## LETTER

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# Quantum correlations in multipartite quantum systems 

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#### Abstract

Quantum entanglement is the most famous type of quantum correlation between elements of a quantum system that has a basic role in quantum communication protocols like quantum cryptography, teleportation and Bell inequality detection. However, it has already been shown that various applications in quantum information theory do not require entanglement. Quantum discord as a new kind of quantum correlations beyond entanglement, is the most popular candidate for general quantum correlations. In this paper, first we find the entanglement witness in a particular multipartite quantum system which consists of a $N$-partite system in $2^{n}$-dimensional space. Then we give an exact analytical formula for the quantum discord of this system. At the end of the paper, we investigate the additivity relation of the quantum correlation and show that this relation is satisfied for a $N$-partite system with $2^{n}$-dimensional space.


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Introduction. - Quantum entanglement is one of the most important features of quantum mechanics $[1,2]$. Many quantum information applications require the sharing of entangled particles. The answer to the question as to whether particles are entangled or not, is an important subject for the researchers. The Peres-Horodecki criterion is a necessary and sufficient condition, for the joint density matrix $\rho$ of two quantum mechanical systems $A$ and $B$, to be separable $[3,4]$. It is also called the PPT criterion, for positive partial transpose, whereas for higher dimensions there exist in general only necessary conditions for separability. The PPT condition is also a necessary and sufficient condition for Hilbert spaces $H_{2} \otimes H_{3}$.

For characterizing entanglement, entanglement witness (EW) $[4,5]$ is important for detecting the presence of entanglement. The EWs are nonpositive Hermitian operators which allow us to distinguish a specific entangled state from the separable ones. In other words, for a given entangled state $\rho$, the entanglement witness is an observable $W$ whose expectation value is non-negative on

[^0]any separable state and strictly negative on the quantum state $\rho$. Convex optimization is a useful tool for quantum optimization problems such as a test for distinguishing an entangled from a separable quantum state [6-11]. In addition to distinguishing the entangled states from the separable ones, another matter is to determine the amount of the correlation between the components of the quantum system. To this end, entanglement is the key resource of quantum information processing. It plays an important role in many quantum processing such as quantum teleportation, quantum key distribution, and quantum algorithm [12]. One can define several types of entanglement measures, for instance, entanglement of formation [13] and the concurrence [14,15]. However, it is shown that quantum entanglement is not the only kind of quantum correlations. Quantum systems without any entanglement can still have another kind of quantum correlation called quantum discord. Quantum discord was first introduced in $[16,17]$. Zhang and Chen in [18] briefly reviewed the concepts and properties of quantum discord. In general, there are several measures of discord including the original measure of discord, Gaussian discord, relative entropybased discord, geometric discord, global discord. In [19] the dynamics of quantum discord under noisy channels,
namely, the Pauli channels $\sigma_{x}, \sigma_{y}, \sigma_{z}$ and the depolarizing channel is studied. In this paper, we choose the quantum discord based on relative entropy as the quantum correlation measure that is distance-based discord and defined as the minimal distance of a quantum state and all states with zero discord $[20,21]$.
In [22] the entanglement witness and quantum discord based on relative entropy are obtained for three-qubit and tripartite systems in the four-dimensional space. In this paper, inspired by the work [22] we find the entanglement witness of a $N$-partite system with $2^{n}$ dimension space. Then using the concept of relative entropy as a distance measure of correlations we give an exact explicit formula for quantum discord of these states. Finally, we investigate the additivity relation of the quantum correlation for the $N$-partite system with $2^{n}$-dimensional space. The organization of this paper is as follows:
In the next section we review the definition of quantum discord based on relative entropy. In the following two section we obtain the entanglement witness and quantum discord based on relative entropy for a $N$-partite system in $2^{n}$-dimension space. Then we investigate the additivity relation of quantum correlations in this system. We conclude in the last section.

Distance measure of correlations. - Briefly, the nonclassical correlations based on relative entropy are defined as [20]

$$
\begin{align*}
\text { Entanglement } E & =\min _{\sigma \in \mathcal{D}} S(\rho \| \sigma),  \tag{1}\\
\text { Discord } D & =\min _{\chi \in \mathcal{C}} S(\rho \| \chi),  \tag{2}\\
\text { Dissonance } Q & =\min _{\chi \in \mathcal{C}} S(\sigma \| \chi),  \tag{3}\\
\text { Classical correlations } C & =\min _{\pi \in \mathcal{P}} S(\chi \| \pi), \tag{4}
\end{align*}
$$

where $\mathcal{P}$ is the set of all product states that have the form $\pi=\pi_{1} \otimes \pi_{2} \otimes \ldots \pi_{N}$ and $\pi_{j}(j=1,2, \ldots, N)$ is the reduced state of the $j$-th subsystem. The set of classical states, $\mathcal{C}$, contains mixtures of locally distinguishable states $\chi=$ $\sum_{v_{n}} p_{v_{1} \ldots v_{N}}\left|v_{1} \ldots v_{N}\right\rangle\left\langle v_{1} \ldots v_{N}\right|$, where $p_{v_{1} \ldots v_{N}}$ is a joint probability distribution and local states $\left|v_{n}\right\rangle$ are an orthonormal basis. The correlations of these states are identified as classical correlations $[20,23,24]$. $\mathcal{D}$ is the set of all separable states of the form $\sigma=\sum_{k} p_{k} \pi_{1}^{k} \otimes \pi_{2}^{k} \otimes \ldots \otimes \pi_{N}^{k}$. The relative entropy between two quantum states $\rho$ and $\chi$ is given by

$$
\begin{equation*}
S(\rho \| \chi) \equiv-\operatorname{tr}(\rho \log \chi)-S(\rho) \tag{5}
\end{equation*}
$$

where $S(\rho)$ is the von Neumann entropy of $\rho$ and $S(\rho) \equiv$ $-\operatorname{tr}(\rho \log \rho)$.

EW of $N$-partite systems in $2^{n}$-dimensional space. - If a pure state $|\psi\rangle \in H_{1} \otimes H_{2} \otimes \ldots \otimes H_{N^{-}} H_{i}$ is the Hilbert space of the $i$-th subsystem which can be written in the form $|\psi\rangle=\left|\psi_{1}\right\rangle \otimes\left|\psi_{2}\right\rangle \otimes \ldots \otimes\left|\psi_{N}\right\rangle$, where $\left|\psi_{i}\right\rangle$ is a pure state of the $i$-th subsystem, it is said to be separable, otherwise it is called entangled. When a system is in an entangled pure state, it is not possible to assign states to


Fig. 1: Nonlinear EW is the envelope of a family of linear EWs.
its subsystems. This will be true, in the appropriate sense, for the mixed state case as well. A mixed state of the composite system is described by a density matrix $\rho$ acting on $H_{1} \otimes H_{2} \otimes \ldots \otimes H_{N} . \rho$ is separable if there exist $p_{k} \geq 0$, $\left\{\rho_{1}^{k}\right\},\left\{\rho_{2}^{k}\right\}, \ldots,\left\{\rho_{N}^{k}\right\}$ which are mixed states of the respective subsystems such that $\rho=\sum_{k} p_{k} \rho_{1}^{k} \otimes \rho_{2}^{k} \otimes \ldots \otimes \rho_{N}^{k}$, where $\sum_{k} p_{k}=1$. Any violation of the condition of separation of the state makes it entangled. The most general approach to study the entanglement of quantum states in higher-dimensional physical systems is based on the notion of entanglement witnesses. In general, a Hermitian operator (or an observable) is called EW if it has a positive expectation value with all separable states while it possesses at least one negative eigenvalue. Mathematically, this means that for the Hermitian operator $W$, we must check the two properties: firstly, $\operatorname{Tr}\left(W \rho_{s}\right) \geq 0$ for all separable states $\rho_{s}$, secondly $\operatorname{Tr}\left(W \rho_{e}\right)<0$ for some entangled state $\rho_{e}$. In fact in quantum information theory, an entanglement witness is a functional which distinguishes a specific entangled state from the separable ones [25]. Each linear EW separates the separable region from the entanglement one. The set of such linear EWs is a family of curves whose envelope can be considered as a nonlinear entanglement witness. Figure 1 shows this nonlinear witness.

In this section, we consider the $N$-partite systems where each of the local dimensions is $2^{n}$ and find the separability condition for these states.

To this end, suppose that the entanglement witness $W$ is in the following form [26]:

$$
\begin{align*}
W= & A_{0} \underbrace{I \otimes I \otimes \ldots \otimes I}_{N-\text {-fold }} \\
& +\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m m} \underbrace{\gamma_{i} \otimes \gamma_{j} \otimes \ldots \otimes \gamma_{l}}_{N-2} \otimes \gamma_{m} \otimes \gamma_{m}, \tag{6}
\end{align*}
$$

where $\left\{\gamma_{i}\right\}$ are Dirac matrices (see appendix) [27] and $I$ denotes the identity matrix of dimension $2^{n}$. Note that here we choose maximally anticommuting sets of $\gamma_{i}$. Throughout the paper, we remove the tensor products. For example, instead of $\gamma_{i} \otimes \gamma_{j} \otimes \ldots \otimes \gamma_{l} \otimes$ $\gamma_{m} \otimes \gamma_{m}$, we use the notation $\gamma_{i} \gamma_{j} \ldots \gamma_{l} \gamma_{m} \gamma_{m}$. Note that $A_{0}$ is a nonzero positive real parameter, $\gamma_{k}$ are
operators with $-1 \leq \operatorname{Tr}\left(\rho_{s} \gamma_{i} \gamma_{j} \ldots \gamma_{l} \gamma_{m} \gamma_{m}\right) \leq 1$ for every separable $\rho_{s}$, and $A_{i j \ldots l m m}$ are real parameters whose ranges must be determined such that $W$ becomes an EW. We define the arbitrary $N$-partite product state as $\rho_{s}=$ $\underbrace{|\alpha\rangle\langle\alpha| \otimes|\beta\rangle\langle\beta| \otimes \ldots \otimes|\gamma\rangle\langle\gamma|}$. For the Hermitian operators $Q \underbrace{i j \ldots l}_{(N-2) \text {-fold }} m m=\gamma_{i} \otimes \gamma_{j} \otimes \ldots \otimes \gamma_{l} \otimes \gamma_{m} \otimes \gamma_{m}$, we consider the maps $P_{i j \ldots l m m}=\operatorname{Tr}\left(\rho_{s} Q_{i j \ldots l m m}\right)$ for any product state such that these maps map the convex set of product states into a bounded convex region which will be named feasible region. In other words, in order for $W$ to be an entanglement witness, it must satisfy the condition $\operatorname{Tr}\left(W \rho_{s}\right) \geq 0$. This condition takes the following form:

$$
\begin{equation*}
\sqrt{\Pi_{11}}+\sqrt{\Pi_{22}}+\ldots+\sqrt{\Pi_{(2 n+1)(2 n+1)}} \leq 1 \tag{7}
\end{equation*}
$$

where

$$
\begin{equation*}
\Pi_{l l}=\sum_{i, j, \ldots, l, m=1}^{2 n+1} P_{i j \ldots l m m}^{2} \tag{8}
\end{equation*}
$$

Note that in the above equation, the number of $i j \ldots l$ is $N-2$. To proof of the inequality eq. (7) assume

$$
\begin{gathered}
\operatorname{Tr}\left(\gamma_{i}^{(1)}|\alpha\rangle\langle\alpha|\right)=a_{i}, \\
\operatorname{Tr}\left(\gamma_{i}^{(2)}|\beta\rangle\langle\beta|\right)=b_{i}, \\
\vdots \\
\operatorname{Tr}\left(\gamma_{i}^{(N-1)}|\delta\rangle\langle\delta|\right)=d_{i}, \\
\operatorname{Tr}\left(\gamma_{i}^{(N)}|\gamma\rangle\langle\gamma|\right)=f_{i}
\end{gathered}
$$

(where the superscripts $1,2, \ldots, N$ in $\gamma_{i}$, denote the first, second, $\ldots, N$-th party, respectively) and

$$
P_{i j \ldots l m m}=\operatorname{Tr}\left(\rho_{s} Q_{i j \ldots l m m}\right)=a_{i} b_{j} \ldots c_{l} d_{m} f_{m}
$$

Hence we get

$$
\begin{align*}
& \sqrt{\Pi_{11}}+\sqrt{\Pi_{22}}+\ldots+\sqrt{\Pi_{(2 n+1)(2 n+1)}}= \\
& \sqrt{\mu_{11}}+\sqrt{\mu_{22}}+\ldots+\sqrt{\mu_{(2 n+1)(2 n+1)}} \tag{9}
\end{align*}
$$

where

$$
\mu_{m m}=\sum_{i, j, \ldots, l, m=1}^{2 n+1} a_{i}^{2} b_{j}^{2} \ldots c_{l}^{2} d_{m}^{2} f_{m}^{2}
$$

Since $\sum_{i=1}^{2 n+1} a_{i}^{2}=1$ and also the similar relations hold for $b_{i}, \ldots, c_{i}, d_{i}$ and $f_{i}$, then we have

$$
\begin{align*}
& \sqrt{\Pi_{11}}+\sqrt{\Pi_{22}}+\ldots+\sqrt{\Pi_{(2 n+1)(2 n+1)}}= \\
& d_{1} f_{1}+d_{2} f_{2}+\ldots+d_{2 n+1} f_{2 n+1}=|\vec{d} \cdot \vec{f}| \leq 1 \tag{10}
\end{align*}
$$

Hence, the convex optimization problem is in the following form:
minimize $\operatorname{Tr}\left(W \rho_{s}\right)=A_{0}+\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m m} P_{i j \ldots l m m}$
subject to $\sqrt{\Pi_{11}}+\sqrt{\Pi_{22}}+\ldots+\sqrt{\Pi_{(2 n+1)(2 n+1)}} \leq 1$.

By solving this problem, we arrive at the following condition:

$$
\begin{align*}
\sum_{i, j, \ldots, l=1}^{2 n+1} A_{i j \ldots l 11}^{2}= & \sum_{i, j, \ldots, l=1}^{2 n+1} A_{i j \ldots l 22}^{2}=\ldots= \\
& \sum_{i, j, \ldots, l=1}^{2 n+1} A_{i j \ldots l(2 n+1)(2 n+1)}^{2} \tag{11}
\end{align*}
$$

So, with the following condition:

$$
\begin{equation*}
A_{0}-\sqrt{\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m}^{2}} \geq 0 \tag{12}
\end{equation*}
$$

$W$ is an entanglement witness.
In the rest of this section, we are going to obtain the detection condition for generic density matrix. To do this, consider the following density matrix:

$$
\begin{equation*}
\rho=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N \text {-fold }}+\sum_{i, j, \ldots, k=1}^{2 n+1} r_{i j \ldots k} \underbrace{\gamma_{i} \gamma_{j} \ldots \gamma_{k}}_{N \text {-fold }}), \tag{13}
\end{equation*}
$$

then we have

$$
\operatorname{Tr}(W \rho)=A_{0}+\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m m} r_{i j \ldots l m m},
$$

and by using eq. (11) and eq. (12), the convex optimization problem takes the form

$$
\begin{gathered}
\operatorname{minimize} A_{0}+\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m m} r_{i j \ldots l m m} \\
\text { subject to } A_{0}-\sqrt{\sum_{i, j, \ldots, l, m=1}^{2 n+1} A_{i j \ldots l m}^{2} \geq 0} \\
F_{11}=F_{22}=F_{(2 n+1)(2 n+1)}
\end{gathered}
$$

where

$$
F_{m m}=\sum_{i, j, \ldots, l=1}^{2 n+1} A_{i j \ldots l m m}^{2}
$$

Using the convex optimization method, one can show that

$$
\begin{align*}
\min \operatorname{Tr}(W \rho)= & \frac{1}{2^{n N}}\left[1-\left(\sqrt{F_{11}}+\sqrt{F_{22}}\right.\right. \\
& \left.\left.+\ldots+\sqrt{F_{(2 n+1)(2 n+1)}}\right)\right] \leq 0 \tag{14}
\end{align*}
$$

where

$$
F_{l l}=\sum_{i, j, \ldots, l . m}^{2 n+1} r_{i j \ldots l m m}^{2}
$$

In order to find the PPT conditions of the density matrix, eq. (13), we consider the following assumptions: In the
general form of a $N$-particles density matrix, the coefficients of $\gamma_{i} \otimes \gamma_{j} \otimes \ldots \otimes \gamma_{k} \otimes \gamma_{l} \otimes \gamma_{m} \otimes \gamma_{n}$ are $r_{i j \ldots k l m n}$, but we choose these coefficients such that if $m \neq n$, then $r_{i j \ldots k l m n}=0$ and also for any $i, j, \ldots, k, l$, we consider $r_{i j \ldots k l m m}=r_{l m}$. So the subscript $m$ appears twice in $\gamma_{i} \otimes \gamma_{j} \otimes \ldots \otimes \gamma_{k} \otimes \gamma_{l} \otimes \gamma_{m} \otimes \gamma_{m}$. The reason for this choice is that we are able to obtain the eigenvalues of the density matrix, otherwise, it is very difficult to gain the eigenvalues of it. So, by these conditions, the density matrix, eq. (13), has the following form:

$$
\begin{equation*}
\rho=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N \text {-fold }}+\sum_{i, j, \ldots, k, l, m=1}^{2 n+1} r_{l m} \underbrace{\gamma_{i} \gamma_{j} \ldots \gamma_{k}}_{N-3} \gamma_{l} \gamma_{m} \gamma_{m}) . \tag{15}
\end{equation*}
$$

Now we are going to find the PPT condition of the density matrix, eq. (15). To this end, we rewrite it as the following form:

$$
\rho=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N \text {-fold }}+B_{i, j, \ldots, k} C_{l m}),
$$

where

$$
C_{l m}=\sum_{l=1}^{2 n+1} \gamma_{l} \sum_{m=1}^{2 n+1} r_{l m} \gamma_{m} \gamma_{m}
$$

and

$$
B_{i, j, \ldots, k}=\sum_{i, j, \ldots, k=1}^{2 n+1} \underbrace{\gamma_{i} \gamma_{j} \ldots \gamma_{k}}_{N-3}
$$

The eigenvalues of $B_{i, j, \ldots, k}$ are $\pm a_{n N}$, where $a_{n N}=$ $\sqrt{(2 n+1)^{N-3}}$., and the eigenvalues of $C_{l m}$ are

$$
\begin{align*}
& {\left[\sum _ { l = 1 } ^ { 2 n + 1 } \left\{(-1)^{i_{1}} r_{l 1}+(-1)^{i_{2}} r_{l 2}+\cdots+(-1)^{i_{2 n}} r_{l 2 n}\right.\right.} \\
& \left.\left.+(-1)^{n}(-1)^{i_{1}+i_{2}+\ldots+i_{2 n}} r_{l 2 n+1}\right\}^{2}\right]^{\frac{1}{2}} \tag{16}
\end{align*}
$$

where $i_{1}, i_{2}, \ldots, i_{2 n} \in\{0,1\}$. Finally, the PPT condition of the density matrix is as follows:

$$
\begin{align*}
& 1 \pm a_{n N}\left[\sum _ { l = 1 } ^ { 2 n + 1 } \left\{(-1)^{i_{1}} r_{l 1}+(-1)^{i_{2}} r_{l 2}+\cdots+(-1)^{i_{2 n}} r_{l 2 n}\right.\right. \\
& \left.\left.+(-1)^{n}(-1)^{i_{1}+i_{2}+\ldots+i_{2 n}} r_{l 2 n+1}\right\}^{2}\right]^{\frac{1}{2}} . \tag{17}
\end{align*}
$$

Furthermore, for the density matrix of the form of eq. (15), the detection condition, eq. (14), is as the following form:

$$
\begin{aligned}
& \operatorname{Tr}(W \rho)=1-a_{n N}\left(\sqrt{\sum_{j=1}^{2 n+1} r_{1 j}^{2}}\right. \\
& \left.+\sqrt{\sum_{j=1}^{2 n+1} r_{2 j}^{2}}+\ldots+\sqrt{\sum_{j=1}^{2 n+1} r_{(2 n+1) j}^{2}}\right) \leq 0 .
\end{aligned}
$$

If $\operatorname{Tr}(W \rho)=0$, i.e.,
$a_{n N}\left(\sqrt{\sum_{j=1}^{2 n+1} r_{1 j}^{2}}+\sqrt{\sum_{j=1}^{2 n+1} r_{2 j}^{2}}+\ldots+\sqrt{\sum_{j=1}^{2 n+1} r_{(2 n+1) j}^{2}}\right)=1$,
then $\rho$ is separable, because we can write it as

$$
\left.\left.\begin{array}{l}
\rho=\frac{a_{n N}}{2^{n N}}(\sqrt{\sum_{j=1}^{2 n+1} r_{1 j}^{2}}(\underbrace{I I \ldots I}_{N}+\beta^{\otimes(N-3)} \vec{n}_{1} \cdot \vec{\gamma} \gamma_{1} \gamma_{1}) \\
+\sqrt{\sum_{j=1}^{2 n+1} r_{2 j}^{2}}(\underbrace{I I \ldots I}_{N}+\beta^{\otimes(N-3)} \overrightarrow{n_{2}} \cdot \vec{\gamma} \gamma_{2} \gamma_{2}) \\
\\
+\ldots+\sqrt{\sum_{j=1}^{2 n+1} r_{(2 n+1) j}^{2}(\underbrace{I I \ldots I}_{N}} \\
\end{array}+\beta^{\otimes(N-3)} \vec{n}_{2 n+1} \cdot \vec{\gamma} \gamma_{2 n+1} \gamma_{2 n+1}\right)\right), ~ l l
$$

with

$$
\overrightarrow{n_{i}} \cdot \vec{\gamma}=\frac{\sum_{j=1}^{2 n+1} r_{i j} \gamma_{j}}{\sqrt{\sum_{j=1}^{2 n+1} r_{i j}^{2}}}
$$

and

$$
\beta=\frac{\gamma_{1}+\gamma_{2}+\ldots+\gamma_{2 n+1}}{\sqrt{2 n+1}}
$$

The terms in parentheses are projection operators and $\rho$ is a convex combination of separable terms, hence $\rho$ is separable.

Quantum discord of a $N$-partite system. - We first recall a result regarding the closest classical states (CCS) and the closest separable states (CSS) of a generic state $\rho$ in [28].

Theorem: Given a generic state $\rho \in \mathcal{H} \backslash I$ and $X \in I$, $\min S(\rho \| X)$ is achieved when $\rho$ and $X$ have common eigenbasis. Here, $I$ is a special subset of the Hilbert space $\mathcal{H}$.

We should mention that obtaining the CCS and CSS of a generic state is still a very difficult problem. In fact, using the above result one can obtain the CCS and CSS of a generic state $\rho$ when the eigenvectors of $\rho$ and $X$ are parallel.

Now, we focus on the density matrix in the form of eq. (15) and find an exact analytical formula for the quantum discord of this density matrix using the concept of relative entropy as a distance measure of correlations (eq. (2)). Now from eq. (2), to determine $D$, we have

$$
\begin{align*}
D=\min _{\chi \in \mathcal{C}} S(\rho \| \chi)= & \min [\operatorname{tr}(\rho \log \rho)-\operatorname{tr}(\rho \log \chi)]= \\
& \sum_{i} \lambda_{i} \log \lambda_{i}-\max (\operatorname{tr}(\rho \log \chi)) . \tag{18}
\end{align*}
$$

The classical states contain mixtures of locally distinguishable states, hence we take the CCS of $\rho$ as

$$
\begin{array}{r}
\chi \in\left\{\frac{1}{2^{n N}}\left(I I \ldots I+r_{l m}^{\prime} \gamma_{i} \gamma_{j} \ldots \gamma_{l} \gamma_{m} \gamma_{m}\right)\right\} \\
(i, j, \ldots, m=1, \ldots, 2 n+1) . \tag{19}
\end{array}
$$

Let us choose $\chi$ as $\chi=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N \text {-fold }}+r_{11}^{\prime} \underbrace{\gamma_{1} \gamma_{1} \ldots \gamma_{1}}_{N \text {-fold }})$, then we obtain

$$
\begin{align*}
& \operatorname{tr}(\rho \log \chi)=\frac{1}{2}\left[\left(1+r_{11}\right) \log \left(1+r_{11}^{\prime}\right)\right. \\
& \left.-\left(-1+r_{11}\right) \log \left(1-r_{11}^{\prime}\right)-2 \log 2^{n N}\right] \tag{20}
\end{align*}
$$

To maximize the above equation, by calculating the gradient of the above equation with respect to $r_{11}^{\prime}$ and make it zero, one can show that $r_{11}^{\prime}=r_{11}$.

Hence quantum discord of $N$-partite systems in $2^{n}$ dimensions is given by

$$
\begin{align*}
D= & \sum_{i} \lambda_{i} \log \lambda_{i}-\frac{1}{2}\left[\left(1+r_{11}\right) \log \left(1+r_{11}\right)\right. \\
& \left.-\left(-1+r_{11}\right) \log \left(1-r_{11}\right)-2 \log 2^{n N}\right] \tag{21}
\end{align*}
$$

Subadditivity relation. - One of the basic question related to the measure of correlations is the additivity of the proposed measures. In this section, we discuss the subadditivity relation for the states given by eq. (15) and show that the subadditivity relation is satisfied for this density matrix. To this end, we use the relative entropy-based definitions of quantum correlations, which is completely applicable for multipartite systems of arbitrary dimensions. Defining the total mutual information as $T_{\rho}=S\left(\rho \| \pi_{\rho}\right)$ and classical correlation as $C_{\sigma}=S\left(\chi_{\sigma} \| \pi_{\sigma}\right)$, the subadditivity relation that is $T_{\rho} \geq E+Q+C_{\sigma}$, can be checked. To simplify the investigation of the subadditivity relation between quantum correlations, we consider the special case that is

$$
\begin{array}{ll}
r_{l m}=0, & l \neq m, \\
r_{l l}, & l=m .
\end{array}
$$

So we arrive at the following density matrix:

$$
\begin{array}{r}
\rho=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N-\text { fold }}+\sum_{l=1}^{2 n+1} r_{l l} \underbrace{\gamma_{i} \gamma_{j} \ldots \gamma_{k}}_{(N-3) \text {-fold }} \gamma_{l} \gamma_{l} \gamma_{l}) \\
(i, j, \ldots, k=1, \ldots, 2 n+1) . \tag{22}
\end{array}
$$

The PPT condition for this density matrix is as follows:

$$
\begin{equation*}
\sum_{l=0}^{2 n+1} r_{l l}^{2} \leq \frac{1}{\sqrt{(2 n+1)^{N-3}}} \tag{23}
\end{equation*}
$$

and then the detection condition of entanglement is $\sum_{l=0}^{2 n+1}\left|r_{l l}\right|>\frac{1}{\sqrt{(2 n+1)^{N-3}}}$. In [28], the analytical
technique for finding the closest separable states (CSS) of $2^{n}$-dimensional Bell diagonal states is provided. Following the calculations of it, we can write the closest separable states (CSS) of the density matrix, eq. (22), as

$$
\begin{array}{r}
\sigma=\frac{1}{2^{n N}}(\underbrace{I I \ldots I}_{N \text {-fold }}+\sum_{l=1}^{2 n+1} r_{l l}^{\prime} \underbrace{\gamma_{i} \gamma_{j} \ldots \gamma_{k}}_{(N-3) \text {-fold }} \gamma_{l} \gamma_{l} \gamma_{l}) \\
(i, j, \ldots, k=1, \ldots, 2 n+1) . \tag{24}
\end{array}
$$

The eigenvectors of $\rho$ and $\sigma$ are parallel [28] and so $r_{l l}^{\prime}=\lambda r_{l l}(l=0, \ldots, 2 n+1)$; on the other hand, since $\sigma$ is a separable state, then $\sum_{l}\left|r_{l l}^{\prime}\right|=\frac{1}{\sqrt{(2 n+1)^{N-3}}}$. Using these two facts, we get

$$
\begin{equation*}
r_{l l}^{\prime}=\frac{r_{l l}}{\sqrt{(2 n+1)^{N-3}} \sum_{l=0}^{2 n+1}\left|r_{l l}\right|}, \quad l=0, \ldots, 2 n+1 \tag{25}
\end{equation*}
$$

For simplicity, we assume that all of $r_{l l}(l=0, \ldots, 2 n+1)$ is positive. Then by some calculations, we obtain the additivity relation between quantum correlations as the following form:

$$
\begin{align*}
& E+Q+C_{\sigma}-T_{\rho}= \\
& -\frac{\left(-1+\sqrt{(2 n+1)^{N-3}} \sum_{l=0}^{2 n+1} r_{l l}\right) \sqrt{\sum_{l=0}^{2 n+1} r_{l l}^{2}}}{\sum_{l=0}^{2 n+1} r_{l l}} \\
& \times \cot ^{-1}\left(\frac{\sum_{l=0}^{2 n+1} r_{l l}}{\sqrt{\sum_{l=0}^{2 n+1} r_{l l}^{2}}}\right), \tag{26}
\end{align*}
$$

then using eq. (23) we have $E+Q+C_{\sigma}-T_{\rho} \leq 0$, that is the correlations of the state in eq. (15) are subadditive.

Conclusion. - In this paper we have studied the system which consists of a $N$-partite system with $2^{n}$-dimensional Hilbert space in each subsystem. In the first step, we have found the entanglement witness for this system, then we have computed the quantum discord based on relative entropy. At the end of the paper, we have found the closest separable state (CSS) for a N partite system of $2^{n}$-dimensional space and have analyzed the subadditivity relation between the quantum correlations of this system.

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Appendix: gamma matrices. - The gamma matrices, $\gamma_{\mu}, \mu=1, \ldots, d$, are the set of $d$ matrices that satisfy the following anticommutation relations:

$$
\begin{equation*}
\gamma_{\mu} \gamma_{\nu}+\gamma_{\nu} \gamma_{\mu}=2 \delta_{\mu \nu} I_{2^{d / 2}} \tag{A.1}
\end{equation*}
$$

where $I_{2^{d / 2}}$ is the identity matrix with dimension $2^{\frac{d}{2}}$. These matrices are also called Dirac matrices and generate a matrix representation of the Clifford algebra with $2^{d}$ dimension. One element of this algebra can be obtained by product all of the $\gamma$ matrices:

$$
\gamma_{s}=i^{-\frac{d}{2}} \gamma_{1} \gamma_{2} \ldots \gamma_{2 n}
$$

that commute with all gamma matrices and $\gamma_{s}^{2}=I_{2^{d / 2}}$. By adding this matrix with other gamma matrices, anticommutation relations have the following form:

$$
\begin{equation*}
\gamma_{i} \gamma_{j}+\gamma_{j} \gamma_{i}=2 \delta_{i j} I_{2^{d / 2}} \quad(i, j=1, \ldots, d, d+1) \tag{A.2}
\end{equation*}
$$

In the case of $d=2$, the Dirac matrices are the Pauli matrices all of which are Hermitian:

$$
\begin{align*}
& \gamma_{1}^{(d=2)} \equiv \sigma_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \\
& \gamma_{2}^{(d=2)} \equiv \sigma_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right),  \tag{A.3}\\
& \gamma_{s}=\gamma_{3}^{(d=2)} \equiv \sigma_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) .
\end{align*}
$$

It is also possible to construct higher-dimensional gamma matrices that generalize to even dimensions as follows:

$$
\begin{align*}
& \gamma_{i}^{(d+2)}=\sigma_{1} \otimes \gamma_{i}^{(d)}=\left(\begin{array}{cc}
0 & \gamma_{i}^{(d)} \\
\gamma_{i}^{(d)} & 0
\end{array}\right) \quad(i=1, \ldots, d+1), \\
& \gamma_{d+2}^{(d+2)}=\sigma_{2} \otimes I^{(d)}=\left(\begin{array}{cc}
0 & -i I_{d} \\
i I_{d} & 0
\end{array}\right),  \tag{A.4}\\
& \gamma_{s}^{d+2}=\gamma_{d+3}^{(d+2)} \equiv \sigma_{3} \otimes I_{d}=\left(\begin{array}{cc}
I_{d} & 0 \\
0 & -I_{d}
\end{array}\right),
\end{align*}
$$

where $I_{d}$ is the unitary matrix in $2^{d / 2}$ dimensions. It can be shown that if $\gamma_{i}$ satisfy eq. (A.2) in $d$-dimension, then the same relation exists for $\gamma_{i}^{d+2}$.

As eq. (A.2) shows, we can indicate the $\gamma$ matrices in odd dimensions through obtaining the $\gamma$ matrices of dimension $d-1$ where $\gamma_{s}$ was added to them. In this case the $\gamma$ matrices are not all algebraically independent, in contrast to the even case.

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