A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

LETTER

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To cite this article: B. H. Yazarloo and H. Mehraban 2016 *EPL* **116** 31004

A Letters Journal Exploring the Frontiers of Physics

November 2016

EPL, **116** (2016) 31004
doi: 10.1209/0295-5075/116/31004

www.epljournal.org

Study of B and B_s mesons with a Coulomb plus exponential type potential

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received 9 September 2016; accepted in final form 25 November 2016
published online 21 December 2016

PACS 12.39.-x – Phenomenological quark models
PACS 12.39.Jh – Nonrelativistic quark model
PACS 12.39.Pn – Potential models

Abstract – In this paper, we have studied the B and B_s mesons spectra and their decays within the framework of nonrelativistic potential model. We have considered a new potential model for the interaction of mesonic systems, the Coulomb plus exponential type potential. We have applied the perturbation approach and reported the total wave function. We have used the Nikiforov-Uvarov (NU) technique to calculate the parent wave function and thereby obtained a series solution for the perturbative wave function. Besides the decay constant and leptonic decay width, we have considered the semileptonic decay width which is related to the Isgur-Wise function. The obtained results are compared with the available experimental and theoretical data.

Introduction. – The investigation and study of bound states and wave functions of quark-antiquark systems are of particular interest. The study of the wave function of the bound state of a quark and an antiquark from the strong interaction between quark and antiquarks in B and D mesons gives important information about the property of strong interaction and the mechanism of heavy-meson decays. From the wave function one can specify the momentum distributions of the quark and antiquark in mesons, which is a significant quantity for computing the amplitude of heavy-meson decays, and other decay properties [1–3]. Because of the importance and application of the wave function in studying hadronic systems, many efforts have been made to obtain the wave function and then investigate the mass spectra and decay properties of hadronic systems. For instance, in the QCD relativistic potential model, Cea *et al.* [4] have obtained the masses and the leptonic decay constants of charmed and beauty pseudoscalar and vector mesons. The mass spectra of a heavy-light quark-antiquark system have been studied by Liu and Yang [5] where they have presented the relativistic generalization of the Schrödinger equation and considered the Cornell potential and spin-dependent interaction. In the relativistic quark model, Ebert *et al.* [6,7] have presented mass spectra and Regge trajectories of light and heavy-light mesons.

In the present work, we intend to explore the wave function of the non-relativistic Schrödinger equation under the Coulomb plus exponential type potential, and then determine the masses, decay constant, leptonic decay width and semileptonic decay width of the B and B_s mesons. The exclusive semileptonic decay processes of heavy mesons generated a great excitement not only in extracting the most accurate values of the Cabbibo-Kobayashi-Maskawa (CKM) matrix elements but also in providing a valuable insight into quark dynamics in the non-perturbative domain of QCD.

The plan of the paper is as follows. In the next section we present the formalism of our work and obtain the wave function of the system under our potential model. From the obtained results of the next section, we begin to calculate the s-wave and P-wave masses of the B and B_s mesons, decay constant, leptonic decay width and semileptonic decay width which are related to the Isgur-wise function in the third section. Finally in the last section we draw our conclusions and present our discussion.

Formalism. –

Potential model. For our study of meson properties, we consider the Coulomb plus exponential type potential, which is defined as

$$V(r) = \frac{a}{r} + be^{\alpha r} + V_0, \tag{1}$$

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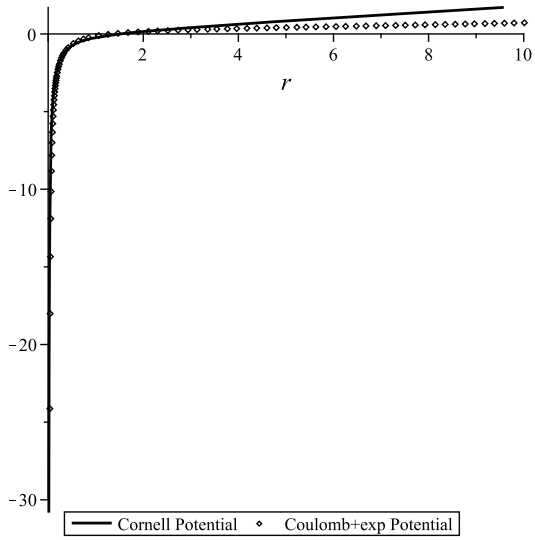

Fig. 1: Comparison of the Cornell potential $(0.18\,r - \frac{0.42}{r})$ [8] and our potential model.

where a, b and V_o are the potential parameters. If we expand the exponential part of the potential, this potential consists of a constant term, a linear term and a harmonic term and others. By considering $\alpha = -\mu$, $a = -4a_C/3$, $b = -\lambda/\mu$ and $V_0 = C + \lambda$ [9], this potential changes into the screened potential model [9]. In fig. 1, we have shown this potential with the Cornell potential. Here, we take a/r as parent part and $be^{\alpha r}$ as perturbation part. We now write the Hamiltonian as

$$H = H_0 + H', \tag{2}$$

where

$$H_0 = -\left(\hbar^2/2\mu\right)\nabla^2 + a/r, \tag{3}$$
$$H' = be^{\alpha r} \tag{4}$$

and $\mu = (m_q m_{\bar{Q}}/m_q + m_{\bar{Q}})$ is the reduced mass of the meson and m_q, $m_{\bar{Q}}$ are the masses of the light and heavy quarks, respectively. Therefore, the two-body Schrödinger equation for the Hamiltonian $H = H_0 + H'$ is

$$H\left|\Psi\right\rangle = \left(H_0 + H'\right)\left|\Psi\right\rangle = E\left|\Psi\right\rangle. \tag{5}$$

The wave function. In order to obtain the masses of B and B_s mesons and other decay properties, we need to obtain the wave function of the systems. To find the unperturbed wave function corresponding to H_0, we use the two-body radial Schrödinger equation (with $\hbar = 1$)

$$\left\{-\frac{1}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{a}{r} + \frac{l(l+1)}{2\mu r^2}\right\} R_{n,l}(r) = E_{n.l}R_{n,l}(r); \tag{6}$$

to obtain the solution of the above equation, we utilize the NU method. According to the appendix below, we have

$$\alpha_1 = 2, \quad \alpha_2 = \alpha_3 = 0, \quad \xi_1 = -2\mu E_{n,l},$$
$$\xi_2 = -2\mu a, \quad \xi_3 = l(l+1) \tag{7}$$

and, therefore, the wave function and the energy read

$$E_{n,l} = -\left\{2\mu a^2/(2n+2l+2)^2\right\}, \tag{8}$$
$$R_{n,l}(r) = Nr^l e^{-\sqrt{-2\mu E_{n,l}}r} L_n^{2l+1}\left(2\sqrt{-2\mu E_{n,l}}r\right). \tag{9}$$

The ground-state solution. For the ground state $n = l = 0$, we have

$$E_{0,0} = -\left(\mu a^2/2\right),\ R_{0,0}(r) = Ne^{-\sqrt{-2\mu E_{0,0}}r}, \tag{10}$$

where N is the normalization constant of the parent wave function. The first-order perturbed eigen function $R'_{0,0}(r)$ can be calculated using the following relation [10]:

$$H_0 R'_{0,0}(r) + H' R_{0,0}(r) = W^0 R'_{0,0}(r) + W' R_{0,0}(r), \tag{11}$$

where $W = E_{0,0}$ and the perturbed eigen energy W' can be computed from

$$W' = 4\pi \int_0^\infty r^2 \left|R_{0,0}(r)\right|^2 H' \mathrm{d}r \tag{12}$$

from eqs. (3), (4) and (10), eq. (11) can be written as

$$\left\{\mathrm{d}^2/\mathrm{d}r^2 + (2/r)\,\mathrm{d}/\mathrm{d}r - (2\mu a/r) + 2\mu E_{0,0}\right\} R'_{0,0}(r) = \left\{2\mu b e^{\alpha r} - 2\mu W'\right\} R_{0,0}(r) \tag{13}$$

by choosing the perturbed wave function as

$$R'_{0,0}(r) = N' Q(r) R_{0,0}(r), \tag{14}$$

where N' is the normalization constant of the perturbed wave function and $Q(r)$ is given as [10]

$$Q(r) = \sum_{l'=0}^{\infty} A_{l'} r^{l'}. \tag{15}$$

Inserting eqs. (10), (14) and (15) in eq. (13) we reach

$$\sum_{l'} A_{l'} l'(l'-1) r^{l'-2} - 2\sqrt{-2\mu E_{0,0}} \sum_{l'} A_{l'} l' r^{l'-1} + 2\sum_{l'} A_{l'} l' r^{l'-2} + \zeta \sum_{l'} A_{l'} r^{l'-1} = \left(2\mu b e^{\alpha r} - 2\mu W'\right) \tag{16}$$

with the help of eq. (16) and equating the corresponding powers of r^{-1}, r, r^2, r^3, r^4 and r^0 on both sides of eq. (16) and the expanding $e^{\alpha r}$ part, we find

$$A_0 = -(2/\zeta)\,A_1, \tag{17a}$$
$$A_1 = \left(2\mu b - 2\mu W' - 6A_2\right)\Big/\left(\zeta - 2\sqrt{-2\mu E_{0,0}}\right), \tag{17b}$$
$$A_2 = \left(2\mu b\alpha - 12A_3\right)\Big/\left(\zeta - 4\sqrt{-2\mu E_{0,0}}\right), \tag{17c}$$
$$A_3 = \left(\mu b\alpha^2 - 20A_4\right)\Big/\left(\zeta - 6\sqrt{-2\mu E_{0,0}}\right), \tag{17d}$$
$$A_4 = \left(\left(\mu b\alpha^3/3\right) - 30A_5\right)\Big/\left(\zeta - 8\sqrt{-2\mu E_{0,0}}\right), \tag{17e}$$
$$A_5 = -(1/12)\left(\mu b\alpha^4/14\mu a\right), \tag{17f}$$

by replacing the coefficients of the $Q(r)$ function (eqs. (17)) and using eqs. (14) and (16), we obtain the following perturbed wave function:

$$R'_{0,0}(r) = N'\left(A_0+A_1r+A_2r^2+A_3r^3+A_4r^4+A_5r^5\right)e^{-\sqrt{-2\mu E_{0,0}}r}. \tag{18}$$

Thus, we obtain the total wave function as

$$R^{total}_{0,0}(r) = N^{total}_{0,0}\left(R'_{0,0}(r) + R_{0,0}(r)\right), \tag{19}$$

where N_{total} is the normalization constant of the total wave function.

The P-wave state solution. For the case $n = 0$, $l = 1$, from eqs. (8) and (9), we find

$$E_{0,1} = -\left(\mu a^2/8\right), \quad R_{0,1}(r) = N_{0,1}r\,e^{-\sqrt{-2\mu E_{0,1}}r}, \tag{20}$$

according to the approach of the previous subsection, the perturbed wave function and the total wave function obtain as

$$R'_{0,1}(r) = N'_{0,1}\left(A'_0r+A'_1r^2+A'_2r^3+A'_3r^4+A'_4r^5+A'_5r^6\right)e^{-\sqrt{-2\mu E_{0,1}}r} \tag{21}$$

$$R^{total}_{0,1}(r) = N^{total}_{0,1}\left(R'_{0,1}(r) + R_{0,1}(r)\right). \tag{22}$$

where

$$A'_0 = -\frac{4A'_1}{\left(-2\mu a - 4\sqrt{-2\mu E_{0,1}}\right)}, \tag{23a}$$

$$A'_1 = \frac{2\mu b - 2\mu W' - 10A'_2}{\left(-2\mu a - 6\sqrt{-2\mu E_{0,1}}\right)}, \tag{23b}$$

$$A'_2 = \frac{2\mu b\alpha - 18A'_3}{\left(-2\mu a - 8\sqrt{-2\mu E_{0,1}}\right)}, \tag{23c}$$

$$A'_3 = \frac{\mu b\alpha^2 - 28A'_4}{\left(-2\mu a - 10\sqrt{-2\mu E_{0,1}}\right)}, \tag{23d}$$

$$A'_4 = \frac{\frac{\mu b\alpha^3}{3} - 40A'_5}{\left(-2\mu a - 12\sqrt{-2\mu E_{0,1}}\right)}, \tag{23e}$$

$$A'_5 = \frac{1}{12}\frac{\mu b\alpha^4}{\left(-2\mu a - 14\sqrt{-2\mu E_{0,1}}\right)}. \tag{23f}$$

The masses and decay properties of mesons. – From the results of the previous section, we are going to compute the mass spectra and decay properties of the B and B_s mesons. For the first step, we calculate the masses of the mesons in the following subsection.

The masses of the mesons. The masses of the B and B_s mesons are defined as

$$M = m_{\bar{q}} + m_Q + E + \langle H_{SS}\rangle + \langle H_{L.S}\rangle + V_0, \tag{24}$$

Table 1: Values of the parameters used in our model.

Parameter	Value	Parameter	Value
m_b	4.812 (GeV)	a	-0.5
m_u	0.35 (GeV)	α_s	0.32
m_s	0.45 (GeV)	b	0.385 (GeV)

Table 2: The calculated masses of s-wave B and B_s mesons ($\alpha = 0.075$ (GeV), $V_0 = 0.192$ (GeV)) in MeV.

	$^{2S+1}L_J$	M (present)	M (others)
$B^\pm$	$^1S_0$	5271.7	5279.29 ± 0.15 [11] 5302 [12]
B^*	$^3S_1$	5327.05	5324.83 ± 0.32 [11] 5356 [12]
B^0_s	$^1S_0$	5384.7	5366.79 ± 0.23 [11] 5340 [12]
B^*_s	$^3S_1$	5408.5	$5415.4^{+1.8}_{-1.5}$ [11] 5384 [12]

where $\langle H_{SS}\rangle$ is the spin-spin interactions, and the form generally used is [13]

$$H_{SS} = \frac{32\pi\alpha_s}{9m_Qm_{\bar{q}}}\vec{S}_Q.\vec{S}_{\bar{q}}\delta(\vec{r}), \tag{25}$$

from the above equation we obtain

$$\langle H_{SS}\rangle = \begin{cases} \dfrac{8\pi\alpha_s}{9m_Qm_{\bar{q}}}\left|\Psi(0)\right|^2, & \text{for } \vec{S} = 1, \\ -\dfrac{8\pi\alpha_s}{3m_Qm_{\bar{q}}}\left|\Psi(0)\right|^2, & \text{for } \vec{S} = 0. \end{cases} \tag{26}$$

To calculate the value of $|\Psi(0)|^2$ we have used the following relation for the ground state:

$$\left|\Psi(0)\right|^2 = \frac{\mu}{2\pi\hbar^2}\left\langle\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right\rangle \tag{27}$$

we have also applied the value of $|\Psi(0)|^2$ to determine various decay rates for the B and B_s mesons. The spin-orbit interactions are defined as

$$H_{L\cdot S} = \left(-\frac{1}{2\mu^2r}\frac{\mathrm{d}V(r)}{\mathrm{d}r}\right)\left(\vec{L}\cdot\vec{S}\right), \tag{28}$$

and we have

$$\left\langle\vec{L}\cdot\vec{S}\right\rangle = \frac{1}{2}\left(j(j+1) - l(l+1) - S(S+1)\right). \tag{29}$$

In table 1, we have reported the values of the parameters used in our model. The calculated masses are represented in tables 2 and 3.

Decay constant. In the non-relativistic limit, the decay constant of the vector and the pseudoscalar mesons

Table 3: The calculated masses of P-wave B and B_s mesons in MeV ($\alpha = 0.02\,(\mathrm{GeV})$, $V_0 = 0.592\,(\mathrm{GeV})$).

	B meson		B_s meson	
States	M (present)	M (others)	M (present)	M (others)
		5743 [11]		5840 [11]
$^3P_2$	5743.1	5741 [7]	5840.44	5842 [7]
		5714 [14]		5820 [14]
		5732 [11]		5833 [7]
$^3P_0$	5745.09	5749 [7]	5842.52	5804 [14]
		5706 [14]		
$^1P_1$	5743.80		5841.13	
$^3P_1$	5744.45		5841.83	

is given by the Van Royen-Weisskopf formula [15] and with QCD radiative corrections taken into account [16]

$$f_{p/v}^2 = \frac{12\,|\Psi(0)|^2}{m_{p/v}}, \tag{30}$$

$$\bar{f}_{p/v}^2 = \frac{12\,|\Psi(0)|^2}{m_{p/v}} C^2(\alpha_s), \tag{31}$$

where

$$C(\alpha_s) = 1 - \frac{\alpha_s}{\pi}\left(\Delta_{p/v} - \frac{m_Q - m_{\bar{q}}}{m_Q + m_{\bar{q}}} \ln \frac{m_Q}{m_{\bar{q}}}\right) \tag{32}$$

and $\Delta_p = 2$ and $\Delta_v = 8/3$. In table 4 the evaluated decay constants of pseudoscalar and vector mesons are represented, respectively. We have also calculated the decay constants including the QCD correction factor ($\bar{f}_{p/v}$) in the table.

Leptonic decay widths. Purely leptonic decays of charged B mesons proceed in the Standard Model (SM) via the W-mediated annihilation tree diagram, with a branching fraction given by [17]

$$Br\left(B^+ \to l^+ \upsilon_l\right) = \Gamma\left(B^+ \to l^+ \upsilon_l\right) \times \tau_B, \tag{33}$$

where

$$\Gamma\left(B^+ \to l^+ \upsilon_l\right) = \frac{G_F^2 M_B}{8\pi} m_l^2 \left(1 - \frac{m_l^2}{M_B^2}\right)^2 f_B^2 \left|V_{ub}\right|^2, \tag{34}$$

and $\tau_{B^+} = (1.638 \pm 0.004) \times 10^{-12}\,(s)$ is the B meson lifetime [11]. To evaluate the leptonic decay width and the branching ratio we have employed the calculated $\bar{f}_{B^+}$ and M_{B^+}, from tables 2 and 4, and tabulate them in table 5.

Semileptonic decay width. The differential semileptonic decay rate $B \to D^* l \bar{\upsilon}$ is calculated to be [18]

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\omega} = \frac{G_F^2}{48\pi^3} \left|V_{cb}\right|^2 M_{D^*}^3 \left(M_B - M_{D^*}\right)^2 \sqrt{(\omega^2 - 1)}\,(\omega + 1)^2 \times \left[1 + \frac{4\omega}{\omega + 1} \frac{1 - 2\omega r^* + r^{*2}}{(1 - r^*)^2}\right] F_{D^*}^2(\omega), \quad r^* = \frac{M_{D^*}}{M_B}, \tag{35}$$

Table 4: Decay constants of pseudoscalar and vector B mesons (in MeV).

	$f_{p/v}$	$\bar{f}_{p/v}$	$\mathrm{Others_{un}}$
$B^\pm$	243.64	250.23	149 [12]
			189 [19]
			195 [20]
B^*	242.37	232.47	238 ± 18 [21]
			151^{+15}_{-13} [22]
B_s^0	179.21	178.56	187 [12]
			218 [19]
			193 [20]
B_s^*	178.82	166.03	272 ± 20 [21]
			236^{+14}_{-11} [22]

where the form factor $F_{D^*}(\omega)$ is given by

$$F_{D^*}(\omega) = h_{A_1}(\omega) \sqrt{\frac{\tilde{H}_+^2(\omega) + \tilde{H}_-^2(\omega) + \tilde{H}_0^2(\omega)}{1 + \frac{4\omega}{\omega+1} \frac{1 - 2\omega r^* + r^{*2}}{(1 - r^*)^2}}}. \tag{36}$$

The helicity amplitudes $\tilde{H}_j(\omega)$

$$\tilde{H}_\pm(\omega) = \frac{\sqrt{1 - 2\omega r^* + r^{*2}}}{(1 - r^*)} \left[1 \mp \sqrt{\frac{\omega - 1}{\omega + 1}} R_1(\omega)\right],$$
$$\tilde{H}_0(\omega) = 1 + \frac{\omega - 1}{1 - r^*}\left[1 - R_2(\omega)\right] r^*, \tag{37}$$

are expressed through form factor ratios

$$R_1(\omega) = \frac{h_V(\omega)}{h_{A_1}(\omega)}, \quad R_2(\omega) = \frac{h_{A_3}(\omega) + r^* h_{A_2}(\omega)}{h_{A_1}(\omega)}. \tag{38}$$

In the limit $m_Q \to \infty$, $R_1 = R_2 = 1$ due to spin-flavour symmetry [18]. In the limit of an infinitely heavy quark all form factors are expressed through the Isgur-Wise function

$$h_{A_1}(\omega) = h_{A_3}(\omega) = h_V(\omega) = \xi(\omega),$$
$$h_{A_2}(\omega) = 0. \tag{39}$$

Table 5: Leptonic decay width and branching ratio of the B^+ meson.

	Γ(GeV)	Br (present)	Br (others)
$B^+ \to e^+\bar{\upsilon}_e$	8.6235×10^{-24}	2.1467×10^{-11}	$< 9.8 \times 10^{-7}$ [11]
			6.22×10^{-12} [12]
			$< 7.7 \times 10^{-6}$ [23]
$B^+ \to \mu^+\bar{\upsilon}_\mu$	3.6852×10^{-19}	9.174×10^{-7}	$< 1.0 \times 10^{-6}$ [11]
			2.63×10^{-7} [12]
			$< 11 \times 10^{-6}$ [23]
			4.8×10^{-7} [24]
$B^+ \to \tau^+\bar{\upsilon}_\tau$	8.1955×10^{-17}	2.040×10^{-4}	$(1.14 \pm 0.27) \times 10^{-4}$ [11]
			0.59×10^{-4} [12]
			$(1.8 \pm 0.6) \times 10^{-4}$ [23]
			$(0.8 \pm 0.12) \times 10^{-4}$ [25]

Table 6: Semileptonic decay width and branching ratio of the B^+ meson.

	Γ (GeV)	Br (present)	Br (others)
$B^+ \to D^{*0}l^+\upsilon$	3.5771×10^{-14}	8.904%	(5.69 ± 0.19) % [11]

The Isgur-Wise function is the overlapping of the wave functions of two hadrons and it can be written as

$$\xi(\omega) = \sqrt{\frac{2}{\omega+1}} \left\langle R^B(r) \mid R^{D^*}(r) \right\rangle. \tag{40}$$

By integrating the differential decay width over the $1 \leq \omega \leq \frac{M_B^2+M_{D^*}^2}{2M_B M_{D^*}}$ range, we have reported the decay width and branching ratio of $B^+ \to D^{0*}l\bar{\upsilon}$ in table 6.

Conclusion and discussion. – In this paper, we have explored B and B_s mesons bound states under the non-relativistic Schrödinger equation with a new potential model. By introducing the spin-spin and spin-orbit interactions, we have calculated the masses of the B and B_s mesons. Our computed masses are in good agreement with experimental results. The calculated masses of s-wave states are $M(B^\pm) = 5271.7\,(\text{MeV})$, $M(B^*) = 5327.05\,(\text{MeV})$, $M(B_s^0) = 5384.7\,(\text{MeV})$ and $M(B_s^*) = 5408.5\,(\text{MeV})$, while the experimental results are $5279.29 \pm 0.15\,(\text{MeV})$, $5324.83 \pm 0.32\,(\text{MeV})$, $5366.79 \pm 0.23\,(\text{MeV})$ and $5415.4^{+1.8}_{-1.5}\,(\text{MeV})$, respectively. Also, we can see that for the P-wave states our obtained results for $^3P_2$ and $^3P_0$ are very close to the experimental and theoretical predictions. By applying the perturbation approach and NU technique we have found the total wave function and the value of the wave function at the origin of the system and obtained some decay properties of the B and B_s mesons with and without QCD correction. By using the Van Royen-Weisskopf formula, we have computed the decay constant of pseudoscalar and vector mesons. In this potential model, we have also calculated the leptonic decay widths and the corresponding branching rations of the B^+ meson. We have seen that the evaluated branching ratios of $Br_{B^+\to e^+\bar{\upsilon}_e} = 2.1467 \times 10^{-11}$, $Br_{B^+\to \mu^+\bar{\upsilon}_\mu} = 9.17403 \times 10^{-7}$ and $Br_{B^+\to \tau^+\bar{\upsilon}_\tau} = 2.040 \times 10^{-4}$ are very close to the experimental values. With the help of the Isgur-Wise function and from eq. (40), the semileptonic decay width and branching ratio of the $B \to D^* l\bar{\upsilon}$ process are computed and represented in table 6. Our calculated $Br = 8.904\%$ is comparable with the experimental value $Br = (5.69 \pm 0.19)\%$.

From our analysis we infer that such studies would provide an impetus to establish better methods for theoretical calculations of bound states using the fundamental QCD.

* * *

The authors wish to thank the referee for their valuable suggestions for improving the paper.

Appendix

The NU method solves many linear second-order differential equations by reducing them to a generalized equation of hypergeometric type. This method has been widely used in the literature to solve various differential equations of quantum mechanics. The NU technique in its parametric form simply solves a differential equations of the form [26–28]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\right]\psi(s) = 0. \tag{A.1}$$

In the NU method, the energy eigenvalues satisfy

$$\begin{aligned}&\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \\ &+ n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0,\end{aligned} \tag{A.2}$$

and the eigenfunctions are

$$\psi(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} p_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{A.3}$$

where

$$\begin{aligned}
\alpha_4 &= \frac{1}{2}(1-\alpha_1), \quad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3),\\
\alpha_6 &= \alpha_5^2 + \xi_1, \quad \alpha_7 = 2\alpha_4\alpha_5 - \xi_2,\\
\alpha_8 &= \alpha_4^2 + \xi_3, \quad \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6,\\
\alpha_{10} &= \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8},\\
\alpha_{11} &= \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \quad \alpha_{12} = \alpha_4 + \sqrt{\alpha_8},\\
\alpha_{13} &= \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}),
\end{aligned} \tag{A.4}$$

with $p_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s)$ being the Jacobi polynomial. For $\alpha_3 = 0$, we have

$$\lim_{\alpha_3\to 0} p_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s) = L_n^{\alpha_{10}-1}(\alpha_{11}s) \tag{A.5}$$

and

$$\lim_{\alpha_3\to\infty}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13}s}, \tag{A.6}$$

therefore, the wave function (A.3) becomes

$$\psi(s) = s^{\alpha_{12}} e^{\alpha_{13}s} L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{A.7}$$

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