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Intermittency, cascades and thin sets in three-dimensional Navier-Stokes turbulence

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Abstract – Visual manifestations of intermittency in computations of three-dimensional Navier-Stokes fluid turbulence appear as the low-dimensional or "thin" filamentary sets on which vorticity and strain accumulate as energy cascades down to small scales. In order to study this phenomenon, the first task of this paper is to investigate how weak solutions of the Navier-Stokes equations can be associated with a cascade and, as a consequence, with an infinite sequence of inverse length scales. It turns out that this sequence converges to a finite limit. The second task is to show how these results scale with integer dimension $D = 1, 2, 3$ and, in the light of the occurrence of thin sets, to discuss the mechanism of how the fluid might find the smoothest, most dissipative class of solutions rather than the most singular.

Introduction. – The most striking visual manifestation of intermittency in three-dimensional incompressible fluid turbulence is the accumulation of vorticity and strain on "thin" or low-dimensional sets. When displayed graphically as iso-surfaces in a cube, these sets typically appear as spaghetti-like entangled tubular filaments: see fig. 1 where snapshots of the energy dissipation field ε, and the Q-field (defined in the caption) of a forced Navier-Stokes flow are displayed. Although differing in the fine detail from case to case, initial three-dimensional vortical structures tend to flatten at intermediate times into quasi–two-dimensional pancakes which subsequently roll up into quasi–one-dimensional tubes, with further iterations of flattening and filamentation resulting in ever finer striations [1–11]: see the recent paper by Elsingha, Ishihara and Hunt [12]. Historically, Batchelor and Townsend [13] were the first to suggest that vorticity and strain are not distributed in a Gaussian fashion across a domain but accumulate on local, intense sets which they identified with intermittency in the energy dissipation [14–18]. In the literature these filamentary structures are loosely referred to as "fractal" because of the roughness of the detail of their evolving fine-scale structure: see fig. 1 for an illustration. In the last generation, various cascade models, such as the beta, bi-fractal and multi-fractal models, explicitly talk about accumulation on sets of non-integer dimension D [16]. Studies in *Fourier decimation* have pursued the idea of intermittency more precisely by projecting a three-dimensional Navier-Stokes velocity field onto a chosen subset of Fourier modes by employing a generalized Galerkin projector [19,20]. Intermittency properties have then been investigated by tuning both the restricted subset and the Reynolds number. Recent work has culminated in making this restricted set fractal [21–24]: in effect, the number of degrees of freedom are limited to a sphere of radius k growing as k^D (for non-integer D) embedded in the three-dimensional space. Reference [24] contains an excellent set of references. However, from the point of view of rigorous Navier-Stokes analysis, the phenomenon is by no means understood, mainly because technical tools exist to pursue analysis only on fixed domains of integer dimension but not on time-evolving fractal sets. In this context, many questions remain outstanding. Is there a universal value of D or are there many disjoint sets of differing dimension? How does this fit in with the idea of a cascade to small scales and the regularity of solutions of the Navier-Stokes equations at these scales? For instance, Biferale and Titi [25] have shown that a helically decimated version of the $3D$ Navier-Stokes equations leads to global regularity. A global theory that answers all these questions still remains elusive: this paper aims to build on what is known rigorously for three-dimensional Navier-Stokes

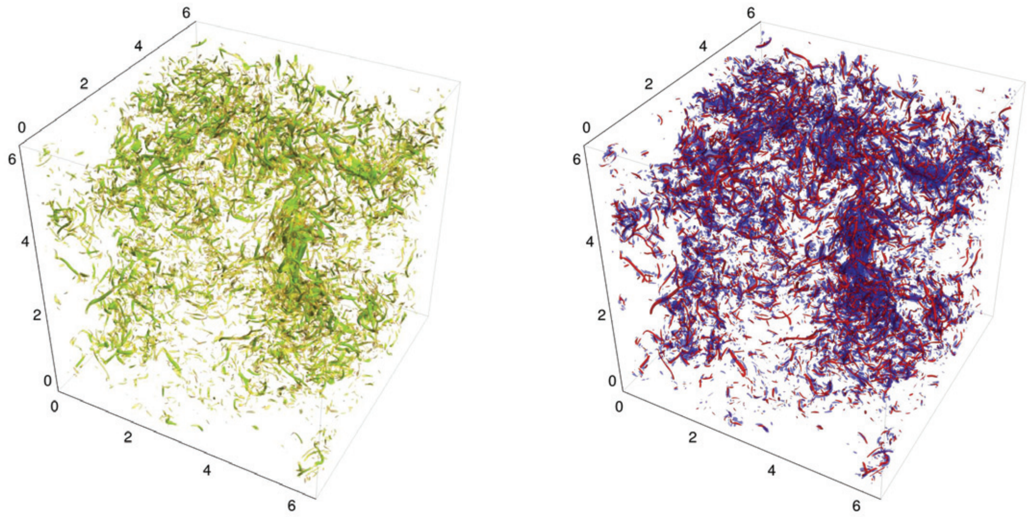

Fig. 1: The left-hand figure is a snapshot of the energy dissipation field $\varepsilon = 2\nu S_{i,j}S_{j,i}$ of a forced 512^3 Navier-Stokes flow at $Re_\lambda = 196$ which is colour-coded such that yellow is 4 times the mean and blue denotes 6 times the mean. The right-hand figure shows the field $Q = \frac{1}{2}(|\boldsymbol{\omega}|^2 - |S|^2)$: the colours correspond to $-2Q_{rms}$ (blue) and $5Q_{rms}$ (red). Plots courtesy of J. R. Picardo and S. S. Ray.

equations to discuss how this fractal set might occur.

Cascades, scaling and weak solution estimates in three dimensions. – A cascade is a sequential process that involves vorticity and strain being driven down to ever smaller length scales in the flow and has long been closely associated with intermittency [16,26–28]. For sufficiently long times a cascade to smaller scales should show up in estimates of both spatially and temporally averaged gradients of a divergence-free velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ that evolve according to the Navier-Stokes equations

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}). \tag{1}$$

The domain $\mathcal{V} = [0,L]^3_{per}$ is chosen to be three-dimensional and periodic. ν is the viscosity and $\boldsymbol{f}$ is an L^2-bounded forcing. Here we show that Leray's weak solutions of the Navier-Stokes equations [29] can be interpreted in terms of a cascade.

We define a doubly labelled set of norms in dimensionless form

$$F_{n,m} = \nu^{-1}L^{1/\alpha_{n,m}}\|\nabla^n\boldsymbol{u}\|_{2m}, \tag{2}$$

where $\alpha_{n,m}$ is defined by

$$\alpha_{n,m} = \frac{2m}{2m(n+1)-3}. \tag{3}$$

The norm notation $\|\cdot\|_{2m}$ in (2) is defined by

$$\|\nabla^n\boldsymbol{u}\|_{2m} = \left(\int_{\mathcal{V}}|\nabla^n\boldsymbol{u}|^{2m}\mathrm{d}V\right)^{1/2m}. \tag{4}$$

Higher values of n allow the detection of smaller scales, while higher values of m account for stronger deviations from the mean, with $m=\infty$ representing the maximum norm.

The Navier-Stokes equations are well known to possess the scale-invariance property

$$\boldsymbol{u}(\boldsymbol{x},t) \to \lambda^{-1}\boldsymbol{u}(\boldsymbol{x}/\lambda, t/\lambda^2), \tag{5}$$

for any value of the dimensionless parameter λ. Under this scaling the $F_{n,m}$ in (2) are invariant in λ and are thus invariant at every length and time scale in the flow. This makes them invaluable as a tool for investigating a cascade of energy through the system. This is further illustrated by the fact that there exists a bounded, weighted, double hierarchy of their time averages

$$\langle F_{n,m}^{\alpha_{n,m}}\rangle_T \le c_{n,m}Re^3 \left\}\begin{array}{ll} n\ge 1, & 1\le m\le\infty,\\ n=0, & 3<m\le\infty,\end{array}\right. \tag{6}$$

as demonstrated in [30]. The angled brackets $\langle\cdot\rangle_T$ are defined by

$$\langle\cdot\rangle_T = T^{-1}\int_0^T \cdot\,\mathrm{d}t, \tag{7}$$

and the Reynolds number Re by

$$Re = LU/\nu \quad \text{with} \quad U^2 = L^{-3}\langle\|\boldsymbol{u}\|_2^2\rangle_T. \tag{8}$$

The physical meaning of the set of inequalities in (6) can be illustrated thus. Consider the time-averaged energy dissipation rate defined in the conventional manner as $\varepsilon_{av} = \nu L^{-3}\langle\|\nabla\boldsymbol{u}\|_2^2\rangle_T$. Then in the case $n=m=1$, (6) becomes

$$\varepsilon_{av} \le c_{1,1}\nu^3L^{-4}Re^3. \tag{9}$$

The upper bound is recognizable as the same result derived by Kolmogorov's theory [16] and, as we shall see below, leads to the well-known $Re^{3/4}$ estimate for the inverse Kolmogorov length. The double hierarchy displayed in (6) furnishes us with bounds which generalize (9) to all derivatives and in every L^{2m}-norm. It is valid for Leray's weak solutions and encapsulates all the known weak solution results in Navier-Stokes analysis [30]. These are distributional in nature and are not unique and thus the result in (6) falls short of a full regularity proof; *i.e.*, existence *and* uniqueness of solutions. It was shown in [30] that to achieve this would require

$$\langle F_{n,m}^{2\alpha_{n,m}} \rangle_T < \infty. \tag{10}$$

While it remains an open problem, there is no evidence that any bounds with the factor of 2 in the exponent exist. Indeed it is possible that weak solutions are all that are available. What has been deduced is that (2) and (3) lead to a definition of a set of inverse length scales $\ell_{n,m}^{-1}$

$$(L\ell_{n,m}^{-1})^{n+1} := F_{n,m}, \tag{11}$$

whose estimated time averages are [30]

$$\langle L\ell_{n,m}^{-1} \rangle_T \leq c_{n,m} Re^{\frac{3}{(n+1)\alpha_{n,m}}} + O(T^{-1}). \tag{12}$$

The exponents of Re in the two cases $n = m = 1$ and $n, m \to \infty$ are

$$\left.\frac{3}{(n+1)\alpha_{n,m}}\right|_{n,m=1} = 3/4, \tag{13}$$

$$\lim_{n,m\to\infty} \frac{3}{(n+1)\alpha_{n,m}} = 3. \tag{14}$$

The first result in (13) is consistent with the inverse Kolmogorov length while the second result in (14) *implies that there exists a finite limit to the cascade process.* However, when Re is large it does so at a level below molecular scales where the Navier-Stokes equations are not valid. Nevertheless it validates Richardson's original assertion that viscosity eventually terminates the cascade process [16,31].

Estimates and scaling in D-dimensions. – Inequalities (6) and (12) are true for weak solutions in a $D = 3$ domain. For integer values of $D = 2$ or $D = 3$ on a periodic domain $\mathcal{V}_D$ the definition of (2) can be generalized to[1]

$$F_{n,m,D} = \nu^{-1} L^{1/\alpha_{n,m,D}} \|\nabla^n \boldsymbol{u}\|_{2m}, \tag{15}$$

where $\alpha_{n,m}$ in (3) and (6) is replaced by

$$\alpha_{n,m,D} = \frac{2m}{2m(n+1) - D}. \tag{16}$$

The $F_{n,m,D}$ in (15) possess the same invariance properties as $F_{n,m}$ in (2). The details of the proof of (6) have been generalized for the integer D-dimensional case using the same methods and results as in three dimensions [30], although the calculation is far from straightforward: see the appendix.

Theorem 1. *For $D = 2, 3$, and for $n \geq 1$ and $1 \leq m \leq \infty$, the equivalent of* (6) *is*

$$\langle F_{n,m,D}^{(4-D)\alpha_{n,m,D}} \rangle_T \leq c_{n,m,D} Re^3. \tag{17}$$

For $D = 1$ the same result holds for Burgers' equation.

More than 40 years ago Fournier and Frisch [32] introduced the idea of turbulence in D dimensions where D is no longer an integer but is restricted to the range $D \geq 2$. They achieved this by analytically continuing the Taylor expansion in time of the energy spectrum $E_k(t)$, assuming Gaussian initial conditions. Since then the idea of a non-integer dimension has taken root in the many papers on the beta, bi-fractal and multi-fractal models [16,17]. Can the Navier-Stokes estimates in (17) be performed on a domain of non-integer dimension? In a fully rigorous sense, the answer is in the negative. For instance, there are no proofs of the Divergence Theorem or the Sobolev inequalities on fractal domains. Thus we can only claim the validity of Theorem 1 for integer values of D. What the result does do, however, is show how the exponent of $F_{n,m,D}$ scales with integer values D. The surprising but crucial factor of $4-D$ in the exponent multiplying $\alpha_{n,m,D}$ deserves some remarks:

1) When $D = 3$, the factor of $4 - D$ is simply unity and (17) reduces to (6).
2) When $n = m = 1$ this factor cancels to make $(4 - D)\alpha_{1,1,D} = 2$ for every value of D, as it should. It also furnishes us with the correct bound on the averaged energy dissipation rate ε_{av}.
3) When $D = 2$ we achieve the $2\alpha_{n,m,2}$ bound required for full regularity, as in (10). Thus the case $D = 2$ is critical for regularity, as is well known [33–35].

As in fig. 1, computations in [7–12] have shown that the process of flattening and filamentation results in ever finer striations as the flow progresses. This would indicate that the set(s) on which vorticity or strain are concentrated has a non-integer and decreasing dimension. While we have no rigorous methods for proving the validity of (17) when D takes non-integer values, it raises the intriguing possibility that this may nevertheless be true. Certainly it is clear that when D decreases in (17) then the exponent $(4 - D)\alpha_{n,m,D}$ of $F_{n,m,D}$ increases, which is the direction of more, not less, regularity. *This suggests that a flow may adjust itself to find the smoothest, most dissipative set, not the most singular, on which to operate.* This runs counter to the traditionally held theory of viscous turbulence in which singularities have been long-standing candidates as the underlying cause of turbulent dynamics [36–38], even though they must be rare events [33–35,39]. Adjustment

[1]When $D = 1$ the Navier-Stokes equations make no sense unless the pressure and divergence-free terms are removed, in which case we have Burgers' equation. The results expressed in D-dimensions with $D = 1$ are valid for this.

to find the smoothest, most dissipative set could be a way of the flow re-organizing and regularizing itself to avoid singularities.

* * *

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Appendix: proof of Theorem 1. – The aim of Theorem 1 is to roll together estimates for the Navier-Stokes equations that are already known individually in both the $D = 2$ and $D = 3$ cases. In addition, Burgers' equation is included, which is appropriate for $D = 1$ when the pressure term and the incompressibility condition have been dropped. The main foundation of the proof of Theorem 1 is the original result of Foias, Guillopé and Temam (FGT) in 3 dimensions [40]. Given that all three results are known separately, we are able to formally manipulate and differentiate the H_n, defined below in (A.1) below, on a periodic domain of integer dimension D.

The FGT result in integer D dimensions. We require the definition

$$H_n = \int_{\mathcal{V}_\mathcal{D}} |\nabla^n \boldsymbol{u}|^2 \mathrm{d}V, \tag{A.1}$$

from which we can write [33]

$$\frac{1}{2}\dot{H}_n \le -\nu H_{n+1} + c_n \|\nabla \boldsymbol{u}\|_\infty H_n. \tag{A.2}$$

For simplicity, we have omitted the forcing. An integer-D-dimensional Gagliardo-Nirenberg inequality gives

$$\|\nabla \boldsymbol{u}\|_\infty \le c_n H_{n+1}^{a/2} H_1^{(1-a)/2} \tag{A.3}$$

with $a = D/2n$ and $n > D/2$. After re-arrangement, (A.2) becomes

$$\begin{aligned}\frac{1}{2}\dot{H}_n &\le -\nu\left(1-\frac{1}{2}a\right)H_{n+1} + c_n\nu^{-\frac{a}{1-a}} H_n^{\frac{2}{2-a}} H_1^{\frac{1-a}{2-a}}\\ &\le -\frac{1}{2}\nu\left(1-\frac{1}{2}a\right)H_{n+1} + c_n\nu^{-\frac{a}{1-a}} H_n^{\frac{4n}{4n-D}} H_1^{\frac{2n-D}{4n-D}}.\end{aligned} \tag{A.4}$$

Divide by $H_n^{n\alpha_{n,1}}$ and time average to give

$$\begin{aligned}\left\langle \frac{H_{n+1}}{H_n^{n\alpha_{n,1}}}\right\rangle_T &\le c_n\nu^{-\frac{1}{1-a}}\left\langle H_n^{\frac{n(4-D)\alpha_{n,1,D}}{4n-D}} H_1^{\frac{2n-D}{4n-D}}\right\rangle_T\\ &\le c_n\nu^{-\frac{1}{1-a}}\langle H_n^{\frac{1}{2}(4-D)\alpha_{n,1,D}}\rangle_T^{\frac{2n}{4n-D}}\langle H_1\rangle_T^{\frac{2n-D}{4n-D}}.\end{aligned} \tag{A.5}$$

Then a Holder inequality gives

$$\begin{aligned}\langle H_{n+1}^{\frac{1}{2}(4-D)\alpha_{n+1,1,D}}\rangle &\le \left\langle\frac{H_{n+1}}{H_n^{n\alpha_{n,1}}}\right\rangle^{\frac{1}{2}(4-D)\alpha_{n+1,1,D}}\\ &\times\left\langle H_n^{\frac{\frac{1}{2}(4-D)n\alpha_{n,1,D}\alpha_{n+1,1,D}}{1-\frac{1}{2}(4-D)\alpha_{n+1,1,D}}}\right\rangle_T^{1-\frac{1}{2}(4-D)\alpha_{n+1,1,D}}.\end{aligned} \tag{A.6}$$

It is then easy to show that the exponent of H_n within the average can be simplified to

$$\frac{\frac{1}{2}(4-D)n\alpha_{n,1,D}\alpha_{n+1,1,D}}{1-\frac{1}{2}(4-D)\alpha_{n+1,1,D}} = \frac{1}{2}(4-D)\alpha_{n,1,D}. \tag{A.7}$$

Taking (A.6) and (A.7) together and using the dimensionless notation of $F_{n,m,D}$, we end up with

$$\begin{aligned}\langle F_{n+1,1}^{(4-D)\alpha_{n+1,1,D}}\rangle_T &\le c_{n,1}\langle F_{n,1}^{(4-D)\alpha_{n,1,D}}\rangle_T\\ &\quad + c_{n,2}\langle F_{1,1,D}^2\rangle_T.\end{aligned} \tag{A.8}$$

To begin an iteration procedure it is necessary to have a bound in the $n = 2$ case because $n > D/2$ and $D = 2, 3$. We repeat the argument above for $n = 2$ only

$$\frac{1}{2}\dot{H}_1 \le -\nu H_2 + \|\boldsymbol{\omega}\|_4^2\|\boldsymbol{\omega}\|_2 \tag{A.9}$$

We note that in D-dimensions $\|\boldsymbol{\omega}\|_4 \le c\|\nabla\boldsymbol{\omega}\|_2^a\|\boldsymbol{\omega}\|_2^{1-a}$ were $a = D/4$. Thus we have

$$\frac{1}{2}\dot{H}_1 \le -\nu\left(1-\frac{1}{4}D\right)H_2 + c\nu^{-\frac{D}{4-D}} H_1^{\frac{6-D}{4-D}}. \tag{A.10}$$

Firstly we consider

$$\begin{aligned}\langle H_2^{\frac{1}{2}\alpha_{2,1,D}}\rangle_T &= \left\langle\left(\frac{H_2}{H_1^\beta}\right)^{\frac{1}{2}\alpha_{2,1,D}} H_1^{\frac{1}{2}\beta\alpha_{2,1,D}}\right\rangle_T\\ &\le \left\langle\frac{H_2}{H_1^\beta}\right\rangle_T^{\frac{1}{2}\alpha_{2,1,D}}\left\langle H_1^{\frac{\frac{1}{2}\beta\alpha_{2,1,D}}{1-\frac{1}{2}\alpha_{2,1,D}}}\right\rangle_T^{1-\frac{1}{2}\alpha_{2,1,D}}.\end{aligned} \tag{A.11}$$

Thus we must choose β to make the exponent of H_1 equal to unity:

$$\beta = 2\alpha_{2,1,D}^{-1} - 1 = \frac{6-D}{4-D} - 1 = \frac{2}{4-D}. \tag{A.12}$$

To see about the ratio we look at (A.9) and divide by H_1^β to obtain

$$\left\langle\frac{H_2}{H_1^\beta}\right\rangle_T \le c\nu^{-\frac{4}{4-D}}\left\langle H_1^{\frac{6-D}{4-D}-\beta}\right\rangle_T = \nu^{-\frac{4}{4-D}}\langle H_1\rangle_T. \tag{A.13}$$

Thus $\langle F_{2,1}^{(4-D)\alpha_{2,1,D}}\rangle_T < \infty$. Then, from (A.8), the result follows for all $n \ge 1$

$$\langle F_{n,1,D}^{(4-D)\alpha_{n,1,D}}\rangle_T \le c_{n,1} Re^3. \tag{A.14}$$

Formally this is the equivalent of the result in [40] when $D = 3$.

The result for $1 \le m \le \infty$. The bound in (A.14) is true for $m = 1$ only. To move up to the $m > 1$ case we use a Gagliardo-Nirenberg inequality in integer D dimensions

$$\|A\|_{2m} \le c\|\nabla^N A\|_2^a\|A\|_2^{1-a}, \tag{A.15}$$

where $2aN = D(m-1)/m$. Therefore, with $A \equiv \nabla^n \boldsymbol{u}$, we use the $F_{n,m,D}$-notation. We also keep in mind the result (A.14) above to find

$$\begin{aligned}
&\langle F_{n,m,D}^{(4-D)\alpha_{n,m,D}} \rangle_T \le c \langle F_{N+n,1,D}^{a(4-D)\alpha_{n,m,D}} F_{n,1,D}^{(4-D)\alpha_{n,m,D}(1-a)} \rangle_T \\
&= c \left\langle (F_{N+n,1,D}^{(4-D)\alpha_{N+n,1,D}})^{\frac{a\alpha_{n,m,D}}{\alpha_{N+n,1,D}}} F_{n,1,D}^{(4-D)(1-a)\alpha_{n,m,D}} \right\rangle_T \\
&\le \langle F_{N+n,1}^{(4-D)\alpha_{N+n,1,D}} \rangle_T^{\frac{a\alpha_{n,m,D}}{\alpha_{N+n,1,D}}} \\
&\times \left\langle F_{n,1,D}^{\frac{(4-D)\alpha_{n,m,D}(1-a)\alpha_{N+n,1,D}}{\alpha_{N+n,1,D} - a\alpha_{n,m,D}}} \right\rangle_T^{1 - \frac{a\alpha_{n,m,D}}{\alpha_{N+n,1,D}}}. \qquad \text{(A.16)}
\end{aligned}$$

Using the fact that $2aN = D(m-1)/m$ and the expression for $\alpha_{n,m,D}$ given in (16), we can then show that the exponent of $F_{n,1,D}$ in the time average satisfies

$$\frac{(4-D)\alpha_{n,m,D}(1-a)\alpha_{N+n,1,D}}{\alpha_{N+n,1,D} - a\alpha_{n,m,D}} = (4-D)\alpha_{n,1,D}. \qquad \text{(A.17)}$$

Using (A.14), we see that both factors on the right-hand side of (A.16) are bounded and finally give (17). ■

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