A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

LETTER

Entropy relations and bounds of regular and singular black holes with nonlinear electrodynamics sources

To cite this article: Juan Wu and Wei Xu 2019 *EPL* **125** 60002

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

March 2019

EPL, **125** (2019) 60002
doi: 10.1209/0295-5075/125/60002

www.epljournal.org

Entropy relations and bounds of regular and singular black holes with nonlinear electrodynamics sources

JUAN WU[(a)] and WEI XU[(b)]

School of Mathematics and Physics, China University of Geosciences - Wuhan 430074, China

received 19 September 2018; accepted in final form 12 March 2019
published online 23 April 2019

PACS 04.70.Dy – Quantum aspects of black holes, evaporation, thermodynamics
PACS 04.50.Kd – Modified theories of gravity
PACS 04.70.-s – Physics of black holes

Abstract – In this paper, we study entropy relations and bounds of regular and singular black holes with nonlinear electrodynamics sources. We focus on the regular and singular Bardeen and Hayward black holes in asymptotically flat and AdS spacetimes. For black holes with vanishing effective mass M_{eff}, it is found that the entropy product is always mass-independent in asymptotically flat spacetime, while the entropy sum takes the mass-independent universality in asymptotically AdS spacetime. For the cases with nonzero effective mass M_{eff} in asymptotically flat and AdS spacetimes, entropy relations are both some functions of mass and charge, hence are always mass-dependent. Besides, the non-linear electrodynamics always affects the entropy bounds of horizons.

Introduction. – Explaining the microscopic origin of black-hole entropy is one of the major challenges in quantum theories of gravity. There is some significant progress in calculating the black-hole entropy, such as the Kerr/CFT correspondence [1]. However, it works only for the special classes of black holes including BPS black holes, where the microscopic degrees of freedom can be explained in terms of a two-dimensional conformal field theory (see the recent review paper [2]). Many works have successfully obtained the microscopic entropy, which mainly focused on the extremal solutions (see [3] for a recent example), while it is still an open question for finding the microscopic entropy of the general black holes. On the other hand, entanglement entropy is introduced to study the microscopic origin of black-hole physics (see the recent review paper [4]), and characterizes some quantum features, which could also coincide with the Bekenstein-Hawking entropy [5]. Although our understanding of the statistical viewpoints of general black-hole entropy remains incomplete, it seems clear that the progress on this subject reflects important features of the underlying quantum mechanical degrees of freedom. The recent progress is generalized to the nonextremal solutions, by a phenomenological approach called universal entropy (area) product [6]. Other entropy relations are also introduced [7–9]. It has been observed that these additional thermodynamic relations always appear to be universal [6–9] and may provide further insight into the quantum physics of black holes [10–12]. These universal entropy relations, including the entropy product [6] and sum [9], are always mass-independent in the sense that they involve thermodynamic quantities defined at multi-horizons, including the event and Cauchy horizons, even the "virtual" horizons. The mass-independent entropy relations should depend solely on the quantized charges, angular momentum and cosmological constant, which indicates that the black-hole entropy might be quantized in a quite specific manner in the black-hole thermodynamic framework, in terms of linear combinations of the electric charges, spin and cosmological constant [7]. For example, based on the mass-independent entropy product, a non-integral horizon-area quantization formula of entropy of Kerr-Newman black hole is derived, in terms of spin (quantized angular momentum) and electric charge [10]. The universality of entropy relations is discussed in many modified theories [13–33]. Besides, there are some viewpoints providing evidence about the relationship between these universal relations and a CFT description [11,34–39]. However, so far no geometric proof and evidence have been found for the universality of entropy relations.

(a) E-mail: wu@cug.edu.cn
(b) E-mail: xuwei@cug.edu.cn (corresponding author)

In the absence of a geometric understanding, it makes sense to explore phenomenologically whether the universality of entropy relations holds in arbitrary gravity theories and spacetimes. Towards this goal, we will consider the entropy relations of a number of regular and singular black holes, which are solutions of the gravities with a nonlinear electric or magnetic field, with the following action [40–42]:

$$I = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}(R + 6\ell^{-2} - \mathcal{L}(\mathcal{F})), \tag{1}$$

where ℓ is the AdS cosmological radius, $F = \mathrm{d}A$ is the field strength of the Maxwell field, $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, and the Lagrangian density $\mathcal{L}$ is a function of $\mathcal{F}$. We will focus on theory with the general Lagrangian density [43]

$$\mathcal{L} = \frac{4\mu}{\alpha}\frac{(\alpha\mathcal{F})^{(\nu+3)/4}}{\left(1+(\alpha\mathcal{F})^{\nu/4}\right)^{(\mu+\nu)/\nu}}. \tag{2}$$

Especially, the regular black-hole solution was firstly constructed by Bardeen, which contains no singularities in the spacetime [44]. Other regular black-hole models in modified gravities have been also proposed later [45–60]. Regular black holes always violate the strong energy condition, hence can break the singularity theorems [61]. Besides, one should note that the nonlinear electric or magnetic fields are not necessary for constructing regular black holes. Actually, there are uncharged regular black holes in the gravity models with other matter sources [60,62,63]. For the discussion about the microscopic origin of regular black-hole entropy, there is only few literature [64,65].

Here we will test the universality of these entropy relations by investigating to what extent they do, or do not, hold in flat/AdS spacetimes with nonlinear electrodynamics sources. We will focus on the regular and singular Bardeen and Hayward black holes in asymptotically flat and AdS spacetimes. It is shown that for black holes with vanishing effective mass M_{eff}, the entropy product is always mass-independent in asymptotically flat spacetime, while the entropy sum is always mass-independent in asymptotically AdS spacetime. In order to keep this universality of the mass-independence, the effect of the "virtual" horizons must be included for both cases. This is consistent with the discussion in [6]. For black holes with nonzero effective mass M_{eff} in asymptotically flat and AdS spacetimes, entropy relations are all some functions of $(M_{ADM}, M_{\mathit{eff}}, Q)$, hence are always mass-dependent.

Based on the entropy relations, one can find some entropy bounds [66–69], which are related to the geometrical bounds [70,71], hence it may lead to a geometrical viewpoint about the universality of entropy relations. It will be shown that the nonlinear electrodynamics always affects the entropy bounds of horizons. Taking the entropy bounds of Bardeen black holes with vanishing effective mass as an example, one can find that the parameter α always diminishes the physical entropy bound for the event horizon, while it enlarges that for the Cauchy horizon; especially, the upper bound of the area for the event horizon is the Penrose-like inequality. For Hayward black holes with vanishing effective mass, the parameter α always enlarges the bounds for Cauchy horizons, while it has no effect on those for the event horizon; the upper bound of the area for the event horizon is the exact Penrose inequality.

The paper is organized as follows. We will firstly study the entropy relations and bounds of regular and singular black holes in asymptotically flat spacetime in the next section. In the third section, we shall generalize the discussion to the cases in asymptotically AdS spacetime. Our conclusions with some remarks are given in the final section.

Entropy relations and bounds for asymptotically flat regular and singular black holes. – In asymptotically flat spacetime, the black-hole solutions read as [43,72]

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2),\\ f(r) &= 1 - \frac{2M_{\mathit{eff}}}{r} - \frac{2\alpha^{-1}q^3r^{\mu-1}}{(r^\nu+q^\nu)^{\mu/\nu}},\\ A &= \frac{q}{\sqrt{2\alpha}}([3-(\nu-3)(q/r)^\nu][1+(q/r)^\nu]^{-(\mu+\nu)/\nu} - 3)\mathrm{d}t. \end{aligned} \tag{3}$$

Here the effective mass M_{eff} takes the following form:

$$M_{\mathit{eff}} = M_{ADM} - \alpha^{-1}q^3, \tag{4}$$

where M_{ADM} describes the ADM mass of a black hole which can be read off from the asymptotic behavior of the metric function $f(r)$, and $M_e = \alpha^{-1}q^3$ is a charged term which is associated with the nonlinear interactions between the graviton and the photon. After calculating the Ricci scalar and other higher-order curvature invariants, one can find that the solutions with $M_{\mathit{eff}} = 0, \mu \geq 3$ correspond to regular black holes, while other cases belong to singular black holes. The electric charge of black holes is $Q = \frac{q^2}{\sqrt{2\alpha}}$. The entropy of the horizons should be

$$S_i = \frac{A_i}{4} = \pi r_i^2, \tag{5}$$

where r_i are the horizons of black holes, and should be roots of $f(r) = 0$, *i.e.*,

$$1 - \frac{2M_{\mathit{eff}}}{r} - \frac{2\alpha^{-1}q^3r^{\mu-1}}{(r^\nu+q^\nu)^{\mu/\nu}} = 0. \tag{6}$$

However, the horizon structure is too complicated to calculate the entropy relations. Hence, we will focus on some reduced cases.

Regular and singular Bardeen black holes. When $\nu = 2$, the solution belongs to the Bardeen family of black holes [42,44]. The horizon function is simplified as

$$f(r) = 1 - \frac{2M_{\mathit{eff}}}{r} - \frac{2\alpha^{-1}q^3r^{\mu-1}}{(r^2+q^2)^{\mu/2}}. \tag{7}$$

For simplicity, we firstly study the case with vanishing effective mass, whose horizon function takes the form

$$f(r) = 1 - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^2+q^2)^{\mu/2}}, \tag{8}$$

for which, the $\mu = 3$ limit is just the Bardeen black hole [44]. Then the equation of horizons could be simplified as

$$(R+q^2)^\mu - 4M_e^2 R^{\mu-1} = 0, \qquad R = r^2. \tag{9}$$

We will view R as the radial variable, then the entropy should be

$$S_i = \pi R_i. \tag{10}$$

In the form of mass and charge, eq. (9) becomes

$$(R+\sqrt{2\alpha}Q)^\mu - 4M_e^2 R^{\mu-1} = 0, \tag{11}$$

which indicates that the maximal number of the horizons is exactly equal to μ, including the "virtual" horizons. By applying the Viète theorem, it is straightforward to find the mass-independent entropy product

$$\prod_{i=1}^{\mu} S_i = (-\sqrt{2\alpha}Q\pi)^\mu \tag{12}$$

and mass-dependent entropy sum

$$\sum_{i=1}^{\mu} S_i = \pi\,(4M_{ADM}^2 - \sqrt{2\alpha}Q\mu), \tag{13}$$

as the black-hole mass is $M_{ADM} = M_e$ now. For this case, the effect of the "virtual" horizons should be included, in order to obtain the universal entropy product. This is consistent with the discussion in [6].

Especially for the singular solution with $M_{eff} = 0$, $\mu = 2$, there are two horizons, *i.e.*, the event horizon and Cauchy horizon $R_{E,C} = (2M_{ADM}^2 - \sqrt{2\alpha}Q) \pm \sqrt{M_{ADM}^2(M_{ADM}^2 - \sqrt{2\alpha}Q)}$. This leads to a condition for the existence of horizons

$$M_{ADM}^2 \geq \sqrt{2\alpha}Q. \tag{14}$$

The entropy relations are simplified as

$$\begin{aligned} &S_E S_C = 2\alpha\pi^2 Q^2,\\ &S_E + S_C = \pi\,(4M_{ADM}^2 - 2\sqrt{2\alpha}Q). \end{aligned} \tag{15}$$

Since $r_E \geq r_C$, *i.e.*, $S_E \geq S_C$, it is easy to obtain these relations $S_C \leq \sqrt{2\alpha}Q\pi \leq S_E$ and $S_C \leq \pi\,(2M_{ADM}^2 - \sqrt{2\alpha}Q) \leq S_E \leq \pi\,(4M_{ADM}^2 - 2\sqrt{2\alpha}Q)$. Then by considering the condition for the existence of horizons together, one can obtain the bounds for the event horizon and the Cauchy horizon,

$$\begin{aligned} &S_C \in (0, \sqrt{2\alpha}Q\pi\,],\\ &S_E \in [2\pi\,M_{ADM}^2, 4\pi\,M_{ADM}^2] \times \left(1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2}\right), \end{aligned} \tag{16}$$

respectively. For gravity with Maxwell source, one can find that the electric charge Q diminishes the physical entropy bound for the event horizon, while it enlarges that for the Cauchy horizon. Besides, considering the effect of nonlinear electrodynamics on entropy bounds, it is shown that the parameter α plays a similar role as the electric charge Q. Furthermore, it is easy to obtain the area bounds

$$\begin{aligned} &\sqrt{\frac{A_C}{16\pi}} \in \left(0, \frac{\sqrt{\sqrt{2\alpha}Q}}{2}\right],\\ &\sqrt{\frac{A_E}{16\pi}} \in \left[\frac{M_{ADM}}{\sqrt{2}}, M_{ADM}\right] \times \left(1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2}\right), \end{aligned} \tag{17}$$

where the upper bound of the area for the event horizon is the Penrose-like inequality. When α is vanishing, it reduces to the exact Penrose inequality [70] which is the first geometrical inequality of a black hole. This provides a clue for the geometric understanding of the universal entropy relations of black holes with nonlinear electrodynamics sources.

Generalizing the study into black holes with nonzero effective mass M_{eff}, the discussion becomes too complicated, hence we take the singular case with $\mu = 2$ as an example. The horizons should be the roots of $f(r) = 0$, *i.e.*,

$$r^3 - 2M_{ADM}r^2 + \sqrt{2\alpha}Qr - 2\sqrt{2\alpha}QM_{eff} = 0, \tag{18}$$

for which there should be one or three positive roots. As we are interested in entropy relations of black holes with multi-horizons, we will only consider the case with three positive horizons r_i ($i = 1, 2, 3$). Applying the Viète theorem, one can obtain some useful relations,

$$\begin{aligned} &r_1 + r_2 + r_3 = 2M_{ADM},\\ &r_1 r_2 + r_2 r_3 + r_1 r_3 = \sqrt{2\alpha}Q,\\ &r_1 r_2 r_3 = 2\sqrt{2\alpha}QM_{eff}. \end{aligned} \tag{19}$$

By using these relations, the entropy relations can be calculated as

$$\begin{aligned} &\prod_{i=1}^{3} S_i = 8\alpha\,Q^2 M_{eff}^2 \pi^3,\\ &\sum_{i=1}^{3} S_i = \pi(4M_{ADM}^2 - 2\sqrt{2\alpha}Q), \end{aligned} \tag{20}$$

which are the functions of (M_{ADM}, M_{eff}, Q), hence depend on mass. Besides, it is easy to observe that the entropy sum is exactly the same, no matter whether the effective mass M_{eff} is vanishing or nonvanishing. On the other hand, after assuming $r_1 \leq r_2 \leq r_3$, *i.e.*, $S_1 \leq S_2 \leq S_3$, one can derive the entropy bounds for three horizons directly from the entropy sum,

$$
\begin{aligned}
S_1 &\in \left[0, \frac{4\pi\, M_{ADM}^2}{3}\right] \times \left(1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2}\right),\\
S_2 &\in \left[0, \frac{8\pi\, M_{ADM}^2}{3}\right] \times \left(1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2}\right),\\
S_3 &\in \left[\frac{4\pi\, M_{ADM}^2}{3}, 4\pi\, M_{ADM}^2\right] \times \left(1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2}\right).
\end{aligned} \tag{21}
$$

This indicates that the parameter α always diminishes the physical entropy bounds for horizons. Note that r_3 is the event horizon, the upper bound of the area for the event horizon is also the Penrose-like inequality $\sqrt{\frac{A_E}{16\pi}} \leq M_{ADM} \times (1 - \frac{\sqrt{2\alpha}Q}{2M_{ADM}^2})$, while the $\alpha \to 0$ limit corresponds to the exact Penrose inequality. One can follow this spirit to find the geometric understanding of the universal entropy relations for black holes with three physical horizons.

Regular and singular Hayward black holes. When $\nu = \mu$, the solution is the Hayward family of black holes [49,72]. The horizon function is reduced to

$$
f(r) = 1 - \frac{2M_{\mathit{eff}}}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\mu + q^\mu)}. \tag{22}
$$

We also begin with the case having vanishing effective mass, whose horizons should be roots of the equation

$$
r^\mu - 2M_e r^{\mu-1} + q^\mu = 0. \tag{23}
$$

Note that the $\mu = 3$ limit is just the Hayward black hole [50]. There exist μ roots, including some "virtual" horizons. We list some relations

$$
\prod_{i=1}^{\mu} r_i = -q^\mu, \qquad \prod_{1\leq i<j\leq \mu} r_i r_j = 0, \qquad \sum_{i=1}^{\mu} r_i = 2M_e, \tag{24}
$$

by which, it is easy to derive the entropy relations, including the mass-independent entropy product

$$
\prod_{i=1}^{\mu} S_i = \pi^\mu \left(\prod_{i=1}^{\mu} r_i\right)^2 = \pi^\mu q^{2\mu} = (\sqrt{2\alpha}Q\pi)^\mu \tag{25}
$$

and mass-dependent entropy sum

$$
\sum_{i=1}^{\mu} S_i = \pi \left(\sum_{i=1}^{\mu} r_i\right)^2 - 2\pi \prod_{1\leq i<j\leq\mu} r_i r_j = 4\pi\, M_{ADM}^2. \tag{26}
$$

In order to observe the effect of nonlinear electrodynamics on entropy bounds, we take the singular case with $M_{\mathit{eff}} = 0, \mu = 2$ as an example. The event horizon and Cauchy horizon are $r_{E,C} = M_{ADM} \pm \sqrt{M_{ADM}^2 - \sqrt{2\alpha}Q}$, which results in the condition for the existence of horizons $M_{ADM}^2 \geq \sqrt{2\alpha}Q$. From the entropy product and sum, we can obtain the relations $0 < S_C \leq \sqrt{2\alpha}Q\pi \leq S_E$ and $0 < S_C \leq 2\pi\, M_{ADM}^2 \leq S_E \leq 4\pi\, M_{ADM}^2$, which lead to the entropy bounds

$$
S_C \in (0, \sqrt{2\alpha}Q\pi], \qquad S_E \in [2\pi\, M_{ADM}^2, 4\pi\, M_{ADM}^2]. \tag{27}
$$

The parameters α and Q may always enlarge the bounds for Cauchy horizons, while they have no effect on that for the event horizon (in the external form). Besides, one can obtain the area bounds

$$
\begin{aligned}
\sqrt{\tfrac{A_C}{16\pi}} &\in \left(0, \tfrac{\sqrt{\sqrt{2\alpha}Q}}{2}\right],\\
\sqrt{\tfrac{A_E}{16\pi}} &\in \left[\tfrac{M_{ADM}}{\sqrt{2}}, M_{ADM}\right],
\end{aligned} \tag{28}
$$

where the upper bound of area for the event horizon is actually the exact Penrose inequality [70].

Then we begin to find the effect of nonzero effective mass M_{eff} on entropy relations. The equation of horizons could be simplified as

$$
r^{\mu+1} - 2M_{ADM}r^\mu + q^\mu\, r - 2M_{\mathit{eff}}q^\mu = 0. \tag{29}
$$

Consider the singular case with $\mu = 1$, the entropy relations should be

$$
\begin{aligned}
&\prod_{i=1}^{2} S_i = 4\sqrt{2\alpha}QM_{\mathit{eff}}^2\pi^2,\\
&\sum_{i=1}^{2} S_i = \pi\left(\left(2M_{ADM} - \sqrt{\sqrt{2\alpha}Q}\right)^2 + 4M_{\mathit{eff}}\sqrt{\sqrt{2\alpha}Q}\right),
\end{aligned} \tag{30}
$$

which are both mass-dependent as well. The corresponding entropy bounds should be

$$
\begin{aligned}
S_C &\in (0, 2\sqrt{\sqrt{2\alpha}Q}M_{\mathit{eff}}\pi],\\
S_E &\in [1/2, 1] \times \pi\left(\left(2M_{ADM} - \sqrt{\sqrt{2\alpha}Q}\right)^2\right.\\
&\qquad \left. + 4M_{\mathit{eff}}\sqrt{\sqrt{2\alpha}Q}\right),
\end{aligned} \tag{31}
$$

where the condition for the existence of a black-hole solution is used. The $\alpha \to 0$ limit is just the exact Penrose inequality. Following the same procedure, one can obtain the mass-dependent entropy relations for the cases with $\mu > 1$

$$
\begin{aligned}
&\prod_{i=1}^{\mu+1} S_i = \pi^{\mu+1}(2M_{\mathit{eff}}q^\mu)^2 = 4M_{\mathit{eff}}^2(\sqrt{2\alpha}Q)^\mu\pi^{\mu+1},\\
&\sum_{i=1}^{\mu+1} S_i = 4\pi\, M_{ADM}^2.
\end{aligned} \tag{32}
$$

Especially for the singular case with $\mu = 2$, one can calculate the entropy bounds from the entropy sum, *i.e.*,

$$\begin{aligned} S_1 &\in \left[0, \frac{4\pi\, M_{ADM}^2}{3}\right], \\ S_2 &\in \left[0, \frac{8\pi\, M_{ADM}^2}{3}\right], \\ S_3 &\in \left[\frac{4\pi\, M_{ADM}^2}{3}, 4\pi\, M_{ADM}^2\right], \end{aligned} \tag{33}$$

for which, the parameter α has no effect on the bounds for horizons (in the external form), while the upper bound of the area for the event horizon is just the exact Penrose inequality.

Totally, in asymptotically flat spacetime, it is shown that for black holes with vanishing effective mass M_{eff}, the entropy product is always mass-independent, while the entropy sum depends on the mass; for the case with nonzero effective mass M_{eff}, they are both mass-dependent and are some functions of (M_{ADM}, M_{eff}, Q). Besides, the nonlinear electrodynamics always affects the entropy bounds of horizons[1]. Especially, the upper bound of the area for the event horizon is the Penrose-like inequality for Bardeen black holes, while it is the exact Penrose inequality for Hayward black holes. These results are useful for understanding the geometric origin of the universal entropy relations of regular and singular black holes with nonlinear electrodynamics sources.

Entropy relations and bounds for asymptotically AdS regular and singular black holes. – In asymptotically AdS spacetime, the black-hole solutions read as [43,73]

$$\begin{aligned} \mathrm{d}s^2 &= -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_k^2, \\ f(r) &= \frac{r^2}{\ell^2} + k - \frac{2M_{eff}}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\nu + q^\nu)^{\mu/\nu}}, \\ A &= \frac{q}{\sqrt{2\alpha}}([3-(\nu-3)(q/r)^\nu][1+(q/r)^\nu]^{-(\mu+\nu)/\nu} - 3)\mathrm{d}t, \end{aligned} \tag{34}$$

where $\mathrm{d}\Omega_k^2$ denotes the metric of the two-dimensional sphere/hyperboloid/torus with constant curvature $k = 1, -1, 0$. The black-hole effective mass and electric charge take the same forms as the case in asymptotically flat spacetime. We will study the entropy relations for some reduced cases.

Regular and singular Bardeen AdS black holes. When $\nu = 2$, the solution reduces to the Bardeen family of AdS black holes with the horizon function

$$f(r) = \frac{r^2}{\ell^2} + k - \frac{2M_{eff}}{r} - \frac{2M_e r^{\mu-1}}{(r^2+q^2)^{\mu/2}}. \tag{35}$$

[1]For some cases in the Hayward family of black holes, there seems to be no effect for the entropy bounds in the external form, while M_{ADM} is actually related to the parameter α.

For simplicity, we only study the case with vanishing effective mass, whose metric function is simplified as

$$f(r) = \frac{r^2}{\ell^2} + k - \frac{2M_e r^{\mu-1}}{(r^2+q^2)^{\mu/2}}. \tag{36}$$

Then the horizons are roots of the equation

$$\left(\frac{R}{\ell^2} + k\right)^2 (R + \sqrt{2\alpha}Q)^\mu - 4M_{ADM}^2 R^{\mu-1} = 0, \tag{37}$$

which contains $(\mu + 2)$ horizons, including the "virtual" horizons. It is straightforward to get the entropy relations

$$\begin{aligned} \prod_{i=1}^{\mu+2} S_i &= \pi^{\mu+2}\prod_{i=1}^{\mu+2} R_i = \ell^4 k^2 (-\sqrt{2\alpha}Q)^\mu \pi^{\mu+2}, \\ \sum_{i=1}^{\mu+2} S_i &= \pi\sum_{i=1}^{\mu+2} R_i = -\pi(\sqrt{2\alpha}Q\mu + 2k\ell^2) \end{aligned} \tag{38}$$

by using the Viète theorem. This indicates that both entropy product and sum are mass-independent only when the effect of the "virtual" horizons is included. Especially for the singular Bardeen AdS black hole with $\mu = 1, k = -1$, one can get the entropy bounds from the entropy sum, *i.e.*,

$$\begin{aligned} S_1 &\in \left[0, \frac{1}{3}\right] \times \pi(2\ell^2 - \sqrt{2\alpha}Q), \\ S_2 &\in \left[0, \frac{2}{3}\right] \times \pi(2\ell^2 - \sqrt{2\alpha}Q), \\ S_3 &\in \left[\frac{1}{3}, 1\right] \times \pi(2\ell^2 - \sqrt{2\alpha}Q), \end{aligned} \tag{39}$$

for which, the parameter α always diminishes the bounds for horizons. These entropy bounds bring a way for the geometrical understanding of entropy relations in AdS spacetime.

Regular and singular Hayward AdS black holes. When $\nu = \mu$, the solution belongs to the Hayward family of AdS black holes. The metric function reduces to

$$f(r) = \frac{r^2}{\ell^2} + k - \frac{2M_{eff}}{r} - \frac{2\alpha^{-1}q^3 r^{\mu-1}}{(r^\mu + q^\mu)}. \tag{40}$$

The equation of horizons could be simplified as

$$\begin{aligned} &\frac{r^{\mu+3}}{\ell^2} + kr^{\mu+1} - 2M_{ADM}r^\mu \\ &+ \frac{q^\mu r^3}{\ell^2} + kq^\mu r - 2M_{eff}q^\mu = 0. \end{aligned} \tag{41}$$

Then one can derive some useful relations for the horizons r_i $(i = 0, \ldots, (\mu+3))$ when $\mu \geq 3$

$$\begin{aligned} &\prod_{i=1}^{\mu+3} r_i = 2M_{eff}q^\mu\ell^2, \qquad \prod_{1\leq i<j\leq(\mu+3)} r_i r_j = k\ell^2, \\ &\sum_{i=1}^{\mu+3} r_i = 0, \end{aligned} \tag{42}$$

which directly lead to entropy relations

$$\prod_{i=1}^{\mu+3} S_i = \pi^{\mu+3}\left(\prod_{i=1}^{\mu+3} r_i\right)^2 = \pi^{\mu+3}(2M_{\mathit{eff}}\, q^\mu \ell^2)^2$$
$$= (2M_{\mathit{eff}}\,\ell^2)^2(\sqrt{2\alpha}Q)^\mu \pi^{\mu+3}, \tag{43}$$

$$\sum_{i=1}^{\mu+3} S_i = \pi\left(\sum_{i=1}^{\mu+3} r_i\right)^2 - 2\pi \prod_{1\le i<j\le(\mu+3)} r_i r_j$$
$$= -2\,k\ell^2\pi. \tag{44}$$

Here, the entropy product depends on the mass, while the entropy sum is mass-independent, which is different from the cases in asymptotically flat spacetime. Especially for singular cases with $\mu = 1, 2$, this property still holds while the results are modified slightly.

In a word, in asymptotically AdS spacetime, it is shown that the entropy sum is always mass-independent when the effect of the "virtual" horizons is included.

Conclusion. – In this paper, we study entropy relations and bounds of regular and singular black holes with nonlinear electrodynamics sources. We calculate them for the regular and singular Bardeen and Hayward black holes in asymptotically flat and AdS spacetimes. There are some interesting features as follows:

– For black holes with vanishing effective mass M_{eff}, the entropy product is always mass-independent in asymptotically flat spacetime, while the entropy sum is always mass-independent in asymptotically AdS spacetime. However, now it is not clear to find the potential physical significance of this opposite dependence of entropy relations. For both cases, the effect of the "virtual" horizons must be included, in order to keep this universality.

– For the cases with nonzero effective mass M_{eff} in asymptotically flat and AdS spacetimes, entropy relations are always mass-dependent and are actually some functions of $(M_{ADM}, M_{\mathit{eff}}, Q)$.

– The nonlinear electrodynamics always affects the entropy bounds of horizons, which result from the entropy relations. Especially in asymptotically flat spacetime, the upper bound of the area for the event horizon is the Penrose-like inequality for Bardeen black holes, while it is the exact Penrose inequality for Hayward black holes. This sheds some lights on the geometric understanding of the universal entropy relations of regular and singular black holes with nonlinear electrodynamics sources.

For the future works, it is interesting to study the universal mass independence for general regular and singular black holes with nonlinear electrodynamics sources, including the rotating black holes, in order to reveal some viewpoints about the microscopic origin of the black-hole entropy. One can also try to find the forms as to how the black-hole entropy could be quantized in the black-hole thermodynamic framework, based on the mass-independent entropy product or sum. Besides, studying the entropy bounds of general (AdS) black holes with nonlinear electrodynamics sources will bring a further understanding of their geometrical properties.

* * *

JW was supported by the National Natural Science Foundation of China (NSFC) under Grant No. U1738130. WX was supported by the National Natural Science Foundation of China (NSFC) under Grants No. 11505065, No. 11374330 and No. 91636111, and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan).

REFERENCES

[1] Guica M., Hartman T., Song W. and Strominger A., *Phys. Rev. D*, **80** (2009) 124008.
[2] Compére G., *Living Rev. Relativ.*, **15** (2012) 11; **20** (2017) 1.
[3] Sinamuli M. and Mann R. B., *JHEP*, **08** (2016) 148.
[4] Solodukhin S. N., *Living Rev. Relativ.*, **14** (2011) 8.
[5] Giaccari S., Modesto L., Rachwal L. and Zhu Y., *Eur. Phys. J. C*, **78** (2018) 459.
[6] Cvetic M., Gibbons G. W. and Pope C. N., *Phys. Rev. Lett.*, **106** (2011) 121301.
[7] Castro A. and Rodriguez M. J., *Phys. Rev. D*, **86** (2012) 024008.
[8] Detournay S., *Phys. Rev. Lett.*, **109** (2012) 031101.
[9] Wang J., Xu W. and Meng X. H., *JHEP*, **01** (2014) 031.
[10] Visser M., *JHEP*, **06** (2012) 023.
[11] Chen B., Liu S. X. and Zhang J. J., *JHEP*, **11** (2012) 017.
[12] Bagchi A., Detournay S., Fareghbal R. and Simón J., *Phys. Rev. Lett.*, **110** (2013) 141302.
[13] Visser M., *Phys. Rev. D*, **88** (2013) 044014.
[14] Toldo C. and VandOREN S., *JHEP*, **09** (2012) 048.
[15] Faraoni V. and Moreno A. F. Z., *Phys. Rev. D*, **88** (2013) 044011.
[16] Townsend P. K. and Zhang B., *Phys. Rev. Lett.*, **110** (2013) 241302.
[17] Castro A., Dehmami N., Giribet G. and Kastor D., *JHEP*, **07** (2013) 164.
[18] Lu H., *JHEP*, **09** (2013) 112.
[19] Cvetic M., Lu H. and Pope C. N., *Phys. Rev. D*, **88** (2013) 044046.
[20] Lü H., Pang Y. and Pope C. N., *JHEP*, **11** (2013) 033.
[21] Anacleto M. A., Brito F. A. and Passos E., *Phys. Lett. A*, **380** (2016) 1105.
[22] Chow D. D. K. and Compére G., *Class. Quantum Grav.*, **31** (2014) 022001.
[23] Xu W., Wang J. and Meng X. H., *Int. J. Mod. Phys. A*, **29** (2014) 1450172.

[24] Wang J., Xu W. and Meng X. H., *Phys. Rev. D*, **89** (2014) 044034.
[25] Xu W., Wang J. and Meng X. H., *Int. J. Mod. Phys. A*, **29** (2014) 1450088.
[26] Xu W., Wang J. and Meng X. H., *Galaxies*, **3** (2015) 53.
[27] Du Y. Q. and Tian Y., *Phys. Lett. B*, **739** (2014) 250.
[28] Zhang Y. and Gao S., *Phys. Rev. D*, **91** (2015) 064032.
[29] Pradhan P., *Eur. Phys. J. C*, **76** (2016) 131.
[30] Page D. N. and Shoom A. A., *Phys. Rev. D*, **92** (2015) 044039.
[31] Pradhan P., *Phys. Lett. B*, **747** (2015) 64.
[32] Liu H. and Meng X. H., *Phys. Lett. B*, **759** (2016) 253.
[33] Yekta Mahdavian D., *Phys. Rev. D*, **95** (2017) 064027.
[34] Chen B. and Zhang J.-J., *JHEP*, **11** (2012) 022.
[35] Chen B. and Zhang J.-J., *JHEP*, **01** (2013) 155.
[36] Chen B. and Zhang J.-J., *Phys. Rev. D*, **87** (2013) 081505.
[37] Chen B., Xue Z. and Zhang J.-J., *JHEP*, **03** (2013) 102.
[38] Chen B., Zhang J.-J., Zhang J.-D. and Zhong D.-L., *JHEP*, **04** (2013) 055.
[39] Chen B. and Zhang J.-J., *Int. J. Mod. Phys. D*, **22** (2013) 1342012.
[40] Ayon-Beato E. and Garcia A., *Phys. Rev. Lett.*, **80** (1998) 5056.
[41] Ayon-Beato E. and Garcia A., *Phys. Lett. B*, **464** (1999) 25.
[42] Ayon-Beato E. and Garcia A., *Phys. Lett. B*, **493** (2000) 149.
[43] Fan Z. Y. and Wang X., *Phys. Rev. D*, **94** (2016) 124027.
[44] Bardeen J. M., in *Conference Proceedings of GR5, Tbilisi, USSR, 1968*, p. 174.
[45] Borde A., *Phys. Rev. D*, **50** (1994) 3692.
[46] Barrabes C. and Frolov V. P., *Phys. Rev. D*, **53** (1996) 3215.
[47] Bogojevic A. and Stojkovic D., *Phys. Rev. D*, **61** (2000) 084011.
[48] Cabo A. and Ayon-Beato E., *Int. J. Mod. Phys. A*, **14** (1999) 2013.
[49] Hayward S. A., *Phys. Rev. Lett.*, **96** (2006) 031103.
[50] Bambi C. and Modesto L., *Phys. Lett. B*, **721** (2013) 329.
[51] Ghosh S. G. and Maharaj S. D., *Eur. Phys. J. C*, **75** (2015) 7.
[52] Toshmatov B., Ahmedov B., Abdujabbarov A. and Stuchlík Z., *Phys. Rev. D*, **89** (2014) 104017.
[53] Junior E. L. B., Rodrigues M. E. and Houndjo M. J. S., *JCAP*, **10** (2015) 060.
[54] Bambi C., Malafarina D. and Modesto L., *Eur. Phys. J. C*, **74** (2014) 2767.
[55] Zhang Y., Zhu Y., Modesto L. and Bambi C., *Eur. Phys. J. C*, **75** (2015) 96.
[56] Bambi C., Malafarina D. and Modesto L., *JHEP*, **04** (2016) 147.
[57] Bambi C., Modesto L. and Wang Y., *Phys. Lett. B*, **764** (2017) 306.
[58] Bambi C., Modesto L., Porey S. and Rachwal L., *JCAP*, **09** (2017) 033.
[59] Bambi C., Modesto L., Porey S. and Rachwal L., *Eur. Phys. J. C*, **78** (2018) 116.
[60] Zhang Q., Modesto L. and Bambi C., *Eur. Phys. J. C*, **78** (2018) 506.
[61] Hawking S. W. and Ellis G. F. R., *The Large Scale Structure of Space-Time* (Cambridge University Press) 1973.
[62] Heydarzade Y. and Darabi F., *Phys. Lett. B*, **771** (2017) 365.
[63] Lin K. and Qian W. L., arXiv:1812.10100 [gr-qc].
[64] Myung Y. S., Kim Y. W. and Park Y. J., *Phys. Lett. B*, **659** (2008) 832.
[65] Myung Y. S., Kim Y. W. and Park Y. J., *Gen. Relativ. Gravit.*, **41** (2009) 1051.
[66] Xu W., Wang J. and Meng X. H., *Phys. Lett. B*, **746** (2015) 53.
[67] Xu W., Wang J. and Meng X. H., *Phys. Lett. B*, **742** (2015) 225.
[68] Liu H., Meng X. H., Xu W. and Zhu B., *EPL*, **119** (2017) 20003.
[69] Debnath U., *Ann. Phys.*, **372** (2016) 449.
[70] Mars M., *Class. Quantum Grav.*, **26** (2009) 193001.
[71] Dain S. and Gabach-Clement M. E., *Living Rev. Relativ.*, **21** (2018) 5.
[72] Fan Z. Y., *Eur. Phys. J. C*, **77** (2017) 266.
[73] Toshmatov B., Stuchlík Z. and Ahmedov B., *Phys. Rev. D*, **98** (2018) 028501.