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Universal description of three two-component fermions

O. I. KARTAVTSEV and A. V. MALYKH

Joint Institute for Nuclear Research - Dubna, 141980, Russia

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Abstract – A quantum-mechanical three-body problem for two identical fermions of mass m and a distinct particle of mass m_1 in the universal limit of zero-range two-body interaction is studied. For the unambiguous formulation of the problem in the interval $\mu_r < m/m_1 \leq \mu_c$ ($\mu_r \approx 8.619$ and $\mu_c \approx 13.607$) an additional parameter b determining the wave function near the triple-collision point is introduced; thus, a one-parameter family of self-adjoint Hamiltonians is defined. The dependence of the bound-state energies on m/m_1 and b in the sector of angular momentum and parity $L^P = 1^-$ is calculated and analysed with the aid of a simple model.

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Low-energy dynamics of few two-species particles has attracted much attention as a basic quantum problem that is closely related to the investigations of ultra-cold binary quantum gases [1–7]. The principal problem is the investigation of few two-species fermions, in particular, the present letter is aimed to study two identical fermions of mass m interacting with a distinct particle of mass m_1 . Since the few-body properties become independent of the particular form of the short-range two-body interaction in the low-energy limit, the universal description is obtained by using the contact or zero-range interaction defined by a single parameter, the two-body scattering length a. As a consequence, one expects that for the properly chosen units the few-body properties depend on a single nontrivial parameter, the mass ratio m/m_1 .

Significant advance was made in [8], where it was demonstrated that for $m/m_1 > \mu_c$ ($\mu_c \approx 13.607$), similarly to the three-boson case, the problem of three two-species fermions is ambiguously defined in the limit of zero-range interaction. For the correct formulation, an additional parameter is needed to define the oscillating wave function near the triple-collision point. By setting this parameter, one comes to the Efimov spectrum, which contains an infinite number of bound states whose binding energies tend to infinity and the ratio of subsequent energies tends to a constant.

For $m/m_1 \leq \mu_c$, one of the important results was the analytic zero-energy solution, which reveals the twohump structure in the low-energy three-body recombination rate dependence on m/m_1 [9]. The three-body energy spectrum and the scattering cross sections for $L^P = 1^$ were studied in [10], where two bound states were disclosed for m/m_1 increasing to μ_c . The conclusions of [10] were confirmed in [11,12] by solving the momentum-space integral equations. The formation of the three-body clusters should affect the properties of fermionic mixtures, in particular, it indicates effective attraction between a diatomic molecule and a light particle in the *p*-wave state, which persists even if the three-body system is unbound. In this respect, a role of the *p*-wave (2+1)-scattering was discussed in [4-6,13] and the molecule-atom p-wave attraction in ${}^{40}\text{K}-{}^{6}\text{Li}$ mixture was detected in [7]. Furthermore, the dynamics of the ultra-cold gas consisting of three-body clusters was investigated [14,15]. Another application to the many-body dynamics was the calculation of the third virial coefficient in the unitary limit $a \to \infty$ [16,17].

In spite of the progress, it is still necessary to correctly formulate the three-body problem for two-species fermions with zero-range two-body interaction in the mass-ratio interval $m/m_1 \leq \mu_c$, as indicated in both physical [18–20] and mathematical [21–24] works. In this respect, the basic question is the unambiguous definition of the wave function in the vicinity of the triple-collision point. In this letter, an additional three-body problem for $\mu_r < m/m_1 \leq \mu_c$ ($\mu_r \approx 8.619$) that corresponds to the construction of a one-parameter family of self-adjoint Hamiltonians. Within the framework of this formulation, comprehensive analytic and numerical study of the three-body bound states is performed. Due to the permutational symmetry of fermions,

the states of unit total angular momentum L and negative parity P are of most interest at low energy; for this reason, the $L^P = 1^-$ sector is considered in this letter.

The Hamiltonian in the centre-of-mass frame is the sixdimensional kinetic-energy operator $H_0 = -\Delta_{\mathbf{x}} - \Delta_{\mathbf{y}}$, where \mathbf{x} and \mathbf{y} are the scaled Jacobi coordinates and the units $\hbar = 2m/(1+m/m_1) = 1$ are used. The two-body interaction is defined by the boundary condition for the wave function Ψ imposed on two hyper-planes corresponding to the zero distance r between either fermion and a distinct particle, $\lim_{r\to 0} \frac{\partial \ln(r\Psi)}{\partial r} = -\frac{1}{a}$. As the wave function is antisymmetric under permutation of fermions, a single condition in one pair of interacting particles is needed [10].

The formal construction of the Hamiltonian does not obviously provide an unambiguous definition of the threebody problem; in particular, one should inspect the solution at the intersection of hyper-planes (triple-collision point). To analyse the wave function, correctly define the three-body problem, and calculate the binding energies, it is suitable to expand the wave function $\Psi = \rho^{-5/2} \sum_{n=1}^{\infty} f_n(\rho) \Phi_n(\rho, \Omega)$ into a set of eigenfunctions $\Phi_n(\rho, \Omega)$ of the auxiliary problem on a hyper-sphere at fixed ρ , where $\rho = \sqrt{x^2 + y^2}$ is a hyper-radius and Ω denotes a set of hyper-angular variables [10]. This leads to an infinite set of coupled hyper-radial equations (HREs),

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{\gamma_n^2(\rho) - 1/4}{\rho^2} + E\right] f_n(\rho) - \sum_{m=1}^{\infty} \left[P_{nm}(\rho) - Q_{nm}(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{\mathrm{d}}{\mathrm{d}\rho}Q_{nm}(\rho)\right] f_m(\rho) = 0, \quad (1)$$

where the eigenvalues of the auxiliary problem $\gamma_n^2(\rho)$ are different branches of the multi-valued function defined for $L^P = 1^-$ by

$$\frac{\rho}{a}\cos\gamma\frac{\pi}{2} = \frac{1-\gamma^2}{\gamma}\sin\gamma\frac{\pi}{2} - 2\frac{\cos\omega\gamma}{\sin2\omega} + \frac{\sin\omega\gamma}{\gamma\sin^2\omega}$$
(2)

and the notation $\sin \omega = 1/(1 + m_1/m)$ is used. The coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ are expressed in the analytical form via $\gamma_n^2(\rho)$ and their derivatives [10,25,26].

Since both eigenfunctions $\Phi_n(\rho, \Omega)$ of the auxiliary problem and the coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ are regular, the wave function Ψ for $\rho \to 0$ is basically determined by one of the channel functions $f_n(\rho)$, which corresponds to the least singular term $(\gamma_n^2 - 1/4)/\rho^2$ in the system of HREs (1), *i.e.*, to the smallest γ_n^2 . For the sake of brevity, the channel index denoting the smallest eigenvalue, $\gamma^2(\rho)$, and the corresponding channel function, $f(\rho)$, will be omitted. To determine the channel function $f(\rho)$ up to the leading-order terms for $\rho \to 0$, one should retain in HRE the singular part $(\gamma^2 - 1/4)/\rho^2 + q/\rho$, where the notations $\gamma \equiv \gamma(0)$ and $q = [\frac{d\gamma^2(\rho)}{d\rho}]_{\rho=0}$ are introduced for brevity. The dependences $\gamma(m/m_1)$ and $q(m/m_1)$ are depicted in fig. 1. Note that the two-body scattering length is taken as a length unit (|a| = 1) and q < 0 (q > 0) for a > 0 (a < 0).



Fig. 1: (Colour online) The dependences $\gamma(m/m_1)$ (solid red line) and $q(m/m_1)$ (dashed blue line). The values μ_r , μ_e , and μ_c correspond to $\gamma = 1, 1/2$, and 0.

Generally, $f(\rho) = C_+ \varphi_+(\rho) + C_- \varphi_-(\rho)$ is a linear combination of two independent solutions, which up to the leading-order terms for $\rho \to 0$ are given by $\varphi_{\pm}(\rho) = \rho^{1/2\pm\gamma}(1+\frac{q\rho}{1\pm2\gamma})$, except $\gamma = 0, 1/2$ when the expressions for $\varphi_{\pm}(\rho)$ contain logarithmic terms.

Consider firstly $\gamma^2 \geq 1 \ (m/m_1 \leq \mu_r \approx 8.619)$, in which case $\varphi_-(\rho)$ is not square-integrable when $\rho \to 0$ and should be excluded, *i.e.*, $C_- = 0$. Thus, one should satisfy the simple condition $f(\rho) \xrightarrow{\to 0} 0$, in other words, the requirement of square integrability of Ψ is sufficient. Conversely, if $\gamma^2 < 1 \ (m/m_1 > \mu_r)$, both $\varphi_+(\rho)$ and $\varphi_-(\rho)$ are square-integrable and an additional boundary condition is needed if $\rho \to 0$. One should further distinguish the case $\gamma^2 < 0 \ (m/m_1 > \mu_c \approx 13.607)$, then $\varphi_{\pm}(\rho)$ oscillate and a standard method to lift ambiguity of the solution is to specify the constant C_-/C_+ , which must satisfy $|C_-/C_+| = 1$ to provide self-adjointness of the Hamiltonian. Thus, one comes to the family of Hamiltonians depending on a single parameter (the phase of C_-/C_+) with the well-known Efimov spectrum of bound states [8].

One of the aims of this letter is the unambiguous formulation of the problem for $1 > \gamma^2 \ge 0$ ($\mu_r < m/m_1 \le \mu_c$), which requires defining the boundary condition for $\rho \to 0$. Again, a standard method is to specify C_-/C_+ , which should be real-valued to provide self-adjointness of the Hamiltonian. It is convenient to define the length $-\infty < b < \infty$ by $-C_-/C_+ = \pm |b|^{2\gamma} \equiv b|b|^{2\gamma-1}$, *i.e.*, \pm refers to the sign of *b*. The boundary condition is straightforwardly written as

$$f(\rho) \underset{\rho \to 0}{\longrightarrow} \rho^{1/2 + \gamma} \mp |b|^{2\gamma} \rho^{1/2 - \gamma} \left[1 + q\rho/(1 - 2\gamma) \right], \quad (3)$$

except for $\gamma = 1/2$ $(m/m_1 = \mu_e \approx 12.313)$. The last term ~ q can be optionally omitted if $1/2 > \gamma > 0$ ($\mu_e < m/m_1 < \mu_c$) and should be retained if $1 > \gamma > 1/2$ ($\mu_r < m/m_1 < \mu_e$), when it exceeds the first term $\rho^{1/2+\gamma}$. If $\gamma = 0$ $(m/m_1 = \mu_c)$, one finds the boundary condition either from eq. (3) in the limit $\gamma \to 0$ or directly from $\varphi_+ \sim \sqrt{\rho} \text{ and } \varphi_- \sim \sqrt{\rho} \ln(\rho),$

$$f(\rho) \underset{\rho \to 0}{\longrightarrow} \rho^{1/2} \ln(\rho/b) , \qquad (4)$$

where only b > 0 is allowed. In the specific case of $\gamma = 1/2$ $(m/m_1 = \mu_e)$ one can take $\varphi_+ \sim \rho$ and $\varphi_- \sim 1 + q\rho \ln \rho$, which leads to the boundary condition

$$f(\rho) \underset{\rho \to 0}{\longrightarrow} \rho - b(1 + q\rho \ln \rho).$$
(5)

One should emphasise that the condition (5) does not follow from (3) in the limit $\gamma \to 1/2$ and the definitions of the parameter b in (5) and (3) do not correspond to each other. Notice that one could substitute $\ln \rho$ with $\ln(\rho/\rho_0)$ in (5) introducing a scale ρ_0 , which simply leads to redefinition of the parameter $\tilde{b} = b/(1 - b \ln \rho_0)$. As all other channel functions $f_n(\rho)$ tend to zero faster than $f(\rho)$ at $\rho \to 0$, it is sufficient to impose the conditions $f_n(0) = 0$ for complete formulation.

Besides the definition of the boundary conditions for the channel function (3)–(5), it is of interest to discuss those for the total wave function Ψ . Generally, eqs. (3)– (5) can be deduced from the boundary condition for Ψ by projecting onto the first-channel eigenfunction on the hyper-sphere $\Phi_1(\rho, \Omega)$ taken in the limit $\rho \to 0$. One can easily find the required expression if $\mu_e < m/m_1 < \mu_c$ (1/2 > $\gamma > 0$), as Ψ is a linear combination of two terms $\sim \rho^{\pm \gamma - 2} \Phi_1(0, \Omega)$ for $\rho \to 0$ and the boundary condition takes a simple form,

$$\lim_{\rho \to 0} \left(\rho^{1-2\gamma} \frac{\mathrm{d}}{\mathrm{d}\rho} \pm \frac{2\gamma}{|b|^{2\gamma}} \right) \rho^{2+\gamma} \Psi = 0, \tag{6}$$

which is similar to (3). On the other hand, for $\mu_r < m/m_1 < \mu_e$ (1 > $\gamma > 1/2$), it is necessary to keep in the expansion of Ψ for $\rho \to 0$ also the term $\sim \rho^{-\gamma-1}$, which includes an additional function of hyper-angles. In this case, the boundary condition becomes cumbersome, as it should contain, besides $\Phi_1(0, \Omega)$, the additional function, whose determination is not necessary for the present letter.

The boundary condition imposed for $\rho \to 0$ is equivalent to including a zero-range three-body potential, while *b* admits an interpretation as the generalised scattering length. This potential represents either the effect of overlapping of the two-body potentials or the true three-body force. For illustration consider the connection of *b* with the parameters of a regularised potential, whose range ρ_0 is allowed to shrink to zero. The simple example is the potential defined as the square well $U(\rho) = -U_0$ for $\rho \leq \rho_0$ and as the diagonal term $U(\rho) = \frac{\gamma^2 - 1/4}{\rho^2} + \frac{q}{\rho}$ for HREs (1) for $\rho > \rho_0$. The channel function $f(\rho) = \cos \kappa \rho$ ($\kappa = \sqrt{U_0 + E}$) for $\rho \leq \rho_0$ and is of the form (3) for $\rho > \rho_0$, which gives the asymptotic relation

$$\kappa\rho_0 \tan \kappa\rho_0 = \gamma - \frac{1}{2} \pm 2\gamma \left(\frac{\rho_0}{|b|}\right)^{2\gamma} - \frac{q\rho_0}{1 - 2\gamma + q\rho_0} .$$
 (7)

Up to the leading-order terms containing b and q, the potential strength U_0 is related to the interaction range ρ_0 as

$$U_{0} = v \left[\frac{1}{\rho_{0}^{2}} \pm \frac{4\gamma}{|b|^{2\gamma}(\gamma^{2} - 1/4 + v)\rho_{0}^{2(1-\gamma)}} + \frac{q}{(\gamma^{2} - 1/4 + v)(\gamma - 1/2)\rho_{0}} \right],$$
(8)

where v is determined by $\sqrt{v} \tan \sqrt{v} = \gamma - 1/2$. Thus, the most singular term $\sim \rho_0^{-2}$ in the dependence $U_0(\rho_0)$ is determined by γ , whereas the parameter b determines less singular terms. With decreasing γ , the higher-order terms containing b ($\sim \rho_0^{-2+2n\gamma}$ for $n \ge 2$) prevail over the term proportional to q, e.g., for $\gamma < 1/4$, the term $\sim \rho_0^{-2+4\gamma}$ is more important than that of q/ρ_0 . For $\gamma = 0$ eqs. (7) and (8) take the following form: $\kappa \rho_0 \tan \kappa \rho_0 =$ $-\frac{1}{2} - \frac{1}{\ln(\rho_0/b)}$ and $U_0 = \frac{v}{\rho_0^2} [1 + \frac{2}{(1/4-v)\ln(\rho_0/b)}]$. If b = 0, relation (7) is not applicable; in this case the form (3) gives $\kappa \rho_0 \tan \kappa \rho_0 + \gamma + \frac{1}{2} = 0$ and $U_0 = \frac{\tilde{v}}{\rho_0^2}$, where $\sqrt{\tilde{v}} \tan \sqrt{\tilde{v}} =$ $-\gamma - 1/2$.

The determination of the three-body bound-state energies is simple in the limit $|a| \to \infty$ due to decoupling of HREs (1), since $\gamma_n^2(\rho)$ in (2) become independent of ρ constants $\gamma_n^2(0)$ and the coupling terms $Q_{nm}(\rho)$ and $P_{nm}(\rho)$ vanish. Picking out one HRE with the smallest $\gamma_n^2(0) \equiv \gamma^2$ from the uncoupled system of HREs (1) one finds for b > 0 that there is one bound state whose energy $E = -4b^{-2}[-\Gamma(\gamma)/\Gamma(-\gamma)]^{1/\gamma}$ and eigenfunction $f(\rho) = \rho^{1/2} K_{\gamma}(\sqrt{-E\rho})$ are expressed in terms of the gamma function and the modified Bessel function. In the limit $b \to \infty$, the bound state goes to the threshold, where it turns to the virtual state, which persists for b < 0 and whose energy is given by the above expression. The above expressions for $|a| \to \infty$ are a good approximation for the properties of the deep state, which exists for $|a|/b \gg 1$. Note also that redefinition of the parameter $\tilde{b} = \frac{b}{2} \left[-\Gamma(-\gamma) / \Gamma(\gamma) \right]^{\frac{1}{2\gamma}}$ gives the usual relation, $E = -\tilde{b}^{-2}$, between the energy and the scattering length.

To elucidate the qualitative features of the problem in connection with the three-body boundary condition, one constructs a simple model that provides reliable dependence of the bound-state energy on b and m/m_1 . The model is based on splitting the Hamiltonian into two parts: the singular one containing terms singular as $\rho \to 0$ and the remaining one describing a smooth dependence on m/m_1 . The former part is defined as one HRE of (1) containing the smallest $\gamma_n^2(\rho)$, moreover, only singular terms $(\gamma^2 - 1/4)/\rho^2 + q/\rho$ are retained, which allows one to obtain the correct behaviour of the solution for $\rho \to 0$ and to reproduce the attraction for finite ρ . The remaining part is defined simply as a constant $\epsilon(m/m_1)$. Explicitly, one comes to the equation $(\frac{d^2}{d\rho^2} - \frac{\gamma^2 - 1/4}{\rho^2} - \frac{q}{\rho} + E - \epsilon) f(\rho) = 0$, whose square-integrable solution is written as $f(\rho) = \rho^{1/2+\gamma} e^{-\kappa\rho} \Psi(1/2 + \gamma + q/(2\kappa), 1 + 2\gamma; 2\kappa\rho)$, where $\kappa = \sqrt{\epsilon - E}$ and $\Psi(a, c; z)$ is the confluent hyper-geometric



Fig. 2: (Colour online) Bound-state energies E as a function of m/m_1 and b for the two-body scattering length a > 0 (panel (a)) and a < 0 (panel (b)) and the energy axis scaled to map $-\infty < E < -1$ (panel (a)) and $-\infty < E < 0$ (panel (b)) to the interval (-1, 0). Values μ_r , μ_e and μ_c correspond to $\gamma = 1$, 1/2 and 0.

function decaying as $z \to \infty$. The eigenenergy equation

$$(2\kappa|b|)^{2\gamma} = \mp \frac{\Gamma(2\gamma)\Gamma\left(1/2 - \gamma + q/(2\kappa)\right)}{\Gamma(-2\gamma)\Gamma\left(1/2 + \gamma + q/(2\kappa)\right)} \tag{9}$$

follows from boundary condition (3) for all $0 < \gamma < 1$ $(\mu_c > m/m_1 > \mu_r)$ except $\gamma = 1/2$ $(m/m_1 = \mu_e)$. The eigenenergy equation for $\gamma = 0$ is obtained either by taking the limit in eq. (9) or from the boundary condition (4) that gives $\ln(2\kappa b) + \psi(\frac{1}{2} + \frac{q}{2\kappa}) + 2\gamma_C = 0$ for b > 0. Hereafter, $\psi(x)$ is the digamma function and $\gamma_C \approx 0.5772$ is the Euler–Mascheroni constant. In the special case of $\gamma = 1/2$ $(m/m_1 = \mu_e)$ the eigenenergy equation $\frac{1}{q}(\frac{1}{b} - \kappa) - \ln(\frac{|q|}{2\kappa}) + \psi(1 + \frac{q}{2\kappa}) + 2\gamma_C - 1 = 0$ comes from (5).

The simple model is equivalent to the generalised Coulomb problem incorporating the zero-range interaction. As follows from eq. (9), the bound-state energies monotonically increase with increasing b; moreover, one bound state appears if b passes through zero. It is helpful to examine two limiting cases of b = 0 and $b \to \infty$, which gives the eigenvalues $\kappa_{nb} = -\frac{q}{2(n+s_b\gamma)+1}$, where nis a non-negative integer and $s_0 = +1$ ($s_{\infty} = -1$). The bound-state energies are

$$E_{nb} = -\frac{q^2}{[2(n+s_b\gamma)+1]^2} + \epsilon,$$
 (10)

where n is restricted by the condition $2(n + s_b\gamma) + 1 > 0$ if a > 0 (q < 0) and $2(n + s_b\gamma) + 1 < 0$ if a < 0 (q > 0). Hereafter it is convenient to take |a| as a length unit that sets the two-body binding energy to unity. A comparison of the ground- and excited-states energies for b = 0 [27] with eq. (10) shows that reasonable agreement could be obtained for ϵ about $-0.4 \div -0.6$. Estimating the constant $\epsilon \approx -0.5$, one finds that for a > 0 there are two branches below the threshold (at $E \leq -1$) if b = 0 and three branches if $b \to \infty$, while for a < 0 there is one branch below the threshold (at $E \leq 0$) if $b \to \infty$ (see fig. 2). For a > 0, from eq. (10) follows degeneracy of the branches E_{n0} and $E_{n\infty}$ (n = 0, 1) as $m/m_1 \to \mu_c$ $(\gamma \to 0)$, E_{00} and $E_{1\infty}$ as $m/m_1 \to \mu_e$ $(\gamma \to 1/2)$, and E_{00} and $E_{2\infty}$ as $m/m_1 \to \mu_r$ $(\gamma \to 1)$. Moreover, from eq. (9) it follows that as $m/m_1 \to \mu_c$ $(\gamma \to 0)$ the energies for any b < 0 converge to either $E_{00} = E_{0\infty}$ or $E_{10} = E_{1\infty}$. As $m/m_1 \to \mu_e$ $(\gamma \to 1/2)$ the energies converge to either of three options, the threshold E = -1, $E_{00} = E_{1\infty}$, and $-\infty$. And as $m/m_1 \to \mu_r$ $(\gamma \to 1)$ the energies converge to either $E_{1\infty}$ or $E_{00} = E_{2\infty}$ as shown in fig. 2. For a < 0, the energies converge to $E_{0\infty}$ as $m/m_1 \to \mu_r$ $(\gamma \to 1)$ and to $-\infty$ as $m/m_1 \to \mu_e$ $(\gamma \to 1/2)$ for $b \neq 0$.

The three-body bound-state energies are determined by numerical solution of the truncated system of HRE (1) complemented by boundary conditions (3), (4), and (5). The numerical method is the same as in [10,27] apart from implementation of the boundary conditions at sufficiently small ρ . Sufficient accuracy of the calculated three-body bound-state energies is achieved by solving up to eight HREs; the results are plotted in fig. 2. The calculated dependences are consistent with the overall predictions of the simple model. The energy dependence on b for fixed m/m_1 is typical of a sum of the finite-range and zero-range potentials, in particular, variation of the parameter b leads to the appearance or disappearance of one bound state.

The calculations for a > 0 show that if $m/m_1 \to \mu_r$ the energies for any b converge either to $E_{1\infty} \sim -4.7473$ or to $E_{00} = E_{2\infty} \sim -1.02090$, if $m/m_1 \to \mu_e$ there is one limit $E_{00} = E_{1\infty} \sim -1.74397$, and if $m/m_1 \to \mu_e$ the energies for any $b \leq 0$ converge either to $E_{00} = E_{0\infty} \to -5.89543$ or to $E_{10} = E_{1\infty} \to -1.13767$. In agreement with [10] it is found that if $m/m_1 \leq \mu_r$, where only b = 0 is allowed, there is one bound state, which arises at $m/m_1 \approx 8.17259$ and naturally continues the branch E_{00} . The calculations for a < 0 show that if $m/m_1 \to \mu_r$ the energies for any b converge to the limit $E_{0\infty} \to -4.7147$. If $m/m_1 \to \mu_r$, the limit $E_{0\infty}$ for a < 0 coincides with the limit $E_{1\infty}$ for a > 0, as predicted by the simple model (10).



Fig. 3: (Colour online) A number of bound states in each domain of the m/m_1 -b plane for the two-body scattering length a > 0 (panels (a) and (b)) and a < 0 (panel (c)). Part of the panel (a) is plotted in the panel (b) to discern details. Solid (red) line: critical three-body parameter $b_c(m/m_1)$ corresponding to the bound-state energy at the threshold. Dashed (green) lines: domain boundaries determined by $m/m_1 = \mu_e$ and b = 0. Values μ_r , μ_e and μ_c correspond to $\gamma = 1$, 1/2 and 0.

Notice that due to discontinuity in the definition of b the limiting values of the bound-state energy for $m/m_1 \rightarrow \mu_e \mp 0$ do not coincide with that calculated exactly at $m/m_1 = \mu_e$. The calculations at $m/m_1 = \mu_e$ show that for a > 0 there are two bound states, one of which disappears for $-.108 < b \le 0$; for a < 0 there is one bound state, which disappears for $-.437 < b \le 0$. In the limit $b \rightarrow \infty$ the bound-state energies tend to -4.319 and -1.061 for a > 0 and to -25.720 for a < 0. For b = 0 definitions (3) and (5) are the same and for a > 0 the bound-state energy takes the value $E_{00} \approx -1.74397$.

Elaborate calculations were carried out to determine the critical parameter $b_c(m/m_1)$, for which the bound-state energy coincides with the threshold. The lines $b_c(m/m_1)$, b = 0, and $m/m_1 = \mu_e$ form boundaries of the domains of the definite number of bound states in the m/m_1 -b plane as presented in fig. 3. Few points of the dependence $b_c(m/m_1)$ are of special interest, viz., one finds for a > 0 that $b_c = 0$ at $m/m_1 \approx 12.91742$, $b_c \to \pm \infty$ at $m/m_1 \approx 10.2948$, $b_c \approx 0.05166$ at $m/m_1 = \mu_c$, and $b_c(m/m_1)$ has a local minimum $b_c \approx -0.01754$ at $m/m_1 \approx 12.550$. Similarly, one finds for a < 0 that $b_c \approx -0.2501$ at $m/m_1 \approx 10.15$.

Until now, in a number of reliable investigations of three two-component fermions (for $m/m_1 \leq \mu_c$) [9–12,27] it was explicitly or implicitly assumed that only one particular form of the wave function near the triple-collision point is allowed, which in terms of this letter means that the three-body parameter b was set to zero. Nonetheless, the problem of two linear-independent square-integrable solutions was mentioned in [9,18,19,28]. A rigorous treatment of few two-component fermions with the contact two-body interactions and the construction of a self-adjoint Hamiltonian was discussed from the mathematical point of view in [21–24]. The approach of [23] was further exploited in the calculation of three-body bound states [29].

It is interesting to discuss the approach used in paper [19], where the two-parameter variety of the threebody problems was defined by introducing the logarithmic derivative of the channel function, $\tan \delta = \rho \frac{d \ln f}{d\rho}$ at small hyper-radius ρ_0 (denoted as R_0 in [19]). The limit $\rho_0 \to 0$ was treated numerically thus resulting in the discontinuous bound-state energy dependence on δ . One expects that the seemingly non-universal description in the framework of the particular model [19] reduces to the one-parameter universal picture of the present letter, at least in the limit $\rho_0 \to 0$. The required relation is readily found by using the asymptotic form of the solution for $\rho \to 0$ (3), which leads to the connection of two parameters δ and ρ_0 and the three-body parameter b of the present letter,

$$|b|^{2\gamma} = \frac{\pm \rho_0^{2\gamma} \left[\tan \delta - \gamma - \frac{1}{2} \right]}{\left[1 + \frac{q\rho_0}{1 - 2\gamma} \right] \tan \delta + \gamma - \frac{1}{2} + q\rho_0 \frac{2\gamma - 3}{2(1 - 2\gamma)}}, \quad (11)$$

except for $\gamma = 1/2$. Thus, the dependence $b(\delta, \rho_0)$ is discontinuous at $\delta = \delta_{cr}$, where

$$\tan \delta_{cr} = \frac{(1-2\gamma)^2 + q\rho_0(3-2\gamma)}{2(1-2\gamma+q\rho_0)},$$
 (12)

which explains why the bound-state energy in [19] is discontinuous. For $\rho_0 \rightarrow 0$, eq. (12) takes a simple form, $\tan \delta_{cr} = 1/2 - \gamma$, which is valid everywhere excluding a small neighbourhood $\sim q\rho_0$ of the point $m/m_1 = \mu_e$ (of the order of $|\gamma - 1/2| < q\rho_0$). To exemplify the correspondence between the model [19] and the present universal description, one compares $\delta_{cr}(m/m_1)$ obtained numerically in [19] and $\delta_{cr}(m/m_1) = \arctan(1/2 - \gamma)$. It is clearly seen from fig. 5 of [19] that both results are in agreement up to $m/m_1 \approx 13$, e.g., $\delta_{cr} \rightarrow -\arctan(1/2) \approx -0.46$ for $m/m_1 \to \mu_r$ and $\delta_{cr} \to 0$ for $m/m_1 \to \mu_e$. On the other hand, the discrepancy arises above $m/m_1 \approx 13$, e.g., the exact expression gives $\delta_{cr} \rightarrow \arctan(1/2) \approx 0.46$ for $m/m_1 \rightarrow \mu_c$, which differs from δ_{cr} in fig. 5 of [19]. Presumably, this discrepancy indicates difficulty of the numerical calculation for $\rho_0 \rightarrow 0$ in this mass-ratio region.

The transition from the infinite Efimov spectrum to the one-parameter spectrum described in this letter under the variation of the mass ratio is a general scenario, which will appear in a number of problems. One should anticipate the same transition for any problem, whose essential properties are determined by the effective potential with the singular part ~ x^{-2} , if its strength depends on a parameter (similar to the mass ratio). Evident example of this kind is the problem of three two-species particles in any L^P sectors [11,12,27]. Similar to the case of the $L^P = 1^$ sector, the three-body parameter should be introduced in the L^- sectors of odd L and in the L^+ sectors of even L > 0 if two identical particles are fermions and bosons, respectively. Also, this scenario will be realised for the three-body problem in the mixed dimensions [30,31] or in the presence of spin-orbit interaction [32,33].

In future studies it is natural to find m/m_1 and b dependences of the scattering cross sections, three-body resonances, and recombination rates. The disclosed dependence on the three-body parameter should be taken into account in many-body properties as well; promising examples are the four-body (3 + 1) [34] and (2 + 2) [35] problems. Furthermore, the three-body parameter will be important in the crossover problem [36], *i.e.*, in the relation of solutions for m/m_1 below and above μ_c ; another interesting point is the crossover of the solutions for m/m_1 above and below μ_r .

* * *

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