LETTER

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Persistent entanglement in a class of eigenstates of quantum Heisenberg spin glasses

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Abstract – The eigenstates of a quantum spin glass Hamiltonian with long-range interaction are examined from the point of view of localisation and entanglement. In particular, low particle sectors are examined and an anomalous family of eigenstates is found that is more delocalised but also has larger inter-spin entanglement. These are then identified as particle-added eigenstates from the one-particle sector. This motivates the introduction and the study of random promoted two-particle states, and it is shown that they may have large delocalisation such as generic random states and scale exactly like them. However, the entanglement as measured by two-spin concurrence displays different scaling with the total number of spins. This shows how for different classes of complex quantum states entanglement can be qualitatively different even if localisation measures such as participation ratio are not.

Introduction. – Consider a Hamiltonian of L spin-(1/2) particles that conserves total spin in some direction; for definiteness, let $\sigma_T^z = \sum_i \sigma_i^z$ be conserved. The Hamiltonian is rendered block-diagonal in the σ^z-basis and the blocks are specifed by the total spin σ_T^z, the most trivial of them being states where all the spins are up or down. The case when m of the spins are up (corresponding to $\sigma_T^z = (L-2m)/2$) is a $\binom{L}{m}$-dimensional subspace. For example, $m = 1$ states are used to transport information across spin-chains [1]. We will refer to states with m up spins as "m-particle states", since similar block-diagonal structure in the Hamiltonian appear in spinless fermion models.

In one-particle states, there is a clear monotonic relationship [2–4] between localisation, for example as measured by the participation ratio (*e.g.*, see [5–8]), and the inter-spin entanglement as measured by, say, concurrence [9,10]: the more the localisation, the less the entanglement. There is no such strict monotonic relationship between localisation and entanglement for states with higher particle-number. However, there are statistically very significant correlations between localisation and entanglement [11–16]. It is shown below that, for two-particle states, in contrast to one-particle states, *on average* (in a way to be defined precisely later), increased localisation implies *enhanced* two-spin entanglement as measured by concurrence. This effect is even more pronounced for two-particle eigenstates of a spin glass Hamiltonian studied below. It should be emphasised that this is entanglement between two spins —other measures such as block entropy may well decrease with localisation.

To study this in the simplest statistical context, *random* states of definite particle-number were considered using an ensemble that was uniformly distributed in such subspaces [17]. It was found that while the expected entanglement between two spins for one-particle states having L spins scales as $1/L$ and that of two-particle states scale as $1/L^2$, in the case of three or more particle states, entanglement is practically absent and is exponentially small in L ($\exp(-L \ln L)$, to be precise). This is consistent with such states having larger multipartite entanglement and the fact that entanglement moves away from being locally shared. In some sense, the "environment" of any two spins is too large for the entanglement between the spins to remain intact.

This scenario is observed in models of many-body localisation, for example, the XXZ model with a random external field [18–20]. In this case, when the interaction dominates, disordered eigenstates in the half-filled ($m = L/2$, $\sigma_T^z = 0$) sector are such that there is vanishing concurrence between two spins. Along with a many-body localisation transition, concurrence also arises to once again slowly disappear when the disorder completely dominates the interaction. The present work must therefore be seen in the larger context of entanglement in disordered interacting quantum systems.

In random states, it is *almost impossible* to find entanglement between subsystems unless the block length (size of the subsystem) is of the order of the size of the (pure) system, typically $\sim L/2$ [21–24]. However, when the states are restricted to be in the subspace of a fixed particle number, then one *can* find entanglement as long as the particle-number does not exceed the block length [17]. To reiterate, for example, concurrence between two spins can be found in a typical two-particle state but not in typical or random three-particle state.

The present work identifies a subset from within the subspace of definite-particle states that have enhanced entanglement. These are simply particle-added states from lower particle-number sectors referred here as "promoted states". One can generate a whole class of *random* promoted states, with high entanglement (compared to generic random definite-particle states), and a different scaling with the total number of spins L.

In order to compare these statistical considerations with physical systems, we study the eigenstates of the infinite-range quantum Heisenberg spin glass. The isotropy of the Hamiltonian implies that total spin along *any* direction is conserved, which in turn allows promoted eigenstates to exist (see later). A subset of two-particle eigenstates is found to have pronounced entanglement. A closer scrutiny shows that these are in fact obtained by promoting one-particle eigenstates. This adds a new dimension to the study of entanglement in many-body systems [25] with spin rotational symmetry. One can expect to find the analysis in this paper to be relevant for generic non-integrable Hamiltonians that have such a symmetry.

Formulation of the problem. –

The infinite-range quantum Heisenberg spin glass. The Hamiltonian considered in this paper is

$$H = \sum_{\substack{j=1 \\ i>j}}^{L} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j, \tag{1}$$

where J_{ij} are independent random variables drawn from the normal distibution $\mathcal{N}(0,1)$. This work concentrates on the eigenstates and therefore the normalisation of the energy is irrelevant. The Hamiltonian takes a block diagonal form, where each block is characterised by a particle-number m, which is also the total number of up-spins in the z-direction.

The m-particle basis states are $|i_m \dots i_1\rangle$, where $i_1 < i_2 < \cdots < i_m$ refer to positions of "up-spins":$|\uparrow\rangle$ in the σ^z-basis, the others being down. For a fixed m, the state with uniform superposition of all the basis states is necessarily an eigenstate, with eigenvalue $S_J = \sum_{i>j} J_{ij}$. This is a direct consequence of the isotropy of the Hamiltonian which implies $[\mathcal{H}, \sigma_T^{\pm}] = 0$, where $\sigma_T^{\pm} = \sum_i \sigma_i^{\pm} = \sum_i (\sigma_i^x \pm \sigma_i^y)$. This in turn implies that repeated action of the σ_T^+ operator on the zero-particle eigenstate $|\downarrow\rangle^{\otimes L}$ will also give eigenstates. We will refer to such eigenstates as "all-one" states of the appropriate particle-number:

$$|\text{all-one state}\rangle_m \propto \left(\sigma_T^+\right)^m |\downarrow\rangle^{\otimes L} \propto \sum_{i_1 < \cdots < i_m} |i_m \dots i_1\rangle . \tag{2}$$

Measures of entanglement and localisation. This work focuses on bipartite entanglement between two spins as measured by concurrence [9]. Concurrence is a simply calculable entanglement monotone within any two-level system which can be in a mixed or pure state. For a two spin state ρ, $C(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, in which $\lambda_1 \geq \lambda_2 \geq \lambda_2 \geq \lambda_4$ are the eigenvalues of the positive matrix $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ with $\tilde{\rho} = (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$. It is known that $0 \leq C(\rho) \leq 1$ and it is 0 iff ρ is a separable state and is 1 iff the state is maximally entangled. For definite-particle states, it is known that the reduced density matrix of any two spins takes the following form [26]:

$$\rho = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & z & 0 \\ 0 & z^* & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix}. \tag{3}$$

In this case, the concurrence is simply given by [26]

$$C(\rho) = \max\left(2(|z| - \sqrt{vy}), 0\right). \tag{4}$$

The quantity $2(|z| - \sqrt{vy})$ is often referred to as *preconcurrence.* Throughout this work, we will refer to "average concurrence" of a state as the average of the concurrence values between all pairs of spins.

The inverse participation ratio (see, *e.g.*, [5–8]) is a basis-dependent quantity that quantifies localisation and is defined as follows. If a state $|\psi\rangle = \sum_\beta a_\beta |\beta\rangle$, where $|\beta\rangle$ are the kets in the computational basis, then IPR $\equiv \sum_\beta a_\beta^4$, from which the participation ratio PR $\equiv 1/\text{IPR}$ is obtained. The range of values that the participation ratio of any state can assume is $[1, D]$, where D is the dimensionality of the Hilbert space. PR $= 1$ occurs when the given state happens to be one of the basis vectors themselves, with all but one of the coefficients being zero and is thus highly localised. PR $= D$ occurs when the magnitude of each of the coefficients is the same and therefore corresponds to a highly delocalised state. Thus, for the all-one state given by eq. (2), PR $= D = \binom{L}{m}$ is the largest possible.

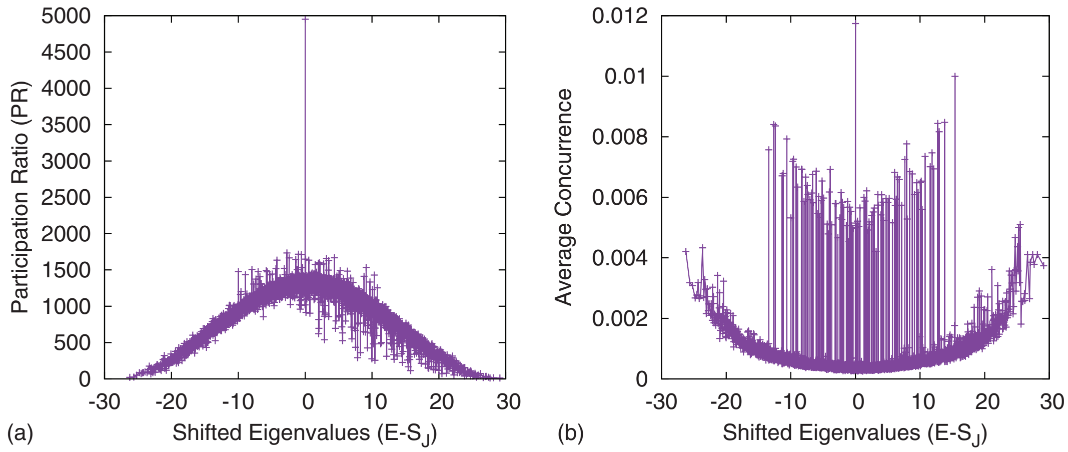

Fig. 1: (Colour online) Plot of (a) the participation ratio and (b) the average concurrence (averaged over all pairs of spins) of all the eigenstates, arranged by their eigenvalues, for a typical realisation of the SK spin glass model with $L = 100$ and $m = 2$. $S_J = \Sigma_{i>j} J_{ij}$ is the energy of the all-one eigenstate. The broad features are similar for *any random* realisation of the SK Hamiltonian.

Concurrence and PR in the eigenstates. – Figure 1 shows the participation ratio and the average concurrence of all the eigenstates of the $m = 2$ sector for one realisation of the Hamiltonian in eq. (1) with $L = 100$. The central spike at $E - S_J = 0$ in each of fig. 1(a) and fig. 1(b) corresponds to the two-particle all-one eigenstate. While no other significant spikes are found in fig. 1(a), several such spikes are found in fig. 1(b). It is now shown that these spikes in fig. 1(b) are in fact the subset of two-particle eigenstates obtained by promoting one-particle eigenstates. Closer scrutiny shows that the small structures present in the particpation ratio do not correspond, by and large, to the well delineated ones in the average concurrence figure, and hence these states with enhanced entanglement are not special as far as localisation is concerned.

Symmetries of the Heisenberg Hamiltonian. The *particle-number operator* $\hat{N}_\uparrow \equiv \sum_k \frac{\sigma_k^z+1}{2}$ has eigenstates which are definite-particle states, with corresponding particle-numbers as their eigenvalues. The definite-particle nature of the Hamiltonian is a result of H commuting with $\sigma_T^z = \sum_{i=1}^L \sigma_i^z$ and hence with $\hat{N}_\uparrow$. Due to the isotropy of the Hamiltonian, it also follows that the operators $\sigma_T^\pm = \sum_{i=1}^L \sigma_i^\pm$ commute with H, where $\sigma_i^\pm = \sigma_i^x \pm i\sigma_i^y$. This implies that if $|\psi\rangle$ is an eigenstate of H, then $\sigma_T^\pm|\psi\rangle$ must also be an eigenstate of H with the same eigenvalue. In addition, if $|\psi\rangle$ has particle-number m, then $\sigma_T^\pm|\psi\rangle$ has particle-number $m \pm 1$. The particle-added state $\sigma_T^+|\psi\rangle$ is referred to as the *promoted* $(m+1)$*-particle state* corresponding to $|\psi\rangle$.

Promoted states. Figure 2 shows a plot of the average concurrence *vs* participation ratio of all the eigenstates of the $m = 2$ sector for one realisation of the Hamiltonian in eq. (1) with $L = 100$. All the eigenstates are marked by a + symbol. They are seen to separate into two "blobs", with the main group at the bottom left, separated from a

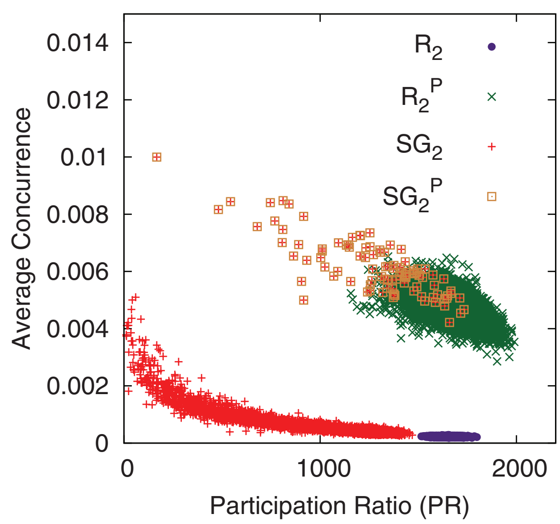

Fig. 2: (Colour online) Scatter plot of average concurrence *vs.* participation ratio (PR) of all the eigenstates in the $m = 2$ sector (SG_2) of *a typical* realisation of the infinite-range quantum Heisenberg spin glass for $L = 100$. The promoted eigenstates (SG_2^P) are seen to form a separate cloud. Data for an equal number (4950) of *random* 2-particle states (R_2) and *random promoted* 2-particle states (R_2^P) are included. The solitary "all-one" eigenstate with a participation ratio of PR = 4950 is not shown.

small set of eigenstates that have larger concurrence and a majority of which are also more delocalised. The latter are shown to correspond to precisely the large spikes in fig. 1(b).

Promoted states are also eigenstates of the operator $\sigma_T^+\sigma_T^-$ with a non-zero eigenvalue, a property we used to identify the promoted two-particle eigenstates amongst the full set of two-particle eigenstates of the Hamiltonian. The promoted eigenstates thus identified have a box enclosing the + symbol in fig. 2. The promoted eigenstates indeed stand out and form a separate cloud with significantly higher average concurrence and predominantly higher delocalisation.

Furthermore, we verified that the three-particle eigenstates promoted from the one-particle and two-particle eigenstates also form distinct clouds when we consider 2-spin and 3-spin entanglement in the three-particle sector (figure not included). Thus, the promoted character of these states is distinguished in these localisation-entanglement plots. As a useful benchmark against which to compare, we also include data for random two-particle states (circles in fig. 2), and "random promoted" two-particle states (marked by ($\times$) symbols in fig. 2). We defer a detailed definition and discussion of these states to the next section, but it is worth pointing out now that while the "genuine" random two-particle states are different from "genuine" two-particle eigenstates of the spin glass Hamiltonian, the promoted two-particle states in these two cases are not all that different.

Random promoted states. – It is naturally of interest to understand the origin of the enhanced entanglement in the promoted states as compared to other states. As a statistical model, states promoted from *random* definite-particle states are easier to study than the promoted eigenstates of random Heisenberg Hamiltonians such as in eq. (1). For the purposes of this work, it suffices to restrict to ensembles states with real coefficients, which are relevant to systems preserving time-reversal symmetry.

The defining characteristic of the random states is that their distribution is isotropic in the associated Hilbert space. To generate a random state, *i.e.*, a state with random orientation in a Hilbert space of dimension D, we form a vector $(r_1, r_2, \dots r_D)$, where r_k's are i.i.d. random variables drawn from a Gaussian distribution $\mathcal{N}(0,1)$. The coefficients of the normalised state are then obtained by dividing the vector by its norm. For example, see [27]. Random definite particles states are those sampled uniformly from the associated definite particle sector of states.

The expectation value of concurrence is calculated as

$$\langle C\rangle = \frac{1}{\binom{L}{2}}\sum_{k>l}\langle C_{k,l}\rangle, \qquad (5)$$

where the averaging for any individual pair of spins comes from an ensemble average, either via a statistical model such as in the case of random states or from an ensemble of Hamiltonians, as in the spin glass case. For random one-particle states, it is known that $\langle C\rangle = 4/(\pi L)$ [2]. Note that for one-particle states, $\langle \mathrm{IPR}\rangle = 3/L$, as the average IPR for any D-dimensional real random states is $3/D$ for large D [28].

It follows that for a random two-particle state $|\psi\rangle = \sum_{i>j} a_{ij}|ij\rangle$, its average IPR ($\sum_{i>j}|a_{ij}|^4$) is

$$\langle \mathrm{IPR}\rangle = \frac{3}{\binom{L}{2}} \approx \frac{6}{L^2}. \qquad (6)$$

To estimate $\langle C\rangle$, it is helpful to first compute the probability of a pair of spins being entangled, $P(C>0)$. For the ensemble of all real random two-particle states, it has been shown [17] that $P(C>0) = 2\sqrt{2}/\sqrt{\pi L}$ and the expectation value of the concurrence is $16/(\pi^{3/2}L^2)$. A procedure similar to the one in [17] can indeed be used to calculate the expectation value of the IPR and concurrence for *promoted* random states.

Random promoted 2-particle states. Consider a random one-particle state $\sum_k a_k|k\rangle$. The a_k's are obtained following the procedure outlined above and satisfy $\sum_k a_k^2 = 1$. A promoted two-particle state (unnormalised) is obtained by action of σ_T^+, on a one-particle state:

$$\sigma_T^+ \sum_k a_k\,|k\rangle = \sum_{i>j}(a_i + a_j)\,|ij\rangle\,. \qquad (7)$$

Thus the special structure of promoted states is apparent here: only L random numbers determine the $\binom{L}{2}$ coefficients of a random promoted two-particle state, while a random two-particle state, in general, depends on $\sim L^2/2$ independent random numbers.

To ease the theoretical calculations, it is convenient to substitute the sum of L random numbers, $\sum_k a_k$, by (L-times) its mean, which is 0. It is a reasonable approximation since this holds exactly for one-particle eigenstates of the above spin glass Hamiltonian (as a direct consequence of orthogonality with respect to the all-one eigenstate). It also follows that $\sum_{i>j}(a_i + a_j) = 0$ if $\sum_k a_k = 0$. The coefficients of the normalised promoted random two-particle states are then $a_{ij} = (a_i + a_j)/\sqrt{L-2}$, as $\sum_{i>j}(a_i+a_j)^2 = L-2$.

The IPR of a promoted two-particle state can then be expressed in terms of the IPR of the corresponding one-particle state:

$$\sum_{i>j} a_{ij}^4 = \frac{1}{(L-2)^2}\left((L-8)\sum_i a_i^4 + 3\right), \qquad (8)$$

where $a_{ij} = (a_i+a_j)/\sqrt{L-2}$, are the normalised coefficients. Note that eq. (8) does not assume L is large but only $\sum_k a_k = 0$. Thus, it implies that when $L<8$, the IPR of the two-particle state decreases with increase in the IPR of the one-particle state and vice versa and the trend changes for $L>8$. We observe in passing the somewhat amusing fact that when $L=8$, whatever may be the one-particle state, the promoted two-particle state has an IPR of exactly $1/12$, again provided that the coefficients sum to zero.

Thus, as $\langle \sum_k a_k^4\rangle = 3/L$ it immediately follows that

$$\langle \mathrm{IPR}\rangle \sim \frac{6}{L^2}. \qquad (9)$$

Thus, to the leading order, this is *identical* to the IPR of "genuine" two-particle random states, as given in eq. (6). Thus as far as localisation is concerned there is typically no difference between promoted and genuine two-particle states, as also confirmed by numerical data in fig. 2. A very different situation is found regarding quantum correlations, such as entanglement, to which we now turn.

Entanglement in promoted two-particle states. The elements of the two-spin reduced density matrix that are involved in the entanglement between the two spins, as quantified by concurrence, are z, v, y (eqs. (3), (4)). For two-particle states, when ρ is the density matrix of spins at positions 1 and 2 (which we consider for simplicity and without any loss of generality), these elements are

$$y = a_{12}^2, \qquad z = \sum_{k=3}^{L} a_{2k}a_{1k}, \qquad v = \sum_{\substack{k,l=3\\k<l}}^{L} a_{kl}^2. \tag{10}$$

Note that we are considering *real* state ensembles. For generic random two-particle states, $\langle v\rangle = \mathcal{O}(1)$, $\langle y\rangle \sim 1/L^2$, $\langle |z|^2\rangle \sim 4/L^3$ and $\langle |z|\rangle^2 = (2/\pi)\langle |z|^2\rangle$. Thus, the negative term in the pre-concurrence $(2(|z|-\sqrt{vy}))$ is typically larger than the positive term, thus resulting in the probability of a positive concurrence *decreasing* with increasing L as $1/\sqrt{L}$ (see [17] for calculations).

However, for two-particle states promoted from one-particle states obeying $\sum_i a_i = 0$, it is straightforward to show using $a_{ij} = (a_i + a_j)/\sqrt{L-2}$ that, up to the leading order,

$$y \approx (a_1+a_2)^2/L, \quad z \approx (1+La_1a_2)/L, \quad v \approx 1. \tag{11}$$

Therefore, although z appears as a sum of order L number of terms in eq. (10), it simplifies for promoted states to this simple form, which implies that both $|z|$ and $\sqrt{y}$ are of the same order of magnitude, namely $1/L$. This follows since $a_i \sim 1/\sqrt{L}$. As $v = \mathcal{O}(1)$, the concurrence (which is proportional to $|z| - \sqrt{vy}$) in promoted two-particle states is always in a fine balance between the two competing terms $|z|$ and $\sqrt{y}$. In contrast, for generic two-particle states, the order of $\sqrt{y}$ is $1/L$ which is much larger than the order of $|z|$ which is $1/L^{3/2}$, resulting in the probability of nonzero concurrence scaling as $1/\sqrt{L}$ [17].

The probability of finding any two spins entangled when the system is in a promoted two-particle state, *i.e.*, $P(C > 0)$ is now estimated. This is approximately same as $P(z^2 > y)$, since $v \approx 1$ (eq. (11)). Introducing the variables $x_i = \sqrt{L}a_i$, and treating the two x_i to be independent[1] and identically distributed random variables drawn from the standard normal distribution $\mathcal{N}(0,1)$, we have

$$\begin{aligned} P(C>0) \approx P(z^2 - y > 0) &= P[(1-x_1^2)(1-x_2^2) > 0] \\ &= \mathrm{erf}^2\left(\frac{1}{\sqrt{2}}\right) + \mathrm{erfc}^2\left(\frac{1}{\sqrt{2}}\right) \approx 0.566, \end{aligned} \tag{12}$$

[1] Strictly speaking, x_1 and x_2 cannot be independent since $\sum_k a_k = 1$, but the dependence between a_1 and a_2 is weak, allowing us to consider x_1 and x_2 to be independent with a small error.

Table 1: A table comparing the *random* promoted two-particle states with *random* one-particle and two-particle states. For the promoted states, $P(C > 0)$ is constant and $\langle C\rangle$ scales as $1/L$, similar to those of one-particle states while the localisation scaling is similar to that of two-particle states.

States	$P(C>0)$	$\langle C\rangle$	$\langle \mathrm{IPR}\rangle$
One-particle	1	$4/\pi L$	$3/L$
Two-particle	$2\sqrt{2}/\sqrt{\pi L}$	$16/\pi^{3/2}L^2$	$6/L^2$
Promoted two-particle	0.566	$0.465/L$	$6/L^2$

where $\mathrm{erfc}(z) \equiv 1-\mathrm{erf}(z)$ is the complementary error function. Thus, $P(C > 0)$ is a constant for random promoted two-particle states and does not decrease with L as it does for generic two-particle states. Note that as $v < 1$ in reality, the above can be expected to underestimate the actual probability. It is also worth recounting that random one-particle states have a probability 1 that the concurrence is nonzero.

The average concurrence of random promoted two-particle states may also be estimated by weighted integration over all x_1, x_2 where $P(C > 0) > 0$:

$$\langle C\rangle \approx \int_{\prod_i (1-x_i^2)>0} 2(|z| - \sqrt{vy})e^{-\frac{(x_1^2+x_2^2)}{2}} \mathrm{d}x_1\mathrm{d}x_2 \approx \frac{0.465}{L}, \tag{13}$$

where eq. (11) and the assumption of independent marginals has been used. The final result was obtained by setting $v = 1$ and factoring out the L dependence, and the L-independent integral was evaluated numerically to obtain 0.465. The $1/L$ behaviour is to be compared with generic two-particle states that have an expectation value of concurrence $\sim 1/L^2$ [17], and that for generic random one-particle states which goes as $\sim 1/L$ [2]. The promoted states have, on average, smaller entanglement than one-particle states, however they are much larger than what may be expected for generic two-particle states.

Interestingly, for the promoted one-particle state, *i.e.*, the all-one state in the one-particle sector, the concurrence between any two spins is $2/L$ and in the two-particle sector, the all-one state has $C = \frac{2}{\binom{L}{2}}\left(L - 2 - \sqrt{\frac{(L^2-5L+6)}{2}}\right)$, which also scales as $1/L$. This deserves a special mention since eq. (11) does not hold for the all-one state, yet the scaling behavior is identical. Thus, on average, the promoted random two-particle states retain the larger entanglement present in the generic one-particle states, while at the same time, they are as delocalised as generic two-particle states. Our results are summarised and compared against generic one-particle and two-particle states in table 1.

One-particle eigenstates of the spin glass Hamiltonian in eq. (1) (considering a large number of realisations of the J_{ij} for various values of L) were obtained by exact diagonalisation. These were then promoted to two-particle

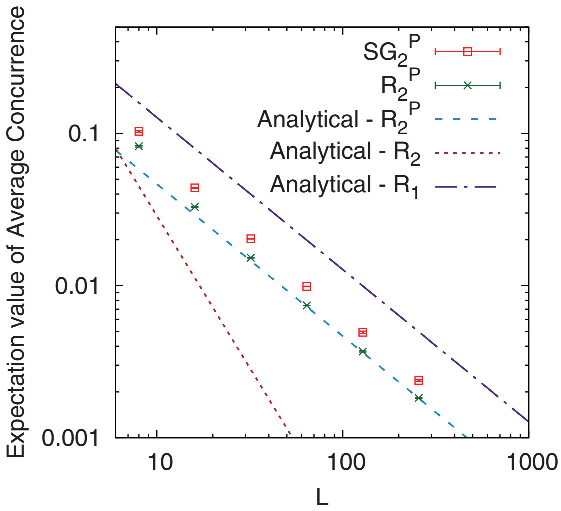

Fig. 3: (Colour online) Plot of expectation value of average concurrence showing the scaling behavior with the size of the system, L. The meaning of the various labels is the same as from the previous figure, while in addition R_1 refers to random one-particle states. For SG_2^p and R_2^p states, error-bars are included by considering an average over many states (obtained from considering many samples of disorder for small L). The error-bars are too tiny to be perceptible.

eigenstates. To compare properties, an equal number of one-particle random states were generated to obtain two-particle promoted random states. Figure 3 shows the expectation value of the concurrence for the promoted two-particle eigenstates of the spin glass Hamiltonian and promoted random two-particle states (SG_2^p and R_2^p). It is seen that R_2^p states mimic the SG_2^p eigenstates rather well, and that they behave differently from the random two-particle states (R_2) [17].

Thus using two reasonable assumptions, that $\sum_k a_k = 0$ and that a_i and a_j ($i \neq j$) are marginally independent, we have obtained analytical results for random promoted two-particle states. While for large L the agreement with the analysis leading to eq. (13) with simulated random states gets better, the same cannot be said for the spin glass eigenstates. This result is not surprising because the spin glass eigenstates carry special structure, which would make them not uniformly distributed on the unit sphere in Hilbert space, while random states are uniformly distributed, by construction. It is also striking that the scaling for generic two particle random states (that goes as $1/L^2$ and is also shown for comparison) is indeed very different. Thus, the statistical analysis of random two-particle promoted states sheds light on the enhanced entanglement observed in certain classes of spin glass eigenstates.

Summary and future directions. – A central finding of this paper is that when a many-body quantum system is governed by a random Heisenberg Hamiltonian with long-range coupling, a special class of eigenstates emerge that are characterised by enhanced entanglement. These special eigenstates, shown to be "promoted eigenstates", display significantly higher average concurrence compared with the rest of the eigenstates.

As a first step to understand the peculiarities of such promoted eigenstates, the properties of random promoted states were studied by a statistical approach. It has been proved analytically and confirmed numerically in this work that random one-particle states, for which the average two-spin entanglement scales as $1/L$, when promoted to have particle-number 2, shows an average two-spin entanglement that is lower but *still scales* as $1/L$. This is to be contrasted with the scaling behaviour of $1/L^2$ for random two-particle states [17]. In contrast, the localisation of a promoted two-particle state, as measured by the inverse participation ratio (IPR), is comparable to that of a typical two-particle state.

Thus, our results provide a small but an important step towards understanding how entanglement is shared across small subsystems in a larger system and providing hints on what kind of states should the quantum system be prepared in, to maximise or minimise entanglement as the application might require. From the point of view of random states, this work has introduced the study of promoted random states that will be found in systems with full rotational symmetry.

Two interesting questions seem natural to pursue further: i) While the random one-particle states and one-particle eigenstates have very different distribution in the concurrence-PR plot (refer to fig. 2), the distinction is largely reduced after the promotion. It remains to investigate if random promoted states are a good approximation to random promoted states for all models of the Heisenberg Hamiltonian and at all particle-numbers. ii) One can see that the zero-particle state when promoted to have a particle-number of 1 or 2 (all-one states), the concurrence still scales as $1/L$. It seems likely that the average two-spin entanglement of promoted one-particle eigenstates will continue to scale as $1/L$ in higher particle sectors as well and that of promoted two-particle eigenstates will continue to scale as $1/L^2$. This implies that one can have half-filled states with an average concurrence much higher than what one would normally expect, something that needs further work for verification.

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