LETTER

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Rich or poor: Who should pay higher tax rates?

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Abstract – A dynamic agent model is introduced with an annual random wealth multiplicative process followed by taxes paid according to a linear wealth-dependent tax rate. If poor agents pay higher tax rates than rich agents, eventually all wealth becomes concentrated in the hands of a single agent. By contrast, if poor agents are subject to lower tax rates, the economic collective process continues forever.

Introduction. – Inequality within human societies is nowadays, more than ever, an important issue [1]. Historical data were collected and analysed in a recent, now famous study [2], concerning individuals, regions, countries, etc. Within this general frame, progressive taxation (rich people paying higher tax rates) is proposed in order to mitigate the observed tendency towards wealth concentration. On the other hand, some argue the contrary, rich people paying less taxes, because somehow their wealth benefits the whole society (the trickle-down argument). Here, we adopt the strategy of summing up the whole wealth of a population formed by a set of agents $n = 1, 2 \ldots N$, and studying the time evolution of the share w_n each individual owns of this total wealth. A simple model is studied via numerical simulation as well as via an analytic, mean-field approach. The model is based on two general ingredients, annual profits and taxation. We focus attention on the final steady-state distribution of wealth. If the share of some particular agent approaches the maximum possible value $w_{\max} = 1$ (this possibility is called *collapse*), the wealth of the entire population lies in hands of a single agent. Economic evolution stops, as this is an absorbing state of the dynamics. This kind of dynamic transition towards possible absorbing states is a recent research field [3], where models as the present one are used in order to study such phenomena as extinction, epidemics, opinion dynamics, vaccination, prevention of fires, etc. In the present case, the general strategy of following the wealth shares (instead of wealth values themselves) distinguishes this study from similar models. This subject was reviewed in [4], describing the different strategies adopted in these models in order to represent the economic dynamics of a society. The simplest one is the pairwise transaction, where two randomly chosen agents exchange money according to some conservative rule. The inclusion of some external entity (a bank or some reservoir of money) allows the inclusion of debts, the total money of the population is no longer conserved. The concept of wealth instead of money considers other individual property, besides money. The total population wealth is not conserved, since the annual production of each agent can increase (or decrease) its wealth. This ingredient is modeled by multiplying the current wealth of each agent by a random factor in the pioneering work of Bouchaud and Mézard [5]. We provide in ref. [6] an incomplete and somewhat arbitrary list of works related to this issue; the interested reader may also wish to consult the works cited in these references. All of them follow the dynamic evolution of the wealth distribution among agents, not the distribution of wealth shares here introduced. Concerning taxes, they were treated in [5] as uniformly applied to each agent independent of its current wealth, the possibility of regressive or progressive tax rates is also introduced here.

Stochastic, computer-simulated version. – Consider a population of N independent agents, each one

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owning a positive wealth W_n(see footnote [1]). The total population wealth is $S = \sum_n W_n$. The share of this total wealth owned by agent n is $w_n = W_n/S$. Consider also the monetary unit taken as the total wealth $S = 1$. Using this unit, wealths W_n or wealth shares w_n are the same quantities.

The dynamic evolution consists of four steps during each "year" (time step). Step I is a multiplicative process with randomness. Each wealth w_n is multiplied by a random, positive factor f_n chosen from a fixed probability distribution. The result is a new set of $W'_n = f_n w_n$ and the corresponding shares $w'_n = W'_n/S'$, where $S' = \sum_n W'_n$. Second, step II, payment of taxes at the end-of-year, according to a tax rate that is a linear function of wealth, $W''_n = (1 - A - p\, w'_n)\, W'_n$, where A and p are fixed parameters obeying the restrictions $0 < A < 1$ and $0 < A + p < 1$. Step III is a partial redistribution procedure: a fraction R of the total collected taxes is uniformly redistributed among all N agents, $W'''_n = W''_n + R \sum_n (A + p\, w'_n)\, W'_n/N$. Numerically, it provides the advantage of restricting from below the dynamic variables w_n, so that they are always strictly positive. This restriction is mandatory, otherwise agents reaching $w_n = 0$ would be removed from the game permanently. The interesting limiting case, however, is $R \to 0$, as we shall see. Finally, step IV is the renormalisation of wealths by a common factor, $w'''_n = W'''_n/S'''$, where $S''' = \sum_n W'''_n$. This step can be interpreted as a simple redefinition of the monetary unit, always kept equal to the total wealth at the beginning of the next year. After this four-step procedure, the new year starts from the set of w'''_n, and so on. Besides the focus on wealth shares instead of wealths, another difference of the current model compared with [5] is the presence of the non-linear (quadratic in W_n) term in the tax paid (for $p \neq 0$).

Anticipating the result, in the quoted limit of no-redistribution, $R \to 0$, there is an absorbing state transition between two possible final steady states, depending on the adopted W-dependent tax rate rule. Collapsed phase occurs when poor agents pay higher tax rates than rich agents, $p < 0$, resulting in one particular agent owning all the wealth at the end, $w_1 = w_{\max} \to 1$ for the richest agent, $w_n \to 0$ for all others. The economy ceases to evolve. Otherwise, an active phase occurs when rich agents pay higher tax rates than poor agents, $p > 0$, the whole economy evolves forever with the wealth distributed among agents. Economic evolution survives. Being a simple global rescaling, parameter A plays no essential role. It is used only to assure all tax rates are positive. Also, the same model with taxes applied only to the annual gains (instead of accumulated wealths) exhibits the same transition.

The same scenario, transition to a collapsed state for $p < 0$, is observed with numerous alternative probability distributions for the multiplicative factors f_n adopted in step I. A simple possibility is to double the current wealth W_n ($f_n = 2$) with probability 50%, and otherwise leave it unaltered ($f_n = 1$). Figure 1 shows the result for this simple choice, in a population of $N = 1000$ agents. Many other rules for selecting the multiplicative factors f_n were tested, all of them giving the same general result, a dynamically induced transition at the critical point $p = p_c = 0$. Notice the transition is indeed critical, as exhibited by the asymptotic power law, central curve in fig. 1.

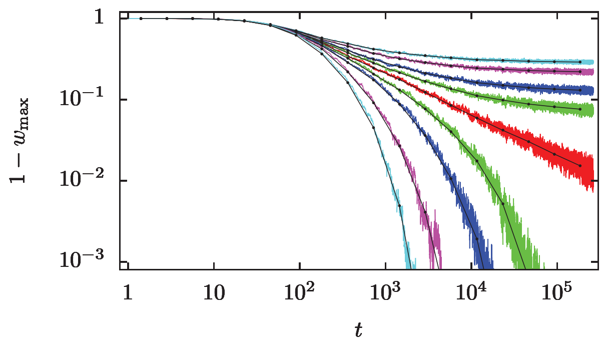

Fig. 1: (Colour online) Sum of all wealth shares but the richest agent as a function of time. The central red curve corresponds to $p = 0$, the critical situation. Black points show the averages in powers-of-2 increasing intervals (bins) of the same data, exhibiting a clear long-term linear behaviour in this log × log plot, thus a power law time dependence. Out from the central curve, green curves correspond to $p = \pm 0.005$, blue curves to $p = \pm 0.01$, purple to $p = \pm 0.02$ and light blue to $p = \pm 0.03$.

Being a multiplicative process, the natural choice for order parameter to describe the transition is $-\log w_{\max}$, which vanishes in the collapsed phase, $p < 0$. We compute this order parameter through a time averaging procedure followed by a sample average. Each sample corresponds to an initially random distribution of wealths, starting from which the above-defined dynamic rule is processed during T years. For each n, the time average of $\log w_n$ is then taken during the last $T/2$ years, where T is large enough to assure convergence (we adopted $T = 32$M, large enough within our numerical precision). The same process is repeated by generating S different initial conditions (we adopted $S = 8$). All time averages are then sample averaged, resulting in a final list $\langle\langle -\log w_n \rangle\rangle$ with N entries and the corresponding error bars. Figure 2 shows one such distribution[2].

The order parameter is

$$m = \langle\langle -\log w_1 \rangle\rangle, \tag{1}$$

[1] The set of wealths is always considered in decreasing order, the so-called Zipf distribution, so index n corresponds to the rank of each agent, not to the specific agent itself.

[2] All algebraic operations were performed within a 113-bit mantissa, by multiplying each fraction w_n by 2^{60} and storing it in an integer variable with 60 bits *plus* an ordinary double precision variable with a 53-bit mantissa. The resulting numerical accuracy on the sum of wealths one needs in order to perform the annual renormalisation of the monetary unit is bounded above 10^{-34} (2^{-113}). Were we to use simply a double precision variable, the accuracy would drop to 10^{-15} (2^{-53}). That is why we restrict the plot to wealth shares larger than 10^{-34}.

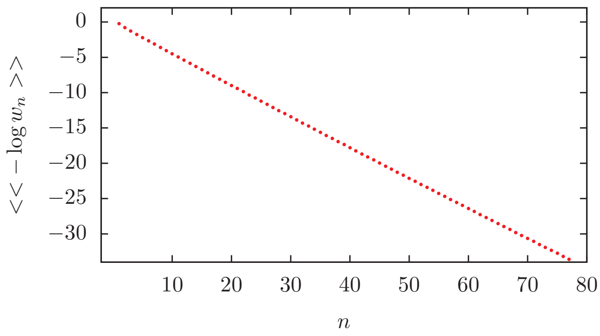

Fig. 2: (Colour online) Steady-state distribution of wealth shares as a function of the rank $n = 1, 2 \ldots N$ (logarithm base 10). Error bars are smaller than the symbols. Here, $N = 1000$ and $p = 0.045$.

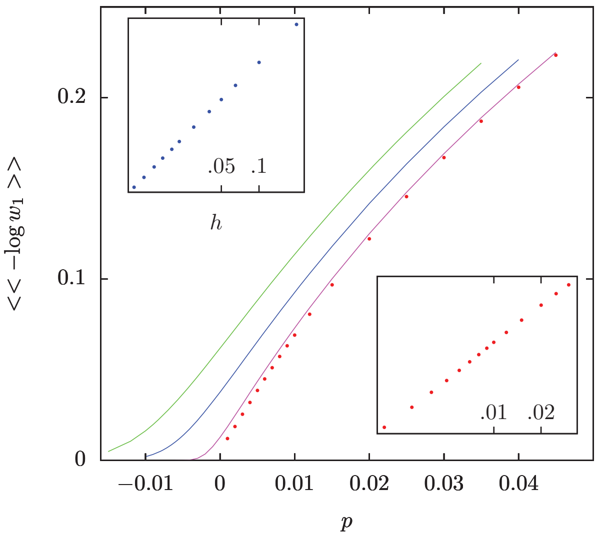

Fig. 3: (Colour online) Order parameter (w_1 refers to the richest agent) as a function of p. Symbols at the main plot are obtained without redistribution, *i.e.*, external field $h \to 0$, exhibiting the characteristic singularity at the critical point $p = 0$. Again, error bars are smaller than the symbols. Smooth, continuous lines correspond to finite external fields $h = 0.05$, $h = 0.03$ and $h = 0.01$ from top down. Insets show the same quantity in $\log \times \log$ plots as function of $p > 0$ (without redistribution, bottom right) or h (at the critical point $p = 0$, upper left): the straight lines confirm the critical character of the transition. Here $N = 10000$, the same points are reproduced for $N = 1000$, within the error bars.

the first entry of the quoted list. It is shown as a function of p in fig. 3 (symbols at the main plot), within the active phase $p > 0$ (for $p < 0$, the result is always $m = 0$ as it should be). Near the transition point, the order parameter m follows a power law $m \propto p^{\beta}$, with $\beta \approx 0.83$ estimated by fitting fig. 3 (bottom right inset) for $p \leq 0.01$. Compared with the corresponding critical exponents β normally observed in equilibrium phase transitions, this value is unusually high. However, equilibrium thermodynamic concepts (like inequalities among critical exponents) cannot be directly applied here. The traditional notation β is used only by analogy.

For completeness, it is interesting to provide some kind of external field acting on all agents on the same footing. The field is expected to smooth the singularities at the transition point in the same way an external magnetic field acting on a ferromagnetic system does. Redistribution plays the role of such an external field h, defined as follows:

$$\frac{1}{h} = -\log R. \tag{2}$$

Full redistribution corresponds to $h \to \infty$, whereas the no-redistribution limit where the transition occurs corresponds[3] to $h \to 0$. Figure 3 shows smooth curves obtained for some values of h. Moreover, at the transition point $p = 0$, the order parameter m follows another power law $m \propto h^{1/\delta}$, fig. 3 (upper left inset). From it, one can estimate $\delta \approx 1.03$. Indeed, $\delta = 1$ is expected by the model definition itself, since there are no interactions between agents. Only the annual redefinition of monetary unit correlates agents to each other. This is different from interacting systems usually treated in equilibrium phase transition studies, where the various elements behave collectively through some prescribed interaction, thus displaying a non-linear response m to the external stimulus h: δ would be larger than unity in such a case.

Deterministic, mean-field approximation. – Let us restrict the treatment to the no-redistribution case where the transition occurs, skipping step III. At time t the fraction of agents whose wealth shares fall between w and $w + \mathrm{d}w$ is $\mathrm{d}w\, g_t(w)$, where

$$\int_0^1 \mathrm{d}w\, g_t(w) = 1 \tag{3}$$

comes from agent counting normalisation, relative to the (large) fixed number N of agents. Another normalisation,

$$S = N < w >_t = N \int_0^1 \mathrm{d}w\, w\, g_t(w) = 1 \tag{4}$$

corresponds to the total wealth considered as monetary unit. Symbol $\langle \ldots \rangle_t$ indicates the average over the current configuration, all agents, at time t. Instead of $g_t(w)$, let us define the ranking function

$$r_t(w) = N \int_w^1 \mathrm{d}w'\, g_t(w'), \tag{5}$$

which is the already quoted Zipf distribution of wealth shares (such as fig. 2, with interchanged horizontal and vertical axes).

[3] In principle, in order to avoid null wealths and keeping the meaning of the system size N, one should set a very small $h > 0$, similar to equilibrium phase transitions where some residual external field is necessary in order to break the symmetry in finite systems. When some particular wealth share w_n becomes smaller than 10^{-318} during the process, it is replaced by a copy of the previous one, w_{n-1}. Thus, the no-redistribution case does not correspond exactly to $R = 0$, but is equivalent to set $R \approx N \times 10^{-318}$.

Consider $P(f)$ the arbitrary but fixed probability distribution for the multiplicative factors f_n. After steps I, II and IV, the new ranking function is

$$r_{t+1}(w) = \int \mathrm{d}f\, P(f)\, r_t(x/f), \tag{6}$$

where

$$x = \frac{(1-A)\langle f\rangle}{2p}\left(1-\sqrt{1-\frac{4pF_t\, w}{1-A}}\right), \tag{6a}$$

$\langle\dots\rangle$ means the average under the fixed probability distribution $P(f)$ and

$$F_t = 1 - p\,\frac{\langle f^2\rangle}{(1-A)\langle f\rangle^2}\,\frac{\langle w^2\rangle_t}{\langle w\rangle_t}. \tag{6b}$$

The averages $\langle w\rangle_t$ and $\langle w^2\rangle_t$ can be determined directly from the ranking function, instead of $g_t(w)$.

The reasoning leading to equations (6) is based on solving backwards relation $W'' = (1-A-pW'/S')W'$, within the mean-field assumption $S' = \langle f\rangle$, *i.e.*, after step I the new total wealth S' is replaced by its average $\langle f\rangle$ over all possible choices of multiplicative factors. Then, we equate the fraction $\mathrm{d}W''\,\overline{g}(W'')$ of agents whose wealths are between W'' and $W''+\mathrm{d}W''$ after taxes to the corresponding fraction $\mathrm{d}W'\,\overline{g}(W')$ of agents whose wealths were between W' and $W'+\mathrm{d}W'$ before taxes.

At this point, after applying eq. (6), the up-to-now continuous ranking function should be discretised along the r-axis in N channels, before starting the next year. This procedure is necessary in order to preserve the system size N, as already discussed. The plot $r \times w$ is divided in N horizontal strips with equal heights, each strip then replaced by a rectangle with the same area. Dynamic evolution, eq. (6) plus discretisation, is then repeated until convergence.

In practice, we divide the w-axis in M channels ($M \gg N$), assigning some tentative guess for the final steady-state ranking function $r(w)$ (any monotonically decreasing form between $r(0) = N$ and $r(1) = 0$). Taking a particular value w, eq. (6) is applied and the resulting r replaces the original $r(w)$. The process is repeated for all channels, again and again, until convergence (the so-called relaxation method). The result is as expected: $w_{\max} = 1$ for $p < 0$.

Returning to the simple case of a binary distribution of multiplicative factors ($f_n = 1$ or 2 with probabilities 50%), the very same mean-field approach can be reformulated as follows.

a) Starting from the current wealth shares w_n at time t, $n = 1, 2\dots, N$, one first performs a copy of them, doubling all values of the copy.

b) Now, one has $2N$ wealths W_n instead of the original N, summing to 3 instead of unity. Then, one applies taxes to these wealths.

c) After that, the resulting $2N$ wealths are listed in decreasing order.

d) Finally, one restores the original population N: the average between the first and the second largest wealths is assigned to the first agent, the average between the third and fourth wealths is assigned to the second agent, so on.

A comment about this new procedure follows. The averaging step (d) highlights the mean-field character of the current deterministic approach, compared with the stochastic formulation where each wealth can be doubled or not. In the stochastic version, at the critical point, $1 - w_{\max}$ slowly vanishes as time goes by as shown in fig. 1, central plot, whereas it converges to $1/2$ in the mean-field approximation generating a first-order transition gap[4]. The transition point $p = 0$, however, remains the same.

Another comment. This simple alternative formulation bypasses the ranking function; its role is automatically performed by the ordering step (c) applied to the set of $2N$ wealths.

Still another comment. One needs only two copies of the wealth shares, step (a), in the particular case when one adopts only two possible wealth multiplicative factors, here 1 or 2. In general, one needs one copy for each possible wealth multiplicative factor f_n, taking into account their probability distribution $P(f)$. In this case, the original formulation by relaxing the ranking function may be more economical for computer calculations. Anyway, the two approaches are completely equivalent.

Conclusions. – The wealth distribution of a population is submitted to a multiplicative process followed by regressive or progressive taxation, where tax rates are higher for poor than rich agents or vice-versa, respectively. In the long term, regressive taxation leads to social collapse, all wealth falls in the hands of a single agent, whereas it remains forever distributed among agents if progressive taxation is adopted instead. This transition is different from that studied in previous works, for instance [5], in which the so-called wealth condensation is the main issue, the concentration of the entire wealth in the hands of a *finite* number of agents within an otherwise infinite population. In some sense, without redistribution, our model always eventually leads to such condensation, even in the active phase. In this case, the wealths listed in decreasing order follow an almost exponential decay, fig. 2, configuring an *effective* finite number of agents participating in the process (inversely proportional to the slope of plots like fig. 2). One way to avoid such condensation is the uniform redistribution of the collected taxes, as in [5], at the end of each year. Doing so, uniformly redistributing a fraction R of the collected taxes, the transition singularities disappear in the same way external fields do in the

[4]Indeed, for $p = 0$ the whole list of wealth shares converges to $w_n = 0.5^n$.

traditional theory of equilibrium phase transitions (curves become smooth, as shown in fig. 3).

As a final comment, we notice that progressive taxation ($p > 0$) alone is not able to solve the inequality problem, since only a finite number of agents effectively participate of the wealth exchange game. Even for progressive taxation, wealth concentration still holds, not in the hands of a single agent but in the hands of a finite number. In order to avoid wealth concentration, economic policy makers should not only adopt progressive taxation, but also implement some kind of redistribution. Just after World War II, policy makers perceived this fact, creating a set of social programs now known as the “welfare state”.

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