A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

LETTER

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To cite this article: M. J. Krawczyk *et al* 2017 *EPL* **118** 58005

A Letters Journal Exploring the Frontiers of Physics

June 2017

EPL, **118** (2017) 58005 www.epljournal.org
doi: 10.1209/0295-5075/118/58005

A small chance of paradise —Equivalence of balanced states

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received 11 April 2017; accepted in final form 24 July 2017
published online 18 August 2017

PACS 89.65.-s – Social and economic systems
PACS 89.75.-k – Complex systems
PACS 02.10.Ox – Combinatorics; graph theory

Abstract – A social network is modeled by a complete graph of N nodes, with interpersonal relations represented by links. In the framework of the Heider balance theory, we prove numerically that the probability of each balanced state is the same. This means in particular, that the probability of the paradise state, where all relations are positive, is 2^{1-N}. The proof is performed within two models. In the first, relations are changing continuously in time, and the proof is performed only for $N = 3$ with the methods of nonlinear dynamics. The second model is the Constrained Triad Dynamics, as introduced by Antal, Krapivsky and Redner in 2005. In the latter case, the proof makes use of the symmetries of the network of system states and it is completed for $3 \leq N \leq 7$.

Introduction. – "War is the father of all things" —this quote from Heraclitus reveals that conflicts are ubiquitous in our life, and peace is rather an exception than a rule. Here we are going to consider the probability of accordance *vs.* conflict as a combinatorial problem, much simplified in comparison with its general formulation. The framework chosen here is the Heider balance, a flagship example of an application of networks in social sciences [1,2]. It is often presented in the form of four statements [3]:

– a friend of my friend is my friend;
– an enemy of my friend is my enemy;
– a friend of my enemy is my enemy;
– an enemy of my enemy is my friend.

The process towards a state defined in this way, the so-called balanced state, can be interpreted as a removal of a cognitive dissonance [4], which we feel when, for example, a friend of our friend turns out to be an unsuitable companion. Various kinds of algorithms have been proposed to model such a process. Here we are going to discuss two of them, one continuous and one discrete. In the continuous one, interpersonal relations are represented by real variables, and their time evolution is governed by nonlinear differential equations [5]. In the discrete one, known as "Constrained Triad Dynamics" (CTD), the relations are just positive or negative (± 1), and their time evolution is modelled by a Monte Carlo method [6].

It is worthwhile to note, that —despite obvious differences— these two processes are inherently similar. The process CTD is designed as to minimize energy U, where U can be defined as a number of unbalanced triads. The continuous evolution of the relation $s(i,j)$ between agents i and j in the simplest form can be derived from the classical mechanical rule $\mathrm{d}s(i,j)/\mathrm{d}t = -\mathrm{d}U/\mathrm{d}s(i,j)$. It is not strange, that the jammed states observed in [6] are also jammed for the continuous dynamics. In such a state, energy has a local minimum and the system remains unbalanced. We should add, however, that the equations of motion used below are designed as to keep the relations $s(i,j)$ in a prescribed range $(-1,1)$, and therefore they cannot be derived with the above method.

In [6], the authors could prove that although the number of jammed states is much larger than the number of balanced states, it is the balanced states that are generic. This means that the basins of attraction of the balanced states are much larger than those of the jammed states. An analogous problem has been raised recently for asymmetric relations, where $s(i,j) \neq s(j,i)$. There the number of jammed states has been found to be at least N times larger than the number of balanced states [7]. Here and there N is the network size. The results of numerical simulations indicate that, contrary to the case of symmetric relations, it is the jammed states that are generic. This difference highlights the role of bassins of attraction of particular stationary states [8].

Here we consider the balanced states for the symmetric relations, where $s(i,j) = s(j,i)$. As proved by [9], for each balanced state the network is divided into two mutually hostile groups; this means, that all relations between members of different groups are negative. On the contrary, all in-group relations are positive (friendly). One of balanced states, the so-called paradise, has special status: all relations are positive there. Our aim here is to check what is the probability of this specific state: is it different from the probability of other balanced states? Again, for a deterministic rule of evolution the state probability is equivalent to the size of its basin of attraction.

In two subsequent sections we provide the details of calculations for the continuous model and the discrete model. In the former case, the analysis is restricted to the simplest case $N = 3$. In the latter, we work on the network of 2^M states, where $M = N(N-1)/2$ is the number of links in the network of relations $s(i,j)$. The calculations are carried out for $N \leq 7$. The last section is devoted to discussion.

The continuous model, N = 3. – The time evolution of the relations is given by [5]

$$\frac{\mathrm{d}s(i,j)}{\mathrm{d}t} = [1 - s^2(i,j)] \sum_{k=1}^{N} s(i,k)s(k,j). \qquad (1)$$

The diagonal components $s(i,i)$ are set to zero. The role of the prefactor $H(s) = 1 - s^2$ is to keep the relations within a prescribed range $(-1,1)$; otherwise they tend to infinity. The price we pay for this limitation is to resign from the analytical solution [10]. For three agents with symmetric relations $s(j,i) = s(i,j)$, the model can be conveniently written with new variables x, y, z. The equations of motion are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= (1 - x^2)yz, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (1 - y^2)zx, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= (1 - z^2)xy. \end{aligned} \qquad (2)$$

It is easy to see that the only stable fixed points of this system are the following four: $(x*, y*, z*) = (-1,-1,1), (-1,1,-1), (1,-1,-1), (1,1,1)$. In these points, the Jacobian is diagonal, and all three eigenvalues are equal to $-2xyz$. Hence all these points are balanced in the sense of Heider, and there is no other balanced points. The last fixed point $(1,1,1)$ is the paradise state (see fig. 1). Below we will refer to these balanced points in the same order as given above.

Now, our problem can be formulated as follows. Suppose that the initial values of x, y, z are taken from a uniform probability in the ranges $(-1,1)$. What is the probability that the trajectory ends at each of the balanced points? It is obvious that the probabilities of the first three balanced points (except the paradise state) are equal by symmetry. Hence we can denote the probabilities as $(p, p, p, 1-3p)$.

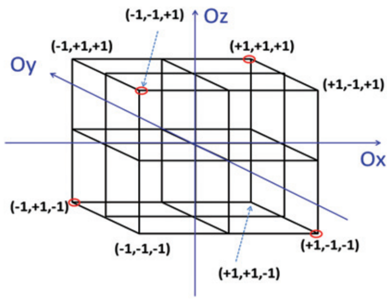

Fig. 1: (Color online) The cube of states for $N = 3$. The balanced states are marked in red. The paradise state is $(+1,+1,+1)$.

To tackle with the problem, let us start with the case when the initial point $x(0), y(0), z(0)$ is placed in the cube $(0,1), (0,1), (0,1)$; in other words, all its coordinates are positive. In this case, $\mathrm{d}x/\mathrm{d}t$ is positive and so are other two, $\mathrm{d}y/\mathrm{d}t$ and $\mathrm{d}z/\mathrm{d}t$. It is clear that the trajectory ends at (1, 1, 1). The contributions from this cube to the four balanced endpoints are then 0, 0, 0, 1/8, because the volume of the considered subspace of the initial points is 1/8 of the total one.

The same scheme of argumentation leads us to the conclusion that all trajectories started in the volume $(0,1), (-1,0), (-1,0)$ end at the point $(1,-1,-1)$. This results from eqs. (2): $\mathrm{sign}(\mathrm{d}x/\mathrm{d}t) = \mathrm{sign}(yz) > 0$, $\mathrm{sign}(\mathrm{d}y/\mathrm{d}t) = \mathrm{sign}(zx) < 0$, and $\mathrm{sign}(\mathrm{d}z/\mathrm{d}t) = \mathrm{sign}(xy) < 0$. The related contributions to the probabilities of the four above endpoints are $0, 0, 1/8, 0$. Similarly, when an initial point is within the volume $(-1,0), (0,1), (-1,0)$, the endpoint must be at $(-1,1,-1)$, and for an initial point at $(-1,0), (-1,0), (0,1)$ the trajectory ends at $(-1,-1,1)$. Summarizing, trajectories which start within the subvolumes which contain the balanced points must end at those balanced points. So far, the distribution of trajectories which end at the endpoints is $(1/8, 1/8, 1/8, 1/8)$.

When the trajectory starts at the cube $(-1,0), (-1,0), (-1,0)$, the trajectory must leave this cube, as there is no balanced points there. Let us take the plane $x = 0$; there $\mathrm{d}x/\mathrm{d}t > 0$, $\mathrm{d}y/\mathrm{d}t = \mathrm{d}z/\mathrm{d}t = 0$, *i.e.*, the trajectory crosses the plane, entering the neighboring volume $(0,1), (-1,0), (-1,0)$. As shown above, such a trajectory ends at $(1,-1,-1)$. By symmetry, the cube of all negative coordinates of the initial points gives equal contributions $(1/3, 1/3, 1/3, 0)/8$ to the probabilities of particular balanced points. The last zero is due to the fact that the cube has no common walls with the cube $(0,1), (0,1), (0,1)$.

At last, we have three remaining cubes which do not contain any balanced points. Consider the one $(0,1),(0,1),(-1,0)$. This cube has common walls with three other cubes with the balanced points. These are: $(-1,0),(0,1),(-1,0)$, $(0,1),(-1,0),(-1,0)$ and $(0,1),(0,1),(0,1)$. By symmetry, the contributions to the first two cubes are equal. Also, they are different from zero: for example, at the plane $x=0$ the trajectory crosses the plane, as $\mathrm{d}x/\mathrm{d}t<0$. Hence the contributions from the cube $(0,1),(0,1),(-1,0)$ can be written as $(0,a,a,b=1-2a)/8$.

To calculate b, we have to refer to the concept of invariant sets [11]. If a trajectory starts at an invariant set, it must remain there. Consequently, if a plane is invariant, a trajectory cannot go through it. Here we have at least six invariant planes: $x=\pm y$, $y=\pm z$, $z=\pm x$. This is seen directly from eqs. (2): once $x=y$, then $\mathrm{d}x/\mathrm{d}t=\mathrm{d}y/\mathrm{d}t$, etc.

Coming back to our exemplary cube $(0,1),(0,1),(-1,0)$, we have three invariant planes cutting it, but only two $z=-x$ and $z=-y$ are relevant here. For initial points where $z>-x$, the endpoint cannot be $(-1,1,-1)$. For initial points where $z>-y$, the endpoint cannot be $(1,-1,-1)$. The argument is that no trajectory can go through the invariant plane, which is between the initial and the final point. Hence, if both these conditions are fulfilled, the trajectory ends at $(1,1,1)$. Note that for $z<-x$ or $z<-y$, the endpoint $(1,1,1)$ cannot be reached. Hence

$$b=\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}y\int_{max(-x,-y)}^0 \mathrm{d}z=\frac{1}{3}, \qquad (3)$$

hence $a=1/3$ as well.

By symmetry, the cube $(0,1),(-1,0),(0,1)$ gives the contributions $(1/3,0,1/3,1/3)/8$, and the cube $(-1,0),(0,1),(0,1)$ gives $(1/3,1/3,0,1/3)/8$. Summarizing, all balanced points are reached with the same probability.

In a supplementary animation (see ref. [12]), we can see a bunch of trajectories started from randomly selected initial points in the cube $(0,1),(0,1),(-1,0)$, the same as discussed above. Each trajectory reaches either $(1,-1,1-)$, $(-1,1,-1)$, or $(1,1,1)$. There is no path to $(-1,-1,1)$. The animation shows three invariant planes: $(x+z=0)$, $(y+z=0)$, and $(x-y=0)$ which cross this cube. We can see that the trajectories never go through the planes.

The discrete model (CTD). – As described in [6], the algorithm used in CTD is as follows. A link $s(i,j)$ is selected randomly, and an attempt is made to change its sign. If the energy U, defined as

$$U=-\sum_{ijk} s(i,j)s(j,k)s(k,i) \qquad (4)$$

decreases, the change is carried out. If U increases, the change is withdrawn. If U remains constant, the change is done with probability 0.5. Therefore, the network of states is weighted and directed, on the contrary to the network of relations $s(i,j)$. Our task here is to compare the probabilities of different balanced states in a fully connected graph of N nodes. However, if the network is not fully connected, this probability distribution is not unique, but it depends on an initial state. Therefore we allow for a finite probability of processes where energy increases.

According to [13], we construct a network of graph states: two states are connected if one state can be attained from the other by a change of sign of one link. There is $M=N(N-1)/2$ links in the initial graph, and the network of states contains 2^M nodes. Some of them are balanced, and their energy U_b is given by the binomial coefficient

$$U_b=-\binom{N}{3}. \qquad (5)$$

Recall that the algorithm by [13] in the case of directed and weighted network, which is such as that we are analyzing here, consists of the following steps. At first we identify nodes of the state network which have the same in- and out-degrees. Next, they must be distinguished as each node is characterized by the energy of the state it represents. This allows us to identify classes of nodes which reflect the structure of connection between nodes. Two nodes are classified to the same class if their neighbors are in the same classes. Now note that the dynamics of the initial graph is equivalent to a random walk of the system in the network of states, with weighted and directed links. The states which belong to the same class have the same probability.

The above procedure is applied to the graph of relations $s(i,j)$. For numerical reasons, the calculations have been performed to graphs of $N=7$ nodes at most; then we have 21 degrees of freedom $s(i,j)$ and 2^{21} nodes in the network of states. The result is that in each case, the paradise state has been found to belong to the class of balanced states; there is only one such class in the whole network of states, and all balanced states belong to this class. Clearly, the energy of each of these balanced states is the same as given in eq. (4). We note that all states in the same class have the same energy.

Summarizing, the probability of each balanced state is equal.

In fig. 2 and subsequent figures 3, 4, 5, 6 we show the obtained networks of classes, for $N=3$, 4, 5, 6 and 7, respectively. For $N=3$ and 4, the networks of states are also presented. Let us consider the simplest case where $N=3$ (fig. 2). On top, we see the network of states. There, positive links are marked with red, and the negative ones with blue color. There are eight possible states, four of them are balanced. In the middle network of fig. 2, the balanced states are marked with stars, and the unbalanced ones with triangles. As we see, each star has three triangles as neighbors, and each triangle has three neighboring stars. This means, that there are two kinds

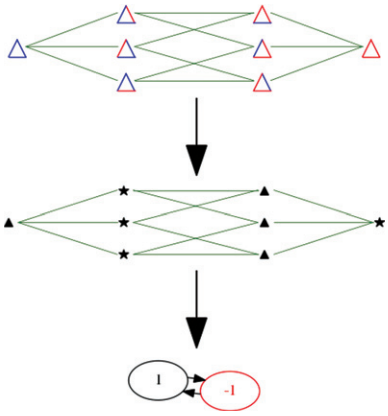

Fig. 2: (Color online) For $N = 3$, we show the network of states (top), where positive links are marked with red, and negative links with blue color. If colors are exchanged (red $\rightarrow$ blue, blue $\rightarrow$ red), the picture is reflected in a vertical mirror. In the middle picture, balanced states are marked by stars. At the bottom, we see the network of classes. In the latter, nodes are marked as circles, with the energy written inside. The class which includes the balanced states is marked with red. For $N = 3$, there are only two classes.

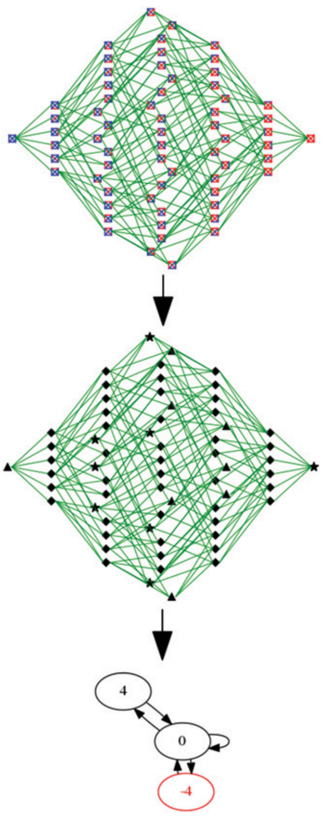

Fig. 3: (Color online) The network of states for $N = 4$. Here a state is marked by a small graph, with positive links red and negative links blue. If colors are exchanged (red $\rightarrow$ blue, blue $\rightarrow$ red), the picture is reflected in a vertical mirror.

of nodes, *i.e.*, two classes. At the bottom of the figure, we show the resulting network of classes. In fig. 3, the same scheme is shown for $N = 4$. For higher N (see figs. 4, 5, 6), only the network of classes are presented.

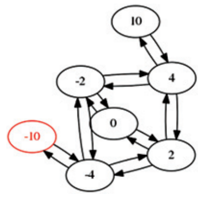

Fig. 4: (Color online) The network of classes for $N = 5$. Note that in this and the other networks of classes, there is only one neighbor class of the balanced class, and its energy is enhanced by $2(N - 2)$ (see the text).

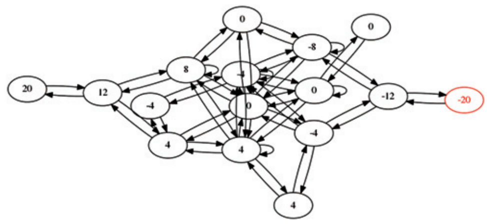

Fig. 5: (Color online) The network of classes for $N = 6$.

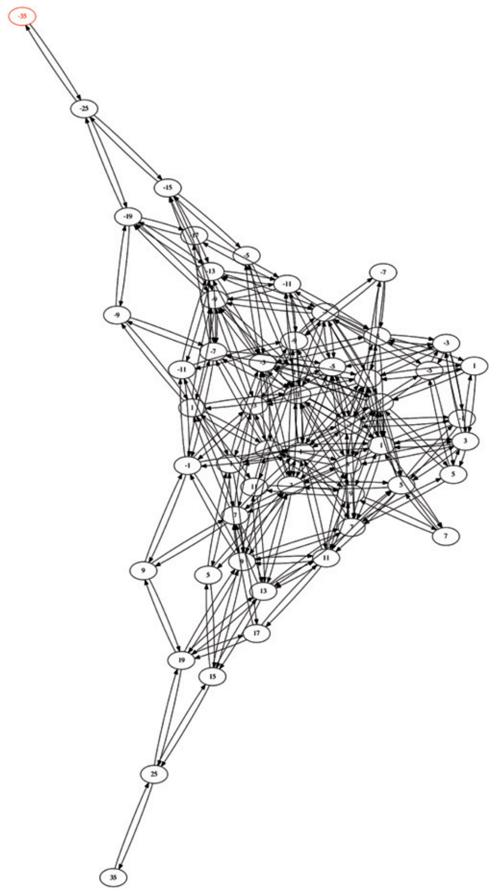

Fig. 6: (Color online) The network of classes for $N = 7$. The original network of states contains $2^{21} = 2097152$ nodes.

Discussion. – It is tempting to postulate that our results could remain valid for larger networks. In the case of the discrete algorithm, a qualitative change of the system behavior could appear for $N = 9$, where jammed states are possible for symmetric relations. This can raise doubts, where the jammed states are placed in the network of classes. We can check it at least for the jammed states identified in [6]. The answer is that even if the energy of states is neglected as a criterion, the local structure of the network of states around a jammed state is different from the one for a balanced state. This is because the latter state has a specific feature: whichever link is selected to be modified, the energy of the system is always increased by the same amount: $\Delta U = 2(N-2)$. In other words, the only unbalanced triads are those which contain the modified link, and their number is equal to $N-2$, *i.e.*, the number of "third" nodes which enter the triad with the modified link. This is never true for the jammed states, as there are more unbalanced triads there. For $N = 9$, we have checked numerically that a modification of different links can lead the system from a jammed state to states of different energies. Yet, direct application of the method for $N = 9$ would demand calculations for a network of 2^{36} states, which —at the moment— remains unfeasible.

On the other hand, when we consider undirected and unweighted networks, the difficulty vanishes. Then, the stationary distribution depends only on the node degree [14,15]. As for each state the number of neighboring states is the same $M = N(N-1)/2$, hence each state has the same probability for any N.

For the continuous model, the case $N = 4$ demands an identification of invariant subspaces in a six-dimensional space of relations. However, a 5-dimensional subspace is not invariant. For example, if we set the condition $s(1,2) = s(1,3)$, then $\mathrm{d}s(1,2)/\mathrm{d}t - \mathrm{d}s(1,3)/\mathrm{d}t = [1 - s^2(1,2)][s(1,4)(s(3,2) - s(4,3))]$ is, in general, different from zero. On the other hand, a subspace of dimension lower than five cannot separate a six-dimensional space into mutually inaccessible parts. Then, either invariant subspaces are more complex for $N > 3$, or they do not exist at all. Neither of those options does undermine the hypothesis that the balanced states are equiprobable; however, the method of the proof must be also more complex than the one applied above.

Concluding, our results indicate that all balanced states are equally probable, and the fraction of initial states which are driven to the paradise state is just 2^{1-N}. In other words, the state of conflict is generic. The proof is limited to small systems; for larger N and directed and weighted networks of states, the result remains hypothetical.

The problem of Heider balance has been applied several times to analyze international conflicts (see [16–20] and references therein). Recently, a proof has been provided that it is possible to control the outcome of the Heider dynamics by the modification of the links of one agent [19,21]. In the context of international relations, potential profits and threads of such a manipulation must be remarkable. Our results indicate that the state of all positive relations should not be expected to appear spontaneously.

* * *

The work was partially supported by the PL-Grid Infrastructure.

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