



LETTER

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Why are tensor field theories asymptotically free?

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Abstract – In this pedagogic letter we explain the combinatorics underlying the generic asymptotic freedom of tensor field theories. We focus on simple combinatorial models with a $1/p^2$ propagator and quartic interactions and on the comparison between the intermediate field representations of the vector, matrix and tensor cases. The transition from asymptotic freedom (tensor case) to asymptotic safety (matrix case) is related to the crossing symmetry of the matrix vertex, whereas in the vector case, the lack of asymptotic freedom (“Landau ghost”), as in the ordinary scalar ϕ_4^4 case, is simply due to the absence of any wave function renormalization at one loop.

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Introduction. – The Grosse-Wulkenhaar model [1] is a renormalizable non-commutative field theory of the Moyal type which can also be considered as an abstract matrix model with a propagator breaking unitary invariance. It is asymptotically safe [2].

In contrast, asymptotic freedom is a generic feature of renormalizable tensor field theories [3,4] discovered by Ben Geloun and his collaborators [5,6]. This remarkable property makes such tensor field theories attractive candidates for quantizing gravity [7]. Just remember how the discovery of asymptotic freedom in non-Abelian gauge theories was the key to our current understanding of particle physics. However princeps papers such as [6] are difficult to read, especially since models with sixth-order interactions are treated, and computations are pushed up to four loops, with subtle issues related to the stability domain and the role of the relevant fourth-order interactions recently underlined [8]. These issues are briefly discussed at the end of this letter. However, our main purpose here is to avoid such technicalities and to better clarify the mechanism responsible for this asymptotic freedom of tensor models, concentrating for pedagogical reasons on the computation of the one-loop β function of the simplest renormalizable combinatorial models with a $1/p^2$ propagator and quartic interactions. We shall compare the intermediate field representation of such quartic models in the vector, matrix and tensor case. Indeed this representation simplifies the combinatorics involved and illuminates why the sign of the one-loop beta coefficient is positive for vectors, zero for matrices and negative for tensors.

Quartic models and their intermediate field representation. – Consider a pair of complex conjugate fields $\phi(\theta)$ and $\bar{\phi}(\bar{\theta})$ on the d -dimensional torus $T_d = S_1^d$, hence $\theta = (\theta_1, \dots, \theta_d)$, $\theta_c \in S_1$. Their Fourier series will be noted as ϕ_p and $\bar{\phi}_{\bar{p}}$ where $p = (p_1, \dots, p_d)$, $\bar{p} = (\bar{p}_1, \dots, \bar{p}_d) \in \mathbb{Z}^d$. The corresponding Hilbert space is $\mathcal{H}_d = L^2(S_1^d) = \ell_2(\mathbb{Z}^d) = \otimes_{c=1}^d \mathcal{H}_c$, where each space $\mathcal{H}_c = L^2(S_1) = \ell_2(\mathbb{Z})$ corresponds to a given component also called *color* in the tensorial context [9,10].

We introduce the familiar Laplacian-based normalized Gaussian measure¹ for d -dimensional Bosonic fields with periodic boundary conditions

$$d\mu_C(\phi, \bar{\phi}) = \left(\prod_{p, \bar{p} \in \mathbb{Z}^d} \frac{d\phi_p d\bar{\phi}_{\bar{p}}}{2i\pi} \right) \text{Det}(C)^{-1} e^{-\sum_{p, \bar{p}} \phi_p C_{p\bar{p}}^{-1} \bar{\phi}_{\bar{p}}} \quad (1)$$

where the covariance C is, up to a field strength renormalization, the inverse of the Laplacian on S_1^d plus a mass term

$$C_{p, \bar{p}} = \frac{1}{Z} \frac{\delta_{p, \bar{p}}}{p^2 + m^2}. \quad (2)$$

Here $p^2 = \sum_{c=1}^d p_c^2$, m^2 is the square of the bare mass, and Z is the so-called wave function renormalization, which can be absorbed into a $(\phi, \bar{\phi}) \rightarrow (Z^{-1/2}\phi, Z^{-1/2}\bar{\phi})$ field strength renormalization. If we restrict the indices p , which should be thought as “momenta”, to lie in $[-N, N]^d$

¹This propagator is the familiar one of ordinary local scalar Bosonic field theory, but is also natural in the more abstract combinatorial context, see, e.g., [11].

rather than in \mathbb{Z}^d we have proper (finite-dimensional) fields. We can consider N as the ultraviolet cutoff, and we are interested in performing the ultraviolet limit $N \rightarrow \infty$.

The generating function for the moments of the model is

$$\mathcal{Z}(g, J, \bar{J}) = \frac{1}{\mathcal{Z}} \int e^{\bar{J} \cdot \phi + J \cdot \bar{\phi}} e^{-\frac{g}{2} V(\phi, \bar{\phi})} d\mu_C(\phi, \bar{\phi}), \quad (3)$$

where $\mathcal{Z} = \mathcal{Z}(g, J, \bar{J})|_{J=\bar{J}=0}$ is the normalization, g is the coupling constant², and the sources J and \bar{J} are dual respectively to $\bar{\phi}$ and ϕ . The generating function for the connected moments is $W = \log \mathcal{Z}(g, J, \bar{J})$.

The quartic vector, matrix and tensor *combinatorial* field theories correspond to choices of the quartic interaction $V(\phi, \bar{\phi})$ which simply differ in the way the momenta indices of the four fields branch at V . They correspond to different symmetry groups, and are not *local* in the usual sense on S_1^d . For instance the tensorial case corresponds to interactions invariant under $U(2N+1)^{\otimes d}$, hence under *independent* change of basis in \mathcal{H}_c for each color c [12,13]. Such combinatorial interactions are interesting in the context of quantization of gravity, since the corresponding Feynman graphs are dual to d -dimensional (colored) triangulations [14] pondered by a discretized form of the Einstein-Hilbert action [15].

The *vector* interaction is the square of the quadratic (mass) term, hence is factorized³ (not connected). It writes

$$V_V = \langle \bar{\phi}, \phi \rangle^2 = \sum_{p,q} (\phi_p \bar{\phi}_p) (\phi_q \bar{\phi}_q), \quad (4)$$

and it is just renormalizable for $d = 4$.

The *matrix* interaction makes sense only for $d = 2r$ even and is obtained by splitting the initial index as a pair (p, q) with $p = (p_1, \dots, p_r)$, $q = (q_1, \dots, q_r)$, hence splitting the space $\mathcal{H}_d = \mathcal{H}_r \otimes \mathcal{H}_r$. The field ϕ is then interpreted as the matrix ϕ_{pq} , with conjugate matrix $\phi^* = {}^t \bar{\phi}$ and the vertex V_M is an invariant trace

$$V_M = \text{Tr} \phi \phi^* \phi \phi^* = \sum_{p,q,p',q'} \phi_{pq} \bar{\phi}_{p'q} \phi_{p'q'} \bar{\phi}_{pq'}. \quad (5)$$

It is just renormalizable for $r = 4$, hence $d = 8$.

Finally the simplest *tensor* interaction V_T is the color-symmetric sum of *melonic* [10,16] quartic interactions [17,18]

$$\begin{aligned} V_T &= \sum_c V_c, \quad V_c(\phi, \bar{\phi}) = \text{Tr}_c (\text{Tr}_c \phi \bar{\phi})^2 \\ &= \sum_{p, \bar{p}, q, \bar{q}} \left[\phi_p \bar{\phi}_{\bar{p}} \prod_{c' \neq c} \delta_{p_{c'} \bar{p}_{c'}} \right] \delta_{p_c \bar{q}_c} \delta_{q_c \bar{p}_c} \left[\phi_q \bar{\phi}_{\bar{q}} \prod_{c' \neq c} \delta_{q_{c'} \bar{q}_{c'}} \right], \end{aligned} \quad (6)$$

where $\text{Tr}_c \phi \bar{\phi}$ means partial trace in $\mathcal{H}_d = \otimes_{c=1}^d \mathcal{H}_c$ over all colors except c , and Tr_c means trace over color c . The corresponding model is just renormalizable for $d = 5$ [19,20],

²The factor $1/2$ in front of g takes care of the \mathbb{Z}_2 symmetry of the quartic vertices considered below, hence simplifies their associated intermediate field representation.

³There are no connected polynomial invariants for vectors beyond the scalar product.

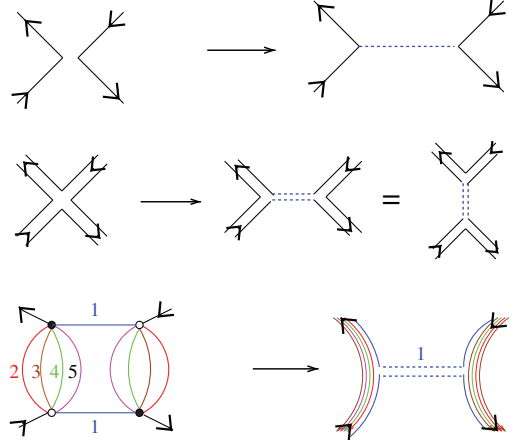


Fig. 1: (Color online) The vertex is cut in two by the intermediate field representation. From top to bottom: the vector, matrix and tensor (melonic, rank 5) case. Incoming and outgoing arrows distinguish ϕ and $\bar{\phi}$.

which we now assume in this case. One of the five melonic interactions at rank 5 is pictured at the bottom of fig. 1, with the particular color $c = 1$ (blue) emphasized. The restriction to quartic melonic interactions is not so important as one may think from the physical point of view of quantizing gravity. Indeed the Feynman perturbation expansion of such restricted models still sums over *all* pseudo-manifolds and admits as boundaries *all* observables of the colored tensor theory [21].

These three different combinatorial models have just renormalizable power counting, like the ordinary “scalar” ϕ^4 . But the class of divergent graphs is more restricted in the combinatorial case.

Remark that in all cases the interactions V are positive for $g > 0$. Hence the models are stable for this sign of the coupling constant, which we now assume. Their perturbative series are certainly at least Borel summable at finite cutoff N , and in some super-renormalizable cases have been shown to remain Borel summable in the $N \rightarrow \infty$ limit [22,23].

We shall now compare the one-loop beta function of these models. For this we first pass to the intermediate field representation⁴. Indeed quartic models, and especially quartic vector and tensor models, simplify in this representation: their dominant graphs as $N \rightarrow \infty$ simply become *trees*⁵. The guiding principle is to introduce an intermediate field σ to split the quartic vertex in two halves, as pictured in fig. 1, through the simple integral representation

$$e^{-\frac{g}{2} V(\phi, \bar{\phi})} = \int d\nu(\sigma) e^{i\sqrt{g}\bar{\phi}\phi\cdot\sigma}. \quad (7)$$

⁴This representation is called the Hubbard-Stratonovic representation in condensed matter.

⁵More precisely, ordinary trees in the vector case and colored trees in the tensorial case [17].

In this formula $d\nu$ is a Gaussian measure with covariance 1 on the intermediate field σ . The three cases (4)–(6) lead to σ fields of different nature and to different combinatorial “rules for the dot” in (7). In the vector case the σ field is a *scalar*, reflecting the already factorized nature of (4). In the matrix and tensor case (5), (6) σ is a *matrix*. More precisely, in the matrix case (5), it is a single matrix with its two arguments in \mathbb{Z}^4 . In the tensor case (6), it is the sum of five different “colored matrices” σ_c with their two arguments in \mathbb{Z} , one for each color c , and should be properly written as [17]

$$\sigma = \sum_c \sigma_c \otimes \mathbb{I}_c = \sigma_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 \otimes \mathbb{I}_5 + \dots + \mathbb{I}_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_3 \otimes \mathbb{I}_4 \otimes \mathbb{I}_5. \quad (8)$$

The advantage of this representation is that the functional integral over $(\bar{\phi}, \phi)$ becomes quadratic, hence can be performed explicitly, yielding

$$\begin{aligned} \mathcal{Z}(g, J, \bar{J}) &= \frac{1}{Z} \int d\nu(\sigma) \int d\mu_C(\phi, \bar{\phi}) e^{i\sqrt{g}\bar{\phi}\phi \cdot \sigma} e^{\bar{J} \cdot \phi + J \cdot \bar{\phi}} \\ &= \frac{1}{Z} \int d\nu(\sigma) e^{\langle \bar{J}, C^{1/2} R(\sigma) C^{1/2} J \rangle} \\ &\quad \times e^{-\text{Tr} \log[\mathbb{I} - i\sqrt{g} C^{1/2} \sigma C^{1/2}]}, \end{aligned} \quad (9)$$

where R is the symmetric *resolvent* operator

$$R(\sigma) \equiv \frac{1}{\mathbb{I} - i\sqrt{g} C^{1/2} \sigma C^{1/2}}. \quad (10)$$

Writing symmetrized expressions with $C^{1/2}$ ’s is a bit heavier but shows the Hermitian nature of $C^{1/2} \sigma C^{1/2}$. It is essential to ensure the non-perturbative existence of the resolvent and logarithm for g in a *cardioid domain* of the complex plane, see, *e.g.*, [17,18,23].

One-loop β -function. – In all cases the one-loop beta function boils down to the same computation, up to subtle differences of purely combinatorial nature. Let us call Γ_{2p} the $2p$ -point vertex function, hence the sum of one-particle irreducible amputated Feynman amplitudes with $2p$ external legs. The renormalized BPHZ prescriptions are defined by momentum space subtractions at zero momentum⁶,

$$\frac{g_r}{2} = -\Gamma_4(0), \quad Z - 1 = \left[\frac{\partial}{\partial p^2} \Gamma_2 \right] (0), \quad (11)$$

where g_r is the renormalized coupling. Performing the field strength renormalization we can rescale to 1 the wave function renormalization at high ultraviolet cutoff at the cost of using a *rescaled* bare coupling $g'_b = Z^{-2} g_b$. The one-loop β_2 coefficient shows how this rescaled bare coupling evolves at fixed g_r when $N \rightarrow \infty$. It writes

$$g'_b = g_r [1 + \beta_2 g_r (\log N + \text{finite}) + O(g_r^2)], \quad (12)$$

⁶In fact we shall restrict these subtractions to *divergent* graphs (*e.g.*, planar in the matrix case, melonic in the tensor case), the difference being a finite renormalization.

where N is the ultraviolet cutoff, and “finite” means a function which is bounded as $N \rightarrow \infty$. As is well known, $\beta_2 > 0$ corresponds to a coupling constant which flows out of the perturbative regime in the ultraviolet (Landau ghost). $\beta_2 < 0$ corresponds to the nice physical situation of asymptotic freedom: the (rescaled) bare coupling flows to zero as $N \rightarrow \infty$, hence the (rescaled) theory tends towards a Gaussian (free) quantum free theory in the ultraviolet regime. $\beta_2 = 0$ indicates the possibility of asymptotic safety (non-trivial ultraviolet fixed point close to the Gaussian one), but is inconclusive as the analysis of the renormalization group flow must be pushed further.

It is easier to compute the bare perturbation theory, as it does not involve any subtraction. Starting from (11), we find that Γ_4 and $Z - 1$ always involve the *same* logarithmically divergent sum, namely (using, *e.g.*, the standard parametric representation [20])

$$\sum_{q \in [-N, N]^4} \frac{1}{(q^2 + m_r^2)^2} = 2\pi^2 \log N + \text{finite}, \quad (13)$$

where $m_r^2 = Zm^2 - \Gamma_2(0)$ is the renormalized mass. However this sum arises with various combinatoric coefficients. More precisely

$$\Gamma_4(0) = -\frac{g_b}{2} \left[1 - a g_b \sum_{q \in [-N, N]^4} \frac{1}{(q^2 + m_r^2)^2} + O(g_b^2) \right], \quad (14)$$

$$\begin{aligned} Z &= 1 + \frac{\partial \Gamma_2}{\partial p^2} \Big|_{p=0} \\ &= 1 + b g_b \sum_{q \in [-N, N]^4} \frac{1}{(q^2 + m_r^2)^2} + O(g_b^2), \end{aligned} \quad (15)$$

where a and b are combinatoric coefficients that depend on the particular case (vector, matrix or tensor) considered.

Since $g'_b = Z^{-2} g_b$, multiplying (12) by Z^2 and taking into account (11)–(15), which imply $g_r = g_b + O(g_b^2)$ and $Z = 1 + O(g_b)$, we find

$$Z^2 \Gamma_4(0) = -\frac{g_b}{2} [1 - \beta_2 g_b (\log N + \text{finite}) + O(g_b^2)], \quad (16)$$

hence in all cases we find that

$$\beta_2 = (a - 2b) 2\pi^2, \quad (17)$$

and we are left with the simple problem of computing the coefficients a and b of the one-loop leading diagrams for Γ_4 and Z .

- In the vector case, $a = 1$. Indeed in the parametric representation the only divergent graph is the one on the left of fig. 2. Resolvents (derivatives with respect to J and \bar{J}), also represented as ciliated vertices in [17], come up with a factor 1 and no symmetry factorial, whether terms from the $\text{Tr} \log$ expansion, also called *loop vertices* [24] and pictured as unciliated vertices in [17] come up with a factor $1/n$ for

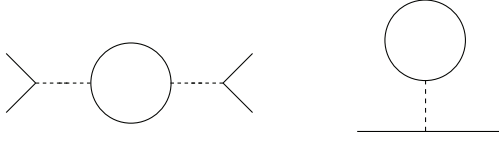


Fig. 2: The (single) one-loop melonic graphs in the tensor case for Γ_4 and Γ_2 are trees for the intermediate field (dashed) lines.

a $\text{Tr}(i\sqrt{g}C^{1/2}\bar{\sigma}C^{1/2})^n$ (because of the Taylor series of the logarithm), plus a symmetry factor $1/k!$ if there are k of them (this factor comes from expansion of the exponential). The combinatoric weight for the tree graph at order g^2 for Γ_4 is therefore 1, which decomposes into a $1/2$ for the single-loop vertex ($n = 2, k = 1$) times a 2 for the two Wick contractions.

In this vector case $b = 0$ since the one-loop tadpole on the right of fig. 2, the only contributing graph at order g , does not have any external momentum dependence. Hence

$$\left[\frac{\partial}{\partial p^2} \Gamma_2 \right] (0) = O(g^2) \Rightarrow b = 0. \quad (18)$$

Hence $\beta_2 = 2\pi^2$ and the theory has no UV fixed point, at least in this approximation.

- In the tensor case, $a = 1$ for the same reasons as in the vector case. Indeed for any of the five melonic interactions there is a single divergent graph of the corresponding color of the type pictured on the left of fig. 2.

But now we also have $b = 1$. Indeed let us remark first that $b > 0$ because two minus signs compensate, one in front of g in (3) and the other coming from the mass subtraction, since

$$\left[\frac{1}{p_c^2} \left(\frac{1}{q^2 + p_c^2 + m_r^2} - \frac{1}{q^2 + m_r^2} \right) \right]_{p_c=0} = - \frac{1}{(q^2 + m_r^2)^2}. \quad (19)$$

The combinatorics is then 1 because there is a single-loop vertex with $n = 1, k = 1$ and a single Wick contraction to branch it on the external resolvent as shown for the graph on the right of fig. 2. Summing over the colors c of this contraction simply reconstructs $\sum_c p_c^2 = p^2$. In conclusion $\beta_2 = -2\pi^2$ and the theory is asymptotically free, in agreement with [5,19,25]. Wave function, or field strength renormalization won over coupling constant renormalization because of the square power in Z^2 .

- In the matrix case, the vertex crossing symmetry means that more terms diverge logarithmically than

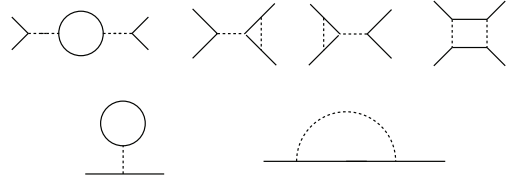


Fig. 3: The dominant one-loop graphs in the matrix case for Γ_4 and Γ_2 are planar in the intermediate field representation.

in the vector and tensor cases, namely those corresponding to planar *maps*⁷ in the intermediate representation. The crossing symmetry is a \mathbb{Z}_2 symmetry, but it acts differently on Γ_4 and Γ_2 . Since the one-loop graph for Γ_4 has *two* vertices, hence two (dotted) σ propagators, the crossing symmetry acts twice independently and generates an orbit of four planar maps, represented in the top part of fig. 3. In contrast the crossing symmetry acts only once on the orbit of the Γ_2 term, generating only the two planar maps pictured in the bottom part of fig. 3. Hence $a = 4$ and $b = 2$ which leads to $\beta_2 = 0!$ This “miracle” persists at all orders: in fact the logarithmically divergent part of $Z^2\Gamma_4(0)$ is exactly 0 at all orders in g , as can be shown through combining a Ward identity with the Schwinger-Dyson equations of the theory [2]. The corresponding theory is asymptotically *safe*.

In conclusion the asymptotic freedom of quartic melonic renormalizable tensor field theories can be traced back to the same combinatorial weight 1 for the melonic terms (rooted trees in the IF representation) contributing to $\Gamma_4(0)$ and Z . It leads to Z^2 winning over $\Gamma_4(0)$. In contrast asymptotic safety for matrix models comes from the crucial crossing symmetry of the vertex. This symmetry boosts the contribution of Z by a factor 2, but the contribution of $\Gamma_4(0)$ by a factor 4, since it acts independently on its two vertices. This restores perfect equilibrium between $\Gamma_4(0)$ and Z^2 .

Some further remarks. – The reader could ask where in this picture sits the ordinary scalar (complex) ϕ_4^4 theory. It corresponds to a vertex which is local in the direct θ representation, hence momentum conserving in the n indices. This vertex in the language of the previous section is therefore, at $d = 4$, hence for $p, \bar{p}, q, \bar{q} \in \mathbb{Z}^4$

$$V_S = \sum_{p, \bar{p}, q, \bar{q}} \phi_p \bar{\phi}_{\bar{p}} \phi_q \bar{\phi}_{\bar{q}} \delta(p - \bar{p} + q - \bar{q}). \quad (20)$$

The symmetry of the vertex no longer distinguishes planar from non-planar Wick contractions. Moreover, $b = 0$, like in the vector case, because the tadpole in this theory is purely local. Again there is no way to have asymptotic freedom or even safety for a stable $g > 0$ coupling, because there is no one-loop wave function renormalization.

⁷As is by now well known, intermediate field graphs are really what mathematicians nowadays call combinatorial maps.

One could also consider real rather than complex models. Again here there are some changes, for instance the crossing symmetry of the scalar vertex has a symmetry of order three, since there are three ways to divide the four equivalent real fields into two pairs. But the conclusions do not change: real scalar and real vector theories are neither asymptotically free nor safe since there is still no one-loop wave function renormalization, whether renormalizable real matrix theories remain asymptotically safe (the combinatorics being the same than for the real matrix Grose-Wulkenhaar model). We expect also real tensor theories to remain asymptotically free, since the quartic vertex pairing still has only one melonic channel.

These features also do not depend as much as one could think of the particular form of the propagator. In particular if we replace \mathbb{Z} by \mathbb{N} and consider linear-type inverse propagators $\frac{1}{n+m^2}$, the conclusions remain similar. The Grosse-Wulkenhaar model is a matrix model with such an inverse-linear propagator and is asymptotically safe at all orders [2], and the quartic melonic tensor model with an inverse-linear propagator is asymptotically free in the dimension $d = 3$ where it is just renormalizable [5].

Finally let us remark that adding Boulatov-type gauge projectors to the tensor propagators, as is natural in tensor group field theory [26–30], *enhances* rather than suppresses their asymptotic safety. In the simplest model of this type which is just renormalizable, namely the quartic melonic $d = 6$ model with propagator $\delta(\sum_{c=1}^6 p_c)(p^2 + m^2)^{-1}$ [29] the basic one-loop integral (13) is changed into

$$\sum_{q \in [-N, N]^5} \frac{\delta(\sum_{c=1}^5 q_c)}{(q^2 + m_r^2)^2} = \frac{2\pi^2}{\sqrt{5}} \log N + \text{finite}. \quad (21)$$

The derivative contributing to Z is also enhanced, so that the ratio between b and a is now $6/5$ instead of 1 (see footnote ⁸). As a result Z^2 wins even *more* over Γ_4 . In the normalization used in the previous section we have for that model

$$\beta_2 = a - 2b = \left(1 - \frac{12}{5}\right) \frac{2\pi^2}{\sqrt{5}} = -\frac{14\pi^2}{5\sqrt{5}}. \quad (22)$$

The asymptotic freedom of renormalizable tensor theories with higher-order interactions (such as the models with sixth-order interactions considered in [3] and in [27]) is a much more delicate issue. First of all, in contrast with the matrix case, and even for models which are symmetric under color permutations, there are at higher order several different tensor invariants, and even several different *melonic* tensor invariants. In particular at order six there are two different melonic invariants B_1 and B_2 with coupling constants g_1 and g_2 (see fig. 4). The computations of [6] show that in the quadrant $g_1 > 0$ and $g_2 > 0$ the two beta-function coefficients for g_1 and g_2 have the right sign for asymptotic freedom. However, in [8] a more

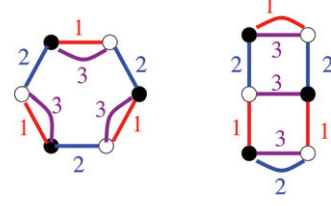


Fig. 4: (Color online) The two melonic observables B_1 and B_2 of order 6 (at rank 3, with a particular coloring).

careful analysis of the RG flow, better including the effect of the fourth-order relevant coupling, showed that in that quadrant, the ultraviolet fixed point is only approximate. After closing towards the Gaussian point, the ultraviolet RG trajectory slightly misses it and apparently enters an unstable region.

However, two important remarks are in order. Firstly, an analog of the Wilson-Fisher ϵ expansion suggests that this type of theories could have an asymptotically safe fixed point [31]. Secondly, the true stability region for sixth-order interactions is certainly larger than the quadrant $g_1 > 0$ and $g_2 > 0$. We should expect the existence of some constants K_1 and K_2 such that $B_1 \leq K_1 B_2$ and $B_2 \leq K_2 B_1$. The stability region is therefore not restricted to the quadrant $g_1 > 0$ and $g_2 > 0$, but includes other regions such as $g_1 < 0, g_2 > 0$ and $|g_1|K_1 < g_2$ or $g_1 > 0, g_2 < 0$ and $|g_2|K_2 < g_1$. These regions may contain asymptotically free trajectories emerging from the ultraviolet Gaussian fixed point.

In conclusion, although the jury is still out on the delicate issue of the ultraviolet RG flow for tensor field theories with several marginal couplings of order higher than four, which clearly requires more analytic and numerical work, the asymptotic freedom of such theories with quartic interactions is a solid mathematical fact, relatively independent of the details of the model. It is deeply rooted in the tensor combinatorics, which is different from the one of matrix models. It would be interesting to study whether, like for non-Abelian gauge theories, this asymptotic freedom survives only to the addition of a bounded number of matter fields.

REFERENCES

- [1] GROSSE H. and WULKENHAAR R., *Commun. Math. Phys.*, **256** (2005) 305 (arXiv:hep-th/0401128).
- [2] DISERTORI M., GURAU R., MAGNEN J. and RIVASSEAU V., *Phys. Lett. B*, **649** (2007) 95 (hep-th/0612251).
- [3] BEN GELOUN J. and RIVASSEAU V., *Commun. Math. Phys.*, **318** (2013) 69 (arXiv:1111.4997).
- [4] BEN GELOUN J., *Renormalizable Models in Rank $d \geq 2$ Tensorial Group Field Theory*, arXiv:1306.1201.
- [5] BEN GELOUN J. and SAMARY D. O., *Ann. Henri Poincaré*, **14** (2013) 1599 (arXiv:1201.0176).
- [6] BEN GELOUN J., *Class. Quantum Grav.*, **29** (2012) 235011 (arXiv:1205.5513).

⁸Beware of a missing factor 2 in the initial version of that computation [29].

- [7] RIVASSEAU V., *AIP Conf. Proc.*, **1444** (2011) 18 (arXiv:1112.5104 [hep-th]); *The Tensor Track: an Update*, arXiv:1209.5284 [hep-th]; *Fortschr. Phys.*, **62** (2014) 81 (arXiv:1311.1461 [hep-th]).
- [8] CARROZZA S., *Ann. Inst. Henri Poincaré D*, **2** (2015) 49 (arXiv:1407.4615 [hep-th]).
- [9] GURAU R., *Commun. Math. Phys.*, **304** (2011) 69 (arXiv:0907.2582 [hep-th]).
- [10] GURAU R. and RYAN J. P., *SIGMA*, **8** (2012) 020 (arXiv:1109.4812 [hep-th]).
- [11] BEN GELOUN J. and BONZOM V., *Int. J. Theor. Phys.*, **50** (2011) 2819 (arXiv:1101.4294 [hep-th]).
- [12] GURAU R., *Ann. Inst. H. Poincaré Probab. Stat.*, **50** (2014) 1474 (arXiv:1111.0519 [math.PR]).
- [13] BONZOM V., GURAU R. and RIVASSEAU V., *Phys. Rev. D*, **85** (2012) 084037 (arXiv:1202.3637 [hep-th]).
- [14] GURAU R., *Class. Quantum Grav.*, **27** (2010) 235023 (arXiv:1006.0714 [hep-th]).
- [15] AMBJORN J., *Simplicial Euclidean and Lorentzian Quantum Gravity*, arXiv:gr-qc/0201028.
- [16] BONZOM V., GURAU R., RIELLO A. and RIVASSEAU V., *Nucl. Phys. B*, **853** (2011) 174 (arXiv:1105.3122 [hep-th]).
- [17] GURAU R., *Commun. Math. Phys.*, **330** (2014) 973 (arXiv:1304.2666).
- [18] DELEPOUVE T., GURAU R. and RIVASSEAU V., arXiv:1403.0170 [hep-th].
- [19] SAMARY D. O., PÉREZ-SÁNCHEZ C. I., VIGNES-TOURNERET F. and WULKENHAAR R., *Correlation functions of just renormalizable tensorial group field theory: The melonic approximation*, arXiv:1411.7213 [hep-th].
- [20] AVOHOU R., RIVASSEAU V. and TANASA A., *Renormalization and Hopf Algebraic Structure of the 5-Dimensional Quartic Tensor Field Theory*, arXiv:1507.03548.
- [21] BONZOM V., LIONNI L. and RIVASSEAU V., *Colored triangulations of arbitrary dimensions are stuffed Walsh maps*, arXiv:1508.03805.
- [22] GURAU R. and RIVASSEAU V., *The Multiscale Loop Vertex Expansion*, arXiv:1312.7226 [math-ph].
- [23] DELEPOUVE T. and RIVASSEAU V., *Constructive Tensor Field Theory: The T_3^4 Model*, arXiv:1412.5091 [math-ph].
- [24] RIVASSEAU V., *JHEP*, **09** (2007) 008 (arXiv:0706.1224 [hep-th]).
- [25] BENEDETTI D., BEN GELOUN J. and ORITI D., *Functional Renormalisation Group Approach for Tensorial Group Field Theory: a Rank-3 Model*, arXiv:1411.3180 [hep-th].
- [26] CARROZZA S., ORITI D. and RIVASSEAU V., *Commun. Math. Phys.*, **327** (2014) 603 (arXiv:1207.6734).
- [27] CARROZZA S., ORITI D. and RIVASSEAU V., *Commun. Math. Phys.*, **330** (2014) 581 (arXiv:1303.6772).
- [28] CARROZZA S., *Tensorial Methods and Renormalization in Group Field Theories, Series: Springer Theses* (Springer) 2014 (arXiv:1310.3736 [hep-th]); REUTER M. and SAUERESSIG F., *New J. Phys.*, **14** (2012) 055022 (arXiv:1202.2274).
- [29] LAHOUCHE V., ORITI D. and RIVASSEAU V., *JHEP*, **04** (2015) 095 (arXiv:1501.02086 [hep-th]).
- [30] LAHOUCHE V. and ORITI D., *Renormalization of a tensorial field theory on the homogeneous space $SU(2)/U(1)$* , arXiv:1506.08393 [hep-th].
- [31] CARROZZA S., *Phys. Rev. D*, **91**, no. 6 (2015) 065023 (arXiv:1411.5385).