LETTER

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To cite this article: Victor Dotsenko 2016 *EPL* **116** 40004

epl A Letters Journal Exploring the Frontiers of Physics

November 2016

EPL, **116** (2016) 40004 www.epljournal.org
doi: 10.1209/0295-5075/116/40004

Two-temperature statistics of free energies in (1 + 1) directed polymers

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received 25 November 2016; accepted in final form 19 December 2016
published online 3 January 2017

PACS 05.20.-y – Classical statistical mechanics
PACS 75.10.Nr – Spin-glass and other random models
PACS 61.41.+e – Polymers, elastomers, and plastics

Abstract – The joint statistical properties of two free energies computed at two different temperatures in *the same sample* of (1+1) directed polymers is studied in terms of the replica technique. The scaling dependence of the reduced free-energies difference $\mathcal{F} = F(T_1)/T_1 - F(T_2)/T_2$ on the two temperatures T_1 and T_2 is derived. In particular, it is shown that if the two temperatures $T_1 < T_2$ are close to each other, the typical value of the fluctuating part of the reduced free-energies difference $\mathcal{F}$ is proportional to $(1 - T_1/T_2)^{1/3}$. It is also shown that the left-tail asymptotics of this free-energy difference probability distribution function coincides with the corresponding tail of the TW distribution.

Introduction. – In this paper we consider the model of one-dimensional directed polymers in terms of an elastic string $\phi(\tau)$ directed along the τ-axes within an interval $[0, t]$ which passes through a random medium described by a random potential $V(\phi, \tau)$. This model is defined in terms of the Hamiltonian

$$H[\phi;\, V] = \int_0^t \mathrm{d}\tau \Big[\frac{1}{2}\big[\partial_\tau \phi(\tau)\big]^2 + V[\phi(\tau), \tau]\Big], \qquad (1)$$

where the disorder potential $V[\phi, \tau]$ is Gaussian distributed with a zero mean $\overline{V(\phi, \tau)} = 0$ and the δ-correlations

$$\overline{V(\phi, \tau)V(\phi', \tau')} = u\delta(\tau - \tau')\delta(\phi - \phi') \qquad (2)$$

with the parameter u describing the strength of the disorder.

This problem, which is equivalent to the one of the Kardar-Parisi-Zhang (KPZ) equation [1] describing the time evolution of an interface in the presence of noise, has been the focus of intense studies during the past three decades [2–26]. At present it is well established that depending on the boundary conditions the fluctuations of the free energy of the model defined by the Hamiltonian (1) are described by the GUE [14–20], GOE [21,22] or GSE [24] Tracy-Widom distribution [27]. The two-point as well as N-point free-energy distribution function which describes joint statistics of the free energies of the directed polymers coming to different endpoints has been derived in [28–32]. Besides, the joint statistical properties of the free energies at two different times has been studied in [33–39].

In the present paper I would like to propose one more "direction" of the studies of this system, namely, joint statistics of the free energies (or the interfaces, in the KPZ language) at two different temperatures defined for the same quenched disorder. In other words, I am going to study the joint probability distribution function of the free energies at two (or more) different temperatures for a given realization of the disorder potential $V[\phi, \tau]$. Some years ago a similar kind of problem (under the name "temperature chaos") has been investigated for spin-glass–like systems [40–43] as well as for directed polymers on a hierarchical lattice [44]. In this paper in terms of the standard replica formalism I derive the general scaling dependence of the difference of two free energies at two different temperatures, eqs. (34) and (29), as well as the left-tail asymptotics of the corresponding universal probability distribution function, eq. (36). In particular, it will be shown that if the two temperatures $T_1 < T_2$ are close to each other, so that $(1 - T_1/T_2) \ll 1$, the difference of the two free energies scales as $(1-T_1/T_2)^{1/3}\, t^{1/3}$, eq. (41).

Replica formalism. – For a fixed boundary conditions, $\phi(0) = \phi(t) = 0$, and for a given realization of

disorder the partition function of the model defined in eqs. (1), (2) is

$$Z(\beta,t)=\int_{\phi(0)=0}^{\phi(t)=0}\mathcal{D}\phi(\tau)\,\mathrm{e}^{-\beta H[\phi;V]}=\exp\bigl(-\beta F(\beta,t)\bigr), \tag{3}$$

where β is the inverse temperature and $F(\beta,t)$ is the (random) free energy. It is well known that in the limit $t\to\infty$ this free energy scales as

$$F(\beta,t)=f_0(\beta)\,t+\frac{1}{2}\,(\beta u)^{2/3}\,t^{1/3}\,f, \tag{4}$$

where $f_0(\beta)$ is the (non-random) selfaveraging free-energy density, and f is a random quantity described by the Tracy-Widom distribution.

For a given realization of the disorder potential $V[\phi,\tau]$ let us consider the above system at two different temperatures $T_1\neq T_2$. More specifically, we are going to study how the two free energies $F(\beta_1,t)$ and $F(\beta_2,t)$ of the same system are related to each other. In the present paper we are going to study the statistical and scaling properties of the quantity

$$\mathcal{F}(\beta_1,\beta_2;t)=\beta_1F(\beta_1,t)-\beta_2F(\beta_2,t), \tag{5}$$

where, in what follows it will be assumed that $\beta_1>\beta_2$ (or $T_1<T_2$). According to the definition (3)

$$\exp\bigl\{-\mathcal{F}(\beta_1,\beta_2;t)\bigr\}=Z(\beta_1,t)Z^{-1}(\beta_2,t). \tag{6}$$

Taking the N-th power of both sides of the above relation and averaging over the disorder we get

$$\int \mathrm{d}\mathcal{F}P_{\beta_1,\beta_2,t}(\mathcal{F})\exp\bigl\{-N\mathcal{F}\bigr\}=\overline{Z^N(\beta_1,t)Z^{-N}(\beta_2,t)}, \tag{7}$$

where $\overline{(\ldots)}$ denotes the averaging over the random potential V and $P_{\beta_1,\beta_2,t}(\mathcal{F})$ is the probability distribution function of the random quantity $\mathcal{F}$, eq. (5). Introducing the replica partition function

$$\mathcal{Z}(M,N;\,\beta_1,\beta_2;\,t)\;=\;\overline{Z^N(\beta_1,t)\,Z^{M-N}(\beta_2,t)} \tag{8}$$

the relation (7) can be formally represented as

$$\int \mathrm{d}\mathcal{F}P_{\beta_1,\beta_2,t}(\mathcal{F})\exp\bigl\{-N\mathcal{F}\bigr\}=\lim_{M\to 0}\mathcal{Z}(M,N;\beta_1,\beta_2;t). \tag{9}$$

Following the standard "logic" of the replica technique, first it will be assumed that both M and N are integers such that $M>N$. Next, after computing the replica partition function $\mathcal{Z}(M,N;\,\beta_1,\beta_2;\,t)$ an analytic continuation for arbitrary (complex) values of the parameters M and N has to be performed and the limit $M\to 0$ has to be taken. After that, the relation (9) can be considered as the Laplace transform of the probability distribution function $P_{\beta_1,\beta_2,t}(\mathcal{F})$ over the parameter N. In the case the function $\mathcal{Z}(0,N;\,\beta_1,\beta_2;\,t)$ had "good" analytic properties in the complex plane of the argument N, this relation, at least formally, would allow to reconstruct by inverse Laplace transform the probability distribution function $P_{\beta_1,\beta_2,t}(\mathcal{F})$. At present, for the considered problem it is possible to derive an explicit expression for the function $\mathcal{Z}(0,N;\,\beta_1,\beta_2;\,t)$ only in the limit $N\gg 1$. Nevertheless, using the relation (9) this allows to reconstruct the left tail ($\mathcal{F}\to-\infty$) of the distribution function $P_{\beta_1,\beta_2,t}(\mathcal{F})$. Moreover, it also allows to derive the scaling dependence of free-energy difference $\mathcal{F}$ on β_1,β_2 and t. Indeed, in the case in which the replica partition function has an exponential asymptotics

$$\mathcal{Z}(0,N\to\infty;\beta_1,\beta_2;t\to\infty)\sim\exp\bigl\{A(\beta_1,\beta_2)tN^{\alpha}\bigr\}, \tag{10}$$

the left tail of the probability distribution function assumes the stretched-exponential form

$$P_{\beta_1,\beta_2,t}(\mathcal{F}\to-\infty)\;\sim\;\exp\bigl\{-B(\beta_1,\beta_2;\,t)\,|\mathcal{F}|^{\omega}\bigr\}. \tag{11}$$

Then the saddle-point estimate of the integral on the l.h.s. of eq. (9) yields

$$\begin{aligned}&\int \mathrm{d}\mathcal{F}\exp\bigl\{-B\,|\mathcal{F}|^{\omega}+N|\mathcal{F}|\bigr\}\sim\\&\exp\bigl\{(\omega-1)\omega^{-\frac{\omega}{\omega-1}}B^{-\frac{1}{\omega-1}}N^{\frac{\omega}{\omega-1}}\bigr\}\\&\sim\exp\bigl\{A\,t\,N^{\alpha}\bigr\}.\end{aligned} \tag{12}$$

From this relation we find that

$$\omega=\alpha/(\alpha-1) \tag{13}$$

and

$$B=(\alpha-1)\alpha^{-\frac{\alpha}{\alpha-1}}\bigl(At\bigr)^{-\frac{1}{\alpha-1}}. \tag{14}$$

Substituting this into eq. (11) we get

$$P_{\beta_1,\beta_2,t}(\mathcal{F}\to-\infty)\sim\exp\left\{-(\alpha-1)\left(\frac{|\mathcal{F}|}{\alpha\bigl(At\bigr)^{1/\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right\}. \tag{15}$$

If we assume that the (unknown) entire probability distribution function has a universal shape, the above asymptotic behavior implies that the considered quantity $\mathcal{F}$ scales as follows:

$$\mathcal{F}=\bigl(A(\beta_1,\beta_2)\bigr)^{1/\alpha}t^{1/\alpha}\,f, \tag{16}$$

where the random quantity $f\sim 1$ is described by some (unknown) probability distribution function $\mathcal{P}(f)$ with the left asymptotics $\mathcal{P}(f\to-\infty)\sim\exp\bigl\{-(\mathrm{const})|f|^{\alpha/(\alpha-1)}\bigr\}$.

Thus, the above speculations demonstrate that even if we know the replica partition function only in the limit $N\gg 1$, we can still derive not only the left tail of the distribution function, but also (supposing that the entire distribution function is universal) the general scaling of the free energy. In the next section we will demonstrate how this replica scheme can be applied for the concrete system under consideration.

Mapping to quantum bosons. – According to eqs. (1)–(3) and (8), after performing the averaging over the random potential we get

$$\mathcal{Z}(M,N;\beta_1,\beta_2;t)=\prod_{a=1}^{M}\int_{\phi_a(0)=0}^{\phi_a(t)=0}\mathcal{D}\phi_a(\tau)\exp\{-H_M[\boldsymbol{\phi}]\}, \tag{17}$$

where $H_M[\boldsymbol{\phi}]$ is the replica Hamiltonian

$$H_M[\boldsymbol{\phi}] = \int_0^t \mathrm{d}\tau\left[\frac{1}{2}\sum_{a=1}^{M}\beta_a\Big(\partial_\tau\phi_a(\tau)\Big)^2 - \frac{1}{2}\,u^2\sum_{a\neq b=1}^{M}\beta_a\beta_b\,\delta(\phi_a-\phi_b)\right] \tag{18}$$

and

$$\beta_a=\begin{cases}\beta_1, & \text{for } a=1,\ldots,N\\ \beta_2, & \text{for } a=N+1,\ldots,M,\end{cases} \tag{19}$$

Introducing

$$\Psi(x_1,\ldots,x_M;t)\equiv\prod_{a=1}^{M}\int_{\phi_a(0)=0}^{\phi_a(t)=x_a}\mathcal{D}\phi_a(\tau)\exp\{-H_M[\boldsymbol{\phi}]\} \tag{20}$$

one can easily show that $\Psi(\mathbf{x};t)$ is the wave function of M-particle boson system with attractive δ-interaction defined by the Schrödinger equation:

$$-\partial_t\Psi(\mathbf{x};t) = \sum_{a=1}^{M}\frac{1}{2\beta_a}\partial_{x_a}^2\Psi(\mathbf{x};t) + \frac{1}{2}u^2\sum_{a\neq b}^{M}\beta_a\beta_b\delta(x_a-x_b)\Psi(\mathbf{x};t) \tag{21}$$

with the initial condition $\Psi(\mathbf{x};0)=\Pi_{a=1}^{M}\delta(x_a)$. According to the definitions (17) and (20),

$$\mathcal{Z}(M,N;\beta_1,\beta_2;t)=\Psi(x_1,\ldots,x_M;t)\Big|_{x_a=0}. \tag{22}$$

The time-dependent wave function $\Psi(\mathbf{x};t)$ of the above quantum problem can be represented in terms of the linear combination of the eigenfunctions $\Psi(\mathbf{x})$ defined by the solutions of the eigenvalue equation

$$2E\Psi(\mathbf{x})=\sum_{a=1}^{M}\frac{1}{\beta_a}\partial_{x_a}^2\Psi(\mathbf{x})+u^2\sum_{a\neq b}^{M}\beta_a\beta_b\delta(x_a-x_b)\Psi(\mathbf{x}). \tag{23}$$

Unlike the case with all β's equal [45–47], for the time being, the general solution of this equation is not known. However, if we do not pretend to derive the exact result for the entire probability distribution function $P_{\beta_1,\beta_2,t}(\mathcal{F})$ but we want to know only its left-tail asymptotics in the limit $t\to\infty$, then it would be enough to get the behavior of the replica partition function $\mathcal{Z}(0,N\to\infty;\beta_1,\beta_2;\,t\to\infty)$ which is defined by the ground-state solution only:

$$\Psi(\mathbf{x};t\to\infty)\;\sim\;\exp\{-E_{g.s.}t\}\,\Psi_{g.s.}(\mathbf{x}). \tag{24}$$

One can easily check that the ground-state solution of eq. (23) is given by the eigenfunction

$$\Psi_{g.s.}(\mathbf{x})\propto\exp\left\{-\frac{1}{2}u\sum_{a,b=1}^{M}\gamma_{ab}\,|x_a-x_b|\right\}, \tag{25}$$

where

$$\gamma_{ab}=\frac{\beta_a^2\,\beta_b^2}{\beta_a+\beta_b}. \tag{26}$$

The corresponding ground-state energy is

$$E_{g.s.}(M,N,\beta_1,\beta_2)=-\frac{1}{2}u^2\sum_{a=1}^{M}\frac{1}{\beta_a}\left(\sum_{b=1}^{a-1}\gamma_{ab}-\sum_{b=a+1}^{M}\gamma_{ab}\right)^2. \tag{27}$$

Note that in the trivial case $\beta_1=\beta_2=\beta$, using eqs. (25)–(27), one easily recovers the well-known ground-state solution $\psi_{g.s.}\propto\exp\{-\frac{1}{4}\,u\beta^3\sum_{a,b=1}^{M}|x_a-x_b|\}$ and $E_{g.s.}=-\frac{1}{24}u^2\beta^5(M^3-M)$. Substituting eqs. (19) and (26) into eq. (27) after simple algebra in the limit $M\to 0$ we obtain

$$E_{g.s.}(0,N,\beta_1,\beta_2)=-\frac{u^2}{24}\lambda(\beta_1,\beta_2)N^3+\frac{u^2}{24}\big(\beta_1^5-\beta_2^5\big)N, \tag{28}$$

where

$$\lambda(\beta_1,\beta_2) = 4\big(\beta_1^5-\beta_2^5\big) - 6\big(\beta_1-\beta_2\big)\frac{2\beta_1^4\beta_2+2\beta_2^4\beta_1+\beta_1^5+\beta_2^5}{\beta_1+\beta_2} + 3\big(\beta_1-\beta_2\big)^2\frac{\beta_1^3(2\beta_2+\beta_1)^2-\beta_2^3(2\beta_1+\beta_2)^2}{(\beta_1+\beta_2)^2}. \tag{29}$$

According to eqs. (22) and (24) we find

$$\mathcal{Z}(0,N\to\infty;\beta_1,\beta_2;t\to\infty)\sim \exp\Big\{\frac{u^2}{24}\lambda(\beta_1,\beta_2)\,N^3t-\frac{u^2}{24}\big(\beta_1^5-\beta_2^5\big)Nt\Big\}. \tag{30}$$

The second (linear-on-N term) in the exponential of the above relation provides the contribution to the self-averaging (non-random) linear in time part of the free-energy variance $\mathcal{F}$. Substituting eq. (30) into eq. (9) and redefining

$$\mathcal{F}=\frac{1}{24}u^2\left(\beta_1^5-\beta_2^5\right)t+\tilde{\mathcal{F}} \tag{31}$$

we find that in the limits $t\to\infty$ and $N\to\infty$ the left tail of the probability distribution function for the random quantity $\tilde{\mathcal{F}}$ (as $\tilde{\mathcal{F}}\to-\infty$) is defined by the relation

$$\int\mathrm{d}\tilde{\mathcal{F}}\,P_{\beta_1,\beta_2,t}(\tilde{\mathcal{F}})\exp\{-N\tilde{\mathcal{F}}\}\sim\exp\Big\{\frac{u^2}{24}\,\lambda(\beta_1,\beta_2)N^3t\Big\}. \tag{32}$$

Redefining

$$N=2(u^2\lambda)^{-1/3}s \tag{33}$$

we find that the free-energy difference $\tilde{\mathcal{F}}$ scales as

$$\tilde{\mathcal{F}} = \frac{1}{2}\, u^{2/3} \left(\lambda(\beta_1, \beta_2)\right)^{1/3} t^{1/3}\, f, \tag{34}$$

where the left tail of the *universal* probability distribution function $\mathcal{P}(f)$ of the random quantity f is defined by the relation

$$\int \mathrm{d}f \mathcal{P}(f) \exp\{-s\, f\} \sim \exp\Big\{\frac{1}{3}\, s^3\Big\}. \tag{35}$$

A simple saddle-point estimate of the above integral (for $s \gg 1$ and $|f| \gg 1$) yields

$$\mathcal{P}(f \to -\infty) \sim \exp\Big\{-\frac{2}{3}\, |f|^{2/3}\Big\}. \tag{36}$$

Note that this tail coincides with the corresponding asymptotics of the usual free-energy TW distribution [27].

Let us consider in more detail the scaling relation (34) which demonstrates the dependence of the typical value of the fluctuating part of the reduced free-energy difference, eq. (5), on the strength of disorder u, on the inverse temperatures β_1 and β_2, and on time t. First of all, one notes that the disorder scaling $\sim u^{2/3}$ as well as the time scaling $\sim t^{1/3}$ coincide with the ones of the usual free-energy scaling in $(1+1)$ directed polymers, which, of course, is not surprising. On the other hand, the dependence on the inverse temperatures β_1 and β_2 turns out to be less trivial.

First of all, using the explicit expression (29) one easily finds that in the limit $\beta_1 \gg \beta_2$ (or $T_1 \ll T_2$)

$$\lambda\big(\beta_1, \beta_2\big)\big|_{\beta_1 \gg \beta_2} \simeq \beta_1^5, \tag{37}$$

so that in this limit the scaling relation (34) turns into the usual one-temperature free-energy scaling

$$\tilde{\mathcal{F}} \simeq \beta_1 \tilde{F}_1 = \frac{1}{2}\big(u^2 \beta_1^5\big)^{1/3} t^{1/3} f. \tag{38}$$

In other words, in this case the free energy F_1 of the polymer with the temperature T_1 is much lower than that of the polymer with the temperature $T_2 \gg T_1$, and the free-energy difference $\tilde{\mathcal{F}}$ is dominated by the free energy F_1 as it should be.

Let us consider now what happens if the two temperature parameters β_1 and β_2 are close to each other. Introducing a small (positive) parameter

$$\epsilon = \frac{\beta_1 - \beta_2}{\beta_1} \ll 1 \tag{39}$$

and substituting $\beta_2 = (1-\epsilon)\beta_1$ into eq. (29) in the leading order in $\epsilon \ll 1$ we get

$$\lambda \simeq 2\beta_1^5 \epsilon. \tag{40}$$

Substituting this into eq. (34) we find that in this case the fluctuating part of the the corresponding free-energy difference $\tilde{\mathcal{F}}$, eq. (5), scales as

$$\tilde{\mathcal{F}} \simeq \frac{1}{2}\big(2u^2 \beta_1^5\big)^{1/3} \left(\frac{\beta_1 - \beta_2}{\beta_1}\right)^{1/3} t^{1/3} f, \tag{41}$$

where the random quantity f is described by a universal distribution function $\mathcal{P}(f)$ whose left-tail asymptotics is given in eq. (36). The above eq. (41) constitutes the main result of the present study.

Conclusions. – In this paper we have studied the joint statistical properties of two free energies computed at two different temperatures in *the same sample* (*i.e.*, for a given realization of the disorder) of $(1+1)$ directed polymers. In particular, it is shown that if the two temperatures $T_1 < T_2$ are close to each other, the typical value of the fluctuating part of the reduced free-energies difference $\mathcal{F} = F(T_1)/T_1 - F(T_2)/T_2$ is proportional to $(1 - T_1/T_2)^{1/3}$, eq. (41), which implies "one more 1/3" exponent in these type of systems. On the other hand, the joint distribution function of these two free energies for the time being remains unknown. The left tail of this free-energy difference probability distribution function, eq. (36), coincides with corresponding tails of the TW distributions (both GUE, GOE and GSE) but this tells nothing about its entire exact shape.

Unfortunately, in real experimental studies of the KPZ-type systems, typically for a given realization of the disorder the measurement of the statistical properties of the evolving interface profile (which by mapping to the directed polymers corresponds to the free energy) can be done only once. In other words, each subsequent measurement implies a new realization of the disorder (see, *e.g.*, [48]). For that reason, at the present stage the possibility of a real experimental study of the effects considered in this paper looks rather problematic. On the other hand, the numerical investigation seems to be quite accessible. Compared with the usual protocol of the previous "one-temperature" studies (see, *e.g.*, [49] and references therein), one has just to repeat each measurement twice: keeping the same realization of the disorder but changing the temperature parameter to another value.

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