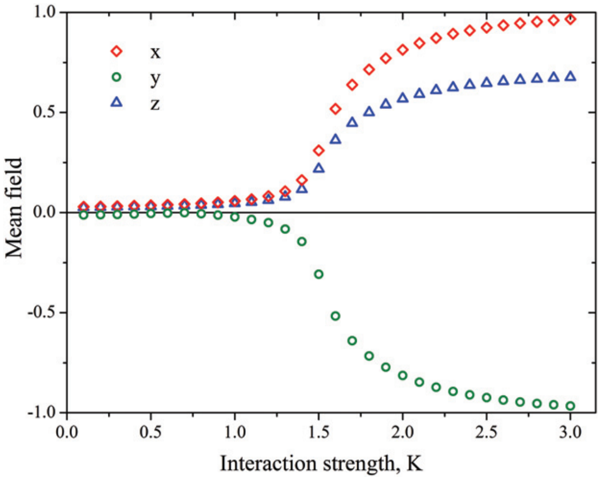

Fig. 1: (Color online) Mean fields of the ATA DMM networks. The diamonds refer to the mean field of cooperators (x), the circles to the mean field of contrarians (y), and the triangles to the global mean field (z). $N = 10^3$ units, $g = 0.01$, and $q = 0.15$. Note that the mean field of contrarians is opposite to the mean field of cooperators.

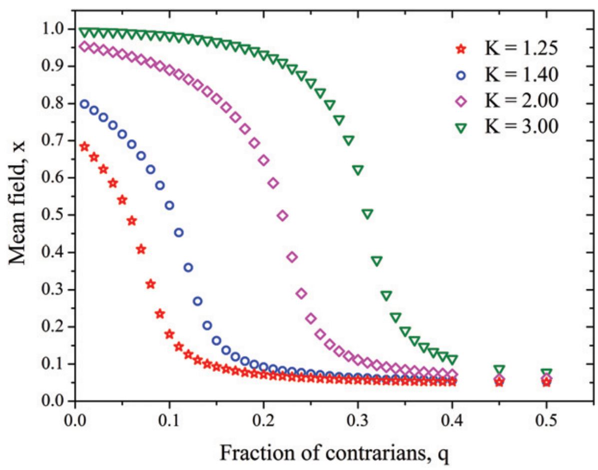

Fig. 2: (Color online) The mean field of cooperators (x) with different fractions of contrarians (q) in the ATA DMM network of 10^3 units with $g = 0.01$. Increasing the fraction of contrarians has the effect of turning the supercritical into critical condition for a convenient fraction of contrarians.

By following the same line of reasoning as that generating eq. (3) we obtain for an ATA DMM of contrarians

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{g}{2}(e^{-Ky} - e^{Ky}) - \frac{g}{2}(e^{Ky} + e^{-Ky})y. \tag{4}$$

We use the variable y to denote the mean field of contrarians for the purpose of distinguishing contrarians from cooperators. Recall that the variable x is associated with individuals that are cooperators.

When all the people in the network are contrarians, the network remains close to the condition of a vanishing mean field. In fact, the Taylor series expansion of the hyperbolic tangent in the solution now yields $y_{eq} = -Ky_{eq}$, which implies that the network remains fixed at the equilibrium value $y_{eq} = 0$, independently of the value K of the interaction strength.

We note however, see fig. 1, that in a network with only a small concentration of contrarians a phase transition occurs with the important symmetry property $y = -x$, indicating that the mean field of contrarians y has the same intensity as the mean field of cooperators x, but with the opposite sign. We are thus led to examining the condition where the global field z, with a fraction q of the individuals being contrarians, is expressed by $z = (1-q)x + qy$, which, using symmetry, becomes $z = (1-2q)x$. The master equation for the cooperators can then be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g\sinh\left[K(1-2q)x\right] - gx\cosh\left[K(1-2q)x\right], \tag{5}$$

and the master equation for the contrarians as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -g\sinh\left[K(1-2q)x\right] - gy\cosh\left[K(1-2q)x\right]. \tag{6}$$

Note that the arguments of the exponential functions coincide with the global field z, which is perceived by both cooperators and contrarians, but contrarians react oppositely to that of the majority reaction to the global field z. In fact, eq. (6), determining the time evolution of y, is obtained by replacing K with $-K$ in the exponential functions of eq. (5) representing the influence of the neighbors on the decisions of single individuals.

We see that, as expected, the equilibrium condition generated by eqs. (5) and (6) is $x_{eq} = \tanh(K'x_{eq}) = -y_{eq}$, where $K' \equiv K(1-2q)$. It is evident that a phase transition occurs following the same mathematical prescription as in the absence of contrarians with the main difference being that the critical value of K is now given by

$$K_c(q) = \frac{1}{1-2q}. \tag{7}$$

Consequently, the critical control parameter increases in value as the fraction of contrarians increases and the interaction effort necessary to make a social decision in the presence of 50% contrarians becomes infinitely large. Therefore, consensus cannot be reached beyond the limit of 50% contrarians.

Figure 2 illustrates the formation of a social decision in a diverse social network, having a mixture of cooperators and contrarians, and the required strength of interaction coupling is in agreement with eq. (7). The figure shows that when the network is in the supercritical condition in the absence of contrarians, the action of an increasing number of contrarians has the effect of shifting the network dynamics down towards the critical point. Inverting eq. (7) to obtain q yields

$$q_c(K) = \frac{1}{2}\left(1 - \frac{1}{K}\right), \tag{8}$$

indicating that if the interaction strength K is a fixed property of the network, there exists a specific fraction of contrarians q_c that will bring the network to the critical point. Thus, the network dynamics can be adjusted to operate at the critical point in three distinct ways: 1) with no contrarians, a subcritical interaction strength can be increased to a critical value; 2) with no contrarians, a supercritical interaction strength can be decreased to a critical value; and 3) for a fixed interaction strength above the critical value, the fraction of contrarians can be increased to the critical point for the network dynamics.

Network-network interaction. – At criticality the DMM mean field fluctuates around the vanishing equilibrium value and the time distance between two consecutive regressions to this "free-will" [20] condition has an inverse power law distribution density with index $\mu = 1.5$ making the free-will events renewal. This is the temporal-complexity [23] property making the transmission of information from a driving complex network B to a driven network A maximally efficient [21].

We consider two identical ATA DMM social networks each with $N = 10^3$ individuals, with their interaction strength fixed at $K = 1.25$. Note, that eq. (8) predicts that an interaction strength with 10% contrarians is necessary to realize criticality. To connect the driving network B to the driven network A, we introduce into network A an additional 20 "lookout" individuals that track the global field of network B. This choice of coupling between the two networks was inspired by a recent experiment [24] where the signal from a few electrodes implanted in the brain of a rat B is the information transmitted directly to the brain of rat A. The fraction of lookout individuals is determined by the theoretical arguments discussed in ref. [21] so as to make the correlation between A and B emerge at the level of a single realization.

Keeping an equal fraction of contrarians q in both networks, we evaluate the cross-correlation between network A and network B for various q values. Figure 3 shows that the transmission of information from network B to network A becomes maximally efficient at $q = 0.09$. The numerical value for the peak of the cross-correlation curve is near the theoretical value $q_c = 0.1$, predicted by eq. (8) on the basis of the conjecture that criticality maximizes the efficiency of the information transport. One possible reason for the slight discrepancy between the numerical results and the value expected from theory is that eq. (8) does not account for the presence of lookout individuals in the DMM network. A fraction of lookout individuals p creates the effective interaction strength $K_p = K(1-p)$ (see eq. (18) below). For the numerical simulations we have $p = 20/1020$ leading to $K_p \approx 1.225$. Replacing K by K_p in eq. (8) yields $q_c = 0.092$ for the critical fraction of contrarians, refining the agreement with the results depicted in fig. 3.

We also see from fig. 3, that when the concentration of contrarians tends to vanish, the transmission of information between networks becomes very small. This is so because the social system falls in the supercritical condition,

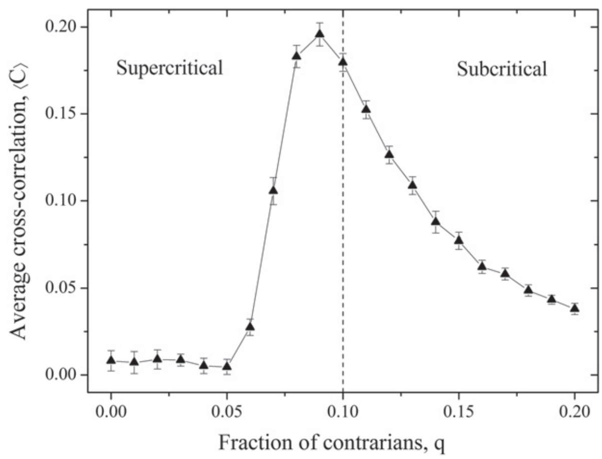

Fig. 3: The average cross-correlation function ($\langle C \rangle$) between two predominantly cooperative networks (the ATA DMM networks with $g = 0.01$ and $K = 1.25$) peaks when there is a critical fraction of contrarians present in both networks. Similar behavior, not shown here, is obtained by assigning to K different values larger than 1.

which is not resilient and is unsuitable to address crucial issues. On the other hand, a fraction of contrarians larger than the critical concentration q_c of eq. (8), realizing the subcritical condition, is still compatible with a significant transmission of information, due to the distinctly asymmetric shape of $\langle C \rangle$ as a function of q.

The process of calculating the cross-correlation involves first generating the mean-field trajectory of the driving system B, $\xi_B(t)$, and then using $\xi_B(t)$ to generate the corresponding mean-field trajectory of the driven system $\xi_A(t)$. Note that each lookout individual in system A adopts the time series $\xi_l(t) = \text{sign}\{\xi_B(t)\}$, influencing the rest of the individuals in system A; but we do not count the lookout individuals when calculating $\xi_A(t)$. Since we are interested in fluctuations around equilibrium, from the mean-field trajectories we calculate the deviation from their time-averaged mean values, $\tilde{X}(t) = |\xi(t)| - \overline{|\xi(t)|}$, where the overline indicates a time-averaged quantity and $|\cdot|$ the absolute value. Finally, the cross-correlation (with zero delay) is expressed as

$$C = \frac{\overline{\tilde{X}_A(t)\tilde{X}_B(t)}}{\left(\overline{\tilde{X}_A(t)^2\tilde{X}_B(t)^2}\right)^{1/2}}. \tag{9}$$

The time averages are calculated over a finite time $T = 10^7$, resulting in the cross-correlation C being a fluctuating quantity. For this reason, in fig. 3 we plot $\langle C \rangle$, where the ensemble average indicated by the brackets was taken over 10 independent realizations of networks A and B, and the error bars represent the standard deviation in the ensemble.

Although C fluctuates, it is important to note that the observation time of $T = 10^7$ is much longer than the time it takes an isolated all-to-all system of size $N = 10^3$ to reach equilibrium [25]. In fact, according to ref. [26] the time necessary to reach equilibrium, T_{eq}, is given by

$$T_{eq} = \frac{1}{1.4\sqrt{(\gamma D)}}, \quad (10)$$

where D is the intensity of finite-size–induced fluctuations. This theoretical prediction is related to the decision making model in the all-to-all condition by setting [25]

$$\gamma = \frac{g}{3} \quad (11)$$

and

$$D \propto \frac{g^2}{N}. \quad (12)$$

With these values we obtain $T_{eq} \approx 4 \times 10^4$. Note that the exact proportionality factor of eq. (12) is unknown, thereby making this estimate of T_{eq} reliable only as far as the order of magnitude is concerned. This leads us to conclude that the adopted value of T is much larger than T_{eq} and that, consequently, we are operating the network dynamics in the ergodic regime where the statistical validity of time averages is ensured. The theory of this paper can be extended to the case $T < T_{eq}$, where important non-ergodic effects have to be taken into account, but this is left as a subject of future investigation.

Considering again the case of an infinite number of units, we notice that when there are l lookout individuals, the mean field of the complete network is

$$\xi = \frac{N_1 - N_2}{N} = \frac{n_1 - n_2 + l\xi_l}{N}, \quad (13)$$

where n_1 and n_2 are the number of ordinary individuals in states $|1\rangle$ and $|2\rangle$, and ξ_l is the state that the lookout individuals adopt. Introducing $p = l/N$ for the fraction of lookout individuals, the mean field becomes $\xi = (1-p)z + p\xi_l$, where we retain the notion that z is the mean field of the ordinary individuals, consisting of a generic mixture of cooperators and contrarians. Recalling that $z = (1-q)x + qy$, we have that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1-q)\frac{\mathrm{d}x}{\mathrm{d}t} + q\frac{\mathrm{d}y}{\mathrm{d}t}. \quad (14)$$

Going back to eq. (5) and eq. (6), in the arguments of the exponential functions we replace $z = (1-2q)x$ with ξ for a diverse network, because the transition rates of the ordinary individuals now depend on the mean field of the full network, with lookout individuals included. We substitute the result into eq. (14) to obtain the master equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (1-2q)g\sinh(K\xi) - gz\cosh(K\xi) \quad (15)$$

describing the mean-field behavior of the ordinary units in the presence of lookout individuals. The equilibrium mean field is defined by the equation

$$z = (1-2q)\tanh(K\xi) \approx (1-2q)\tanh\left(K(1-p)z\right), \quad (16)$$

where the approximation ignores the contribution to ξ from $p\xi_l$, under the assumption that there are only a few lookout individuals so that $p \ll 1$. Evaluating the value of the interaction strength at the critical point by a Taylor expansion gives

$$K_c(q,p) = \frac{1}{1-p}\frac{1}{1-2q}. \quad (17)$$

Alternatively, inverting the expression to obtain the fraction of contrarians yields

$$q_c(K,p) = \frac{1}{2}\left(1 - \frac{1}{K(1-p)}\right). \quad (18)$$

Looking back to eq. (8), we see that the effects of introducing lookout individuals into the network can be accounted for by redefining an effective interaction strength $K_p = K(1-p)$ among the ordinary people.

Finite-size effect. – The recent work of ref. [27] raised the important issue of the effects that a finite size may have on the statistical prediction. The model under study in this case is the q-voter model [28]. The authors of this paper noticed that three different groups [29–31] got analytically and numerically a continuous function for the exit probability, namely, the probability that in the long-time limit all the units of the social system share the opinion of a small fraction of units with a given opinion. The q-voter model shares the DMM assumption that each unit is led to adopting the opinion of the majority of its neighbors. The authors of refs. [29–31] found for the exit probability a continuous function in sharp contrast with the step function predicted in the earlier work of ref. [32]. The authors of ref. [27] pointed out that this striking difference is a consequence of the fact that the numerical work is limited to finite-size systems, the number of units of numerical simulation being limited to $N = 10^3$, which is also the number of units considered in this letter. On the other hand, these authors stress that the social systems do not fit the condition of physical systems, with N being the Avogadro number, thereby advocating the need to understand the finite-size effects. The importance of understanding the finite-size influence, as well as the effects of the structure of the network, has been supported also by the authors of ref. [33]. We want to stress that the central result of this manuscript is a consequence of the finite size of the system. In fact, as made clear by eq. (12), in the ideal case of infinite size the stochastic force responsible for temporal complexity [23] vanishes, and with it the transfer of information from one social network to another is annihilated. As explained in ref. [26], as a consequence of finite size, the correlation of the mean-field fluctuations around equilibrium is not stationary and this departure from the ordinary Poisson condition is the source of the efficiency of information transport [20,21].

Concluding remarks. – While the results of the present paper lend support to the attractive discovery of the neural benefits of inhibitory links in cognitive networks [4], our findings have a distinct sociological significance as well. They suggest a kind of equivalence between two apparently quite different forms of complexity, one of neurophysiological interest [4] and the other of sociological interest, the latter pertaining to the ATA DMM used herein.

The peaking of the cross-correlation function in fig. 3 indicates that the concentration of contrarians within a network can be used to establish a form of resonance between a driven and a driving network, a central result of this paper. When the concentration q is assigned the critical value forcing the network to transition from a disordered state to the condition when consensus is possible, the two networks establish a kind of synchronization.

As a potential impact of the results of this paper, we notice that the intensity of the cross-correlation becomes negligible for values of q smaller than the critical value, with the network freezing in a rigid consensus with a locked-in dependence among individuals, this being a condition of flawed democracy [34]. We believe that criticality corresponds to the condition of full democracy that, according to the authors of ref. [34], would be necessary to promote the energy sustainability of future generations.

As earlier pointed out, finite-size–induced temporal complexity is the main ingredient behind the transfer of information from one to another social network, this being in fact the main difference compared to the pioneer work of Galam and Moscovici [19]. We hope that this letter may attract the attention of the researchers on this important finite-size effect.

* * *

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