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LETTER

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Aharonov-Bohm effect on a generalized Klein-Gordon oscillator with uniform magnetic field in a spinning cosmic string space-time

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Abstract – In this work, we study a generalized Klein-Gordon oscillator field on the background space-time induced by spinning cosmic string coupled to a homogeneous magnetic field including a magnetic quantum flux. We solve the generalized Klein-Gordon oscillator equation in the considered system and obtain the energy eigenvalues and eignfunctions and analyze a relativistic analogue of the Aharonov-Bohm effect.

Introduction. – The one-dimensional spinning cosmic strings are characterized by a wedge parameter α and an angular momentum J. Spinning cosmic strings are the counterpart of static cosmic strings that would arise in the early universe [1] and were studied in the context of Cartan-Einstein's theory [2,3] and teleparallel gravity [4].

The relativistic wave equations have been studied in physics [5–8]. Landau levels of a particle localized in spinning cosmic strings space-time were investigated in [9,10]. Several authors have studied the spinning cosmic string space-time, such as the Dirac oscillator [11], scalar charged particle with an external field and potential [12], Klein-Gordon oscillator with an external field [13], Klein-Gordon scalar field with a Cornell-type potential [14], motion of a quantum particle [15], spin-0 relativistic scalar particle [16]. On the other hand, static cosmic string space-time has also been studied in the relativistic quantum system (*e.g.*, [17–21]).

Our motivation is to analyze a relativistic analogue of the Aharonov-Bohm effect for bound states [22,23] of a relativistic scalar particle in a spinning cosmic string space-time subject to a homogeneous magnetic field. The generalized Klein-Gordon oscillator field is coupled covariantly with an electromagnetic field including a magnetic quantum flux and we solve this equation which was not studied earlier [13,20,24,25]. We solve the generalized Klein-Gordon oscillator subject to a homogeneous magnetic field including a magnetic quantum flux in the spinning cosmic string space-time and evaluate the energy eigenvalues and eigenfunctions.

Generalized KG-oscillator in spinning cosmic string space-time. – The relativistic quantum dynamics of a spin-0 particle of mass m is described by the following equation [19]:

$$\frac{1}{\sqrt{-g}}\, D_\mu(\sqrt{-g}\, g^{\mu\nu}\, D_\nu\, \Psi) = \left(\frac{m\, c}{\hbar}\right)^2 \Psi, \tag{1}$$

where $D_\mu = \partial_\mu - \frac{i\, e}{\hbar\, c}\, A_\mu$, e is the electric charge and A_μ is the electromagnetic four-vector potential.

We consider the electromagnetic four-vector potential $A_\mu = (0, 0, A_\phi, 0)$ with [12,26–28]

$$A_\phi = -\frac{1}{2}\, \alpha\, B_0\, r^2 + \frac{\Phi_B}{2\, \pi}, \tag{2}$$

such that the applied magnetic field is $\vec{B} = -B_0\, \hat{k}$. Here Φ_B=const is the internal magnetic quantum flux [29,30] through the core of topological defects [30,31]. It is noteworthy that the Aharonov-Bohm effect has been investigated in several branches of physics, such as in graphene [32], Newtonian theory [33], bound states of massive fermions [34], scattering of dislocated wavefronts [35],

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torsion effects on a relativistic position-dependent mass system [27,28], the Kaluza-Klein theory [24,36–41].

Consider the following spinning cosmic-string space-time [10–12,15,42–44]:

$$ds^2 = -(c\,dt + a\,d\phi)^2 + \alpha^2\,r^2\,d\phi^2 + dr^2 + dz^2. \qquad (3)$$

Here $a = \frac{4\,G\,J}{c^3}$ is the rotation parameter and has units of distance, J is the angular parameter, and $\alpha = 1 - \frac{4\,\mu\,G}{c^2}$ is the wedge parameter which determines the angular deficit, $\nabla\,\phi = 2\,\pi\,(1-\alpha)$. The letters c, $\hbar$, G, and μ stand for the speed of light, Planck constant, gravitational Newton constant, and linear mass density of the string.

The determinant of the corresponding metric tensor (3) is

$$\det g = -c^2\,r^2\,\alpha^2. \qquad (4)$$

The co-variant and contra-variant form of the metric tensor are

$$g_{\mu\,\nu} = \begin{pmatrix} -c^2 & 0 & -a\,c & 0 \\ 0 & 1 & 0 & 0 \\ -a\,c & 0 & -a^2 + r^2\,\alpha^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g^{\mu\,\nu} = \begin{pmatrix} \frac{a^2 - r^2\,\alpha^2}{c^2\,r^2\,\alpha^2} & 0 & -\frac{a}{c\,r^2\,\alpha^2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{a}{c\,r^2\,\alpha^2} & 0 & \frac{1}{r^2\,\alpha^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (5)$$

Now, let us consider a Klein-Gordon oscillator [45,46] coupled to this background. A Klein-Gordon oscillator is obtained from the Klein-Gordon equation by the replacement of four-momentum as [25,47]

$$p_\mu \to (p_\mu + i\,M\,\omega\,X_\mu), \qquad (6)$$

where $X_\mu = (0, r, 0, 0)$ with $\vec{r} = r\,\hat{r}$, r being the distance from the particle to the string. Thus the covariant form of the Klein-Gordon oscillator is given by

$$\Big[\frac{1}{\sqrt{-g}}\,(D_\mu + m\,\omega\,X_\mu)\,\{\sqrt{-g}\,g^{\mu\nu}\,(D_\nu - m\,\omega\,X_\nu)\} - \Big(\frac{m\,c}{\hbar}\Big)^2\Big]\,\Psi = 0, \qquad (7)$$

where ω is the oscillator frequency.

To generalize the above Klein-Gordon oscillator field, we replace r by a function $f(r)$ into the vector X_μ defined as [41,48–50]

$$X_\mu = (0, f(r), 0, 0), \qquad (8)$$

where we have chosen the function [41,48–50]

$$f(r) = c_1\,r + \frac{c_2}{r}, \qquad c_1 > 0, \qquad c_2 > 0. \qquad (9)$$

For the line element (3), KG-oscillator (7) with eq. (9) becomes

$$\Big[-\Big(\frac{1}{c}\partial_t\Big)^2 + \Big\{\frac{a}{cr\alpha}\partial_t - \frac{1}{\alpha r}\Big(\partial_\phi - \frac{ie}{c\hbar}A_\phi\Big)\Big\}^2 + \frac{1}{r}\partial_r(r\partial_r) - 2m\omega\,c_1 - m^2\omega^2\Big(c_1 r + \frac{c_2}{r}\Big)^2 + \partial_z^2 - \Big(\frac{mc}{\hbar}\Big)^2\Big]\,\Psi = 0. \qquad (10)$$

Since the metric is independent of t, ϕ, z, one can choose the following ansatz for the function Ψ

$$\Psi(t, r, \phi, z) = e^{i\,(-\frac{E}{\hbar}\,t + l\,\phi + k\,z)}\,\psi(r), \qquad (11)$$

where E is the total energy, and $l = 0, \pm 1, \pm 2, \ldots$ are the eigenvalues of the z-component of the angular momentum operator, and k is a constant.

Using the above ansatz eq. (11), we obtain the following equation:

$$\psi''(r) + \frac{1}{r}\psi'(r) + \Big[\frac{E^2}{c^2\hbar^2} - \frac{1}{\alpha^2 r^2}\Big(\frac{aE}{c\hbar} + l - \frac{e}{c\hbar}A_\phi\Big)^2 - 2m\omega c_1 - m^2\,\omega^2\,\Big(c_1\,r + \frac{c_2}{r}\Big)^2 - k^2 - \Big(\frac{m\,c}{\hbar}\Big)^2\Big]\,\psi(r) = 0. \qquad (12)$$

Considering the angular component of four-vector potential (2) into the eq. (12), we obtain the following differential equation for $\psi(r)$:

$$\psi''(r) + \frac{1}{r}\,\psi(r) + \Big[\lambda - m^2\,\Omega^2\,r^2 - \frac{\tilde{j}^2}{r^2}\Big]\,\psi(r) = 0, \qquad (13)$$

where

$$\begin{aligned} \lambda &= \Big(\frac{E}{c\,\hbar}\Big)^2 - \Big(\frac{m\,c}{\hbar}\Big)^2 - k^2 - \frac{2\,m\,\omega_c}{\hbar\,c}\,j \\ &\quad -2\,m\,\omega\,c_1 - 2\,m^2\,\omega^2\,c_1\,c_2, \\ \Omega &= \sqrt{\omega^2\,c_1^2 + \frac{\omega_c^2}{\hbar^2\,c^2}}, \\ \tilde{j} &= \sqrt{j^2 + m^2\,\omega^2\,c_2^2}, \\ j &= \frac{1}{\alpha}\,\Big(\frac{a\,E}{\hbar\,c} + l - \frac{\Phi}{\hbar\,c}\Big), \\ \Phi &= \frac{\Phi_B}{(2\,\pi/e)}, \\ \omega_c &= \frac{e\,B_0}{2\,m} \end{aligned} \qquad (14)$$

is called the cyclotron frequency of the particle moving in the magnetic field.

Let us introduce a new variable $x = m\,\Omega\,r^2$, then eq. (13) becomes [51]

$$\psi''(x) + \frac{1}{x}\,\psi'(x) + \frac{1}{x^2}\,(-\xi_1\,x^2 + \xi_2\,x - \xi_3)\,\psi(x) = 0, \qquad (15)$$

where

$$\xi_1 = \frac{1}{4}, \qquad \xi_2 = \frac{\lambda}{4\,m\,\Omega}, \qquad \xi_3 = \frac{\tilde{j}^2}{4}. \qquad (16)$$

By comparing eq. (15) with (B.1) in appendix B in [47,48], we get

$$
\begin{aligned}
&\alpha_1 = 1,\ \ \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0, \quad \alpha_5 = 0,\ \ \alpha_6 = \xi_1,\\
&\alpha_7 = -\xi_2, \quad \alpha_8 = \xi_3, \quad \alpha_9 = \xi_1, \quad \alpha_{10} = 1 + 2\sqrt{\xi_3},\\
&\alpha_{11} = 2\sqrt{\xi_1}, \quad \alpha_{12} = \sqrt{\xi_3}, \quad \alpha_{13} = -\sqrt{\xi_1}.
\end{aligned} \tag{17}
$$

Therefore, the second degree energy eigenvalues equation using eqs. (14), (15) into eq. (B.8) in appendix B in [47,48] is given by

$$
\begin{aligned}
&\left(\frac{E_{n,l}}{c\,\hbar}\right)^2 - \frac{2\,m\,\omega_c}{\alpha\,\hbar^2\,c^2}\,(\hbar\,c\,l - \Phi + a\,E_{n,l})\\
&-\frac{2\,m\,\Omega}{\hbar\,c}\,|\hbar\,c\,l - \Phi + a\,E_{n,l}| = \left(\frac{m\,c}{\hbar}\right)^2\\
&+k^2 + 2\,m\,\Omega\,(2\,n+1)\\
&+2\,m\,\omega\,c_1 + 2\,m^2\,\omega^2\,c_1\,c_2,
\end{aligned} \tag{18}
$$

where $n = 0, 1, 2, \ldots$.

Equation (18) is the compact expression of the relativistic energy eigenvalues of a generalized Klein-Gordon oscillator particle subject to a uniform magnetic field including magnetic quantum flux in the spinning cosmic string space-time.

The corresponding wave-function is given by

$$
\begin{aligned}
\psi_{n,l}(x) &= x^{\frac{j}{2}}\,e^{-\frac{x}{2}}\,L_n^{(j)}(x)\\
&= x^{\frac{1}{2}\sqrt{\frac{1}{\alpha^2}\,(l-\frac{\Phi}{\hbar\,c}+\frac{a\,E_{n,l}}{\hbar\,c})^2+m^2\,\omega^2\,c_2^2}}\,e^{-\frac{x}{2}}\\
&\quad\times L_n^{(\sqrt{\frac{1}{\alpha^2}\,(l-\frac{\Phi}{\hbar\,c}+\frac{a\,E_{n,l}}{\hbar\,c})^2+m^2\,\omega^2\,c_2^2})}(x).
\end{aligned} \tag{19}
$$

For zero rotation, $a \to 0$, the energy eigenvalues eq. (18) becomes

$$
\begin{aligned}
E_{n,l} = \pm\hbar c\Bigg\{&\frac{2m\omega_c}{\alpha\hbar c}\left(l - \frac{\Phi}{\hbar c}\right) + 2m\Omega\left(2n+1+\left|l - \frac{\Phi}{\hbar\,c}\right|\right)\\
&+\left(\frac{mc}{\hbar}\right)^2 + k^2 + 2\,m\,\omega\,c_1\,(1 + m\,\omega\,c_2)\Bigg\}^{\frac{1}{2}}.
\end{aligned} \tag{20}
$$

Equation (20) is the relativistic energy eigenvalues of a generalized Klein-Gordon oscillator field in the presence of a uniform magnetic field including a magnetic quantum flux in the static cosmic string space-time.

Special case. Now we discuss a special case corresponding to $c_1 \to 1$ and $c_2 \to 0$. In that case, the considered system reduces to the Klein-Gordon oscillator field in a spinning cosmic string space-time.

The radial wave equation for $\psi(r)$ becomes

$$
\psi''(r) + \frac{1}{r}\,\psi(r) + \left[\lambda_0 - m^2\,\Omega_0^2\,r^2 - \frac{j^2}{r^2}\right]\,\psi(r) = 0, \tag{21}
$$

where

$$
\begin{aligned}
\lambda_0 &= \left(\frac{E}{c\,\hbar}\right)^2 - \left(\frac{m\,c}{\hbar}\right)^2 - k^2 - \frac{2\,m\,\omega_c}{\hbar\,c}\,j - 2\,m\,\omega,\\
\Omega_0 &= \frac{1}{\hbar\,c}\sqrt{\hbar^2\,c^2\,\omega^2 + \omega_c^2}.
\end{aligned} \tag{22}
$$

Following a similar technique as done earlier, we obtain the following second degree algebraic equation for $E_{n,l}$:

$$
\begin{aligned}
&\left(\frac{E_{n,l}}{c\,\hbar}\right)^2 - \frac{2\,m\,\omega_c}{\alpha\,\hbar^2\,c^2}\,(\hbar\,c\,l - \Phi + a\,E_{n,l})\\
&-\frac{2\,m\,\Omega_0}{\alpha\,\hbar\,c}\,|\hbar\,c\,l - \Phi + a\,E_{n,l}| = \left(\frac{m\,c}{\hbar}\right)^2\\
&+k^2 + 2\,m\,\Omega_0\,(2\,n+1) + 2\,m\,\omega
\end{aligned} \tag{23}
$$

with its solution

$$
E_{n,l} = -\left(\frac{m\,\omega_c\,a}{\alpha} - \frac{\hbar\,c\,a\,|l - \frac{\Phi}{\hbar\,c}|}{\alpha\,(l - \frac{\Phi}{\hbar\,c})}\,m\,\Omega_0\right) \pm \sqrt{\delta_1 + \delta_2}, \tag{24}
$$

where $n = 0, 1, 2, \ldots$ and

$$
\begin{aligned}
\delta_1 &= \frac{1}{4}\left(\frac{2\,m\,\omega_c\,a}{\alpha} - \frac{2\,\hbar\,c\,a\,|l - \frac{\Phi}{\hbar\,c}|}{\alpha\,(l - \frac{\Phi}{\hbar\,c})}\,m\,\Omega_0\right)^2\\
&\quad +2\,m\,\omega\,\hbar^2\,c^2 + \hbar^2\,c^2\,k^2 + m^2\,c^4,\\
\delta_2 &= 2\,\hbar^2\,c^2\,(2\,n+1)\,m\,\Omega_0\\
&\quad +\left(-\frac{2m\omega_c\hbar c(l - \frac{\Phi}{\hbar c})}{\alpha} + 2\hbar^2 c^2\,\frac{|l - \frac{\Phi}{\hbar\,c}|^2}{\alpha(l - \frac{\Phi}{\hbar\,c})}\,m\,\Omega_0\right).
\end{aligned} \tag{25}
$$

We can see that the energy eigenvalues $E_{n,l}$ depend explicitly on the rotational parameter a, the wedge parameter α which characterizes the metric in a spinning cosmic string space-time, and the magnetic quantum flux Φ_B.

The corresponding wave function is given by

$$
\begin{aligned}
\psi_{n,l}(x) &= x^{\frac{1}{2\,\hbar\,c\,\alpha}\,(\hbar\,c\,l - \Phi + a\,E_{n,l})}\\
&\quad\times e^{-\frac{x}{2}}\,L_n^{\left(\frac{1}{\hbar\,c\,\alpha}\,(\hbar\,c\,l - \Phi + a\,E_{n,l})\right)}(x),
\end{aligned} \tag{26}
$$

where $L_n^{(\beta)}(x)$ is the generalized Laguerre polynomial.

For zero magnetic quantum flux, $\Phi_B \to 0$, the energy eigenvalues in eqs. (24), (25) reduce to the result obtained in [13]. Thus, we can see that the energy eigenvalues in eqs. (24), (25) are the extended result in comparison to those in [13] due to the presence of a magnetic quantum flux.

For zero rotation of the space-time, $a \to 0$, the energy eigenvalues from eq. (23) become

$$
\begin{aligned}
E_{n,l} = \pm\hbar c\Bigg\{&\frac{2m\omega_c}{\alpha\hbar c}\left(l - \frac{\Phi}{\hbar c}\right) + \left(\frac{mc}{\hbar}\right)^2 + k^2\\
&+2m\Omega_0\left(2n+1+\frac{|l - \frac{\Phi}{\hbar c}|}{\alpha}\right) + 2m\omega\Bigg\}^{\frac{1}{2}}.
\end{aligned} \tag{27}
$$

For $\Phi_B \to 0$, the energy eigenvalues eq. (27) reduces to the result obtained in [20]. For $B_0 \to 0$ (or $\omega_c \to 0$) and $\Phi_B \to 0$, the energy eigenvalues eq. (27) reduces to the result in [20] and also in [24] provided $\lambda = 0$ there. For $B_0 \to 0$, $\Phi_B \to 0$ and $\alpha \to 1$, one will recover from eq. (27) the energy spectrum of a Klein-Gordon oscillator field in flat space metric [25]. Thus, we can see that the energy eigenvalues eq. (27) are the modified result due to the presence of a magnetic quantum flux Φ_B, the external magnetic field B_0 as well as the wedge parameter α which causes shifts in the energy levels.

We can see that the relativistic energy eigenvalues obtained above depend on the geometric quantum phase [29,30]. Thus, we have that $E_{n,l}(\Phi_B + \Phi_0) = E_{n,l\mp\tau}(\Phi_B)$ where, $\Phi_0 = \pm\frac{2\pi c\hbar}{e}\tau$ with $\tau = 0, 1, 2 \ldots$. This dependence of the relativistic energy eigenvalues on the geometric quantum phase Φ gives rise to a relativistic analogue of the Aharonov-Bohm effect [22,23].

Conclusions. – In this paper, we study a generalized Klein-Gordon oscillator with electromagnetic field (B_0) including a magnetic quantum flux (Φ_B) in a spinning cosmic string space-time. In the second section, we have solved the generalized Klein-Gordon oscillator equation under the considered systems and obtained a compact expression of the relativistic energy eigenvalues, eq. (18), and eigenfunctions, eq. (19). There, we have obtained the energy eigenvalues in eq. (20) in a static cosmic string space-time. We have shown that the energy eigenvalues depend explicitly on the rotational parameter a, and the wedge parameter α which characterize the global structure of the metric. Furthermore, we have discussed a special case corresponding to $c_1 \to 1$ and $c_2 \to 0$ in this system. We have solved the equation and obtained the relativistic energy eigenvalues in eqs. (24), (25). For zero magnetic quantum flux, $\Phi_B \to 0$, the energy eigenvalues in eqs. (24), (25) reduce to the result obtained in [13] (see eq. (13) in [13]). For zero rotation of the space-time, we have obtained the energy eigenvalues in eq. (27) and have seen that for zero magnetic flux, $\Phi_B \to 0$, these eigenvalues reduce to the result obtained in [20]. Also, for zero rotation of the space-time, $a \to 0$, no external magnetic field, $B \to 0$, and zero magnetic quantum flux, $\Phi_B \to 0$, these relativistic energy eigenvalues reduce to the results in [20] and also in [24].

We have seen that the relativistic energy eigenvalues depend on the geometric quantum phase [29,30]. Thus, we have that $E_{n,l}(\Phi_B + \Phi_0) = E_{n,l\mp\tau}(\Phi_B)$, where $\Phi_0 = \pm\frac{2\pi\hbar c}{e}\tau$ with $\tau = 0, 1, \ldots$. This dependence of the relativistic energy eigenvalues on the geometric quantum phase gives rise to a relativistic analogue of the Aharonov-Bohm effect for bound states [22,23]. It has also been shown that the presence of topological defects of the space-time, and magnetic quantum flux shifted the energy levels of the quantum system in comparison to the results known in the literature.

* * *

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