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A new model of regular black hole in $(2+1)$ dimensions

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Abstract – We provide a new regular black hole solution in $(2+1)$ dimensions with the presence of matter fields in the energy momentum tensor. The inclusion of our proposed energy density plus a negative cosmological constant allows that the solution can have both flat as de Sitter or Anti-de Sitter core. This latter is a proper characteristic of our solution, because other models of regular black holes have only a single type of core. Since the first law of thermodynamics for regular black holes is modified by the presence of the matter fields, we provide a new version of the first law, where a local definition of the variation of energy is defined, and where the entropy and temperature are consistent with the ones previously known in the literature. At the hypothetical limit when the horizon radius $r_+ \to \infty$ the usual first law $\mathrm{d}M = T\mathrm{d}S$ is recovered. The effectiveness of the formalism used to compute the mass of our regular black holes in $(2+1)$ dimensions suggests the potential applicability of this method to calculate the mass of other models of regular black holes in $d \geq 4$ dimensions.

editor's choice

Introduction. – The recent detection of gravitational waves through the collision of two rotating black holes [1,2], together with the also recent assignment of the Nobel Prize, have positioned the black holes as one of the most interesting and intriguing objects in gravitation. In this connection, the fact that the black holes, due to quantum fluctuations, emit as black bodies, where their temperature is related to their surface gravity [3], shows that in these objects the geometry and thermodynamics are directly connected.

The Schwarzschild black hole has a central singularity where the laws of physics cease to operate. From the classical point of view, Bardeen in ref. [4] proposed the first model of regular black holes (RBHs). In this model the singularity is avoided by changing the mass parameter in the Schwarzschild solution by a radial mass function such that near the origin the function behaves as $f \approx 1 - Cr^2$, *i.e.*, the solution has a de Sitter core. After this, several models of regular black holes have been constructed, where the Einstein field equations are coupled to non-linear electromagnetic sources. Examples of this can lead to RBHs with de Sitter core [5] or flat core [6,7].

Furthermore, it is possible to construct RBHs solutions including matter fields in the energy momentum tensor. Examples of this are the models of refs. [8,9]. These models are solutions of General Relativity and have de Sitter core. We can see a review of these types of RBHs in ref. [10]. Other models of RBHs based on the presence of matter fields in the energy momentum tensor for higher curvature theories, with a de Siter core, can be found in refs. [11–13].

It is well known that the first law of thermodynamics is modified for RBHs due to the presence of matter fields in the energy momentum tensor. In ref. [14] a way of writing the first law was shown including a correction factor, which corresponds to an integration of the radial coordinate up to infinity. However, the first law of thermodynamics for RBHs is still an open question of physical interest.

On the other hand, models of gravity in $(2+1)$ dimensions have drawn high attention in the last years, due to the simplicity of its equations of motion. These models can be viewed as toy models, with the hope that its results can help the understanding of $(3+1)$- and higher-dimensional models of gravity [15]. Models of black holes in $(2+1)$ dimensions also have been widely studied in the last years. The BTZ model [16,17] is a vacuum black hole solution of the Einstein field equations in $(2+1)$ dimensions, which has been studied to find some conceptual issues in quantum gravity, string theory and AdS/CFT correspondence [18]. In this solution the curvature invariants are

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regular everywhere. Regular black holes in $(2+1)$ dimensions coupled to a non-linear electrodynamics were studied in refs. [19,20]. Nevertheless, in the literature there are also solutions in $(2+1)$ dimensions with a central singularity, as for example the solutions of ref. [21] for Massive Gravity [22], or the solution given in [23]. See refs. [24–26] for a generalization of the BTZ model with other fields and refs. [27,28] for an interesting higher-dimensional generalization of BTZ and geodesic analysis. We can see more recent studies in refs. [29–34].

Thus, it is undoubtedly interesting the search of new black hole solutions in $(2+1)$ dimensions with matter fields in the energy momentum tensor and the study of its thermodynamics properties. In this work we will provide a new model of energy density in $(2+1)$ dimensions, which will lead to a new regular black hole solution. Furthermore we will propose a structure for the first law of thermodynamics for RBHs and compute the mass of the solution using a definition of conserved charge appeared recently in the literature [35,36]. Furthermore we will study the stability of our solution.

$(2+1)$ **Einstein field equations.** – The Einstein field equation are given by

$$G^{\mu}_{\nu} + \Lambda\delta^{\mu}_{\nu} = 8\pi\bar{G}T^{\mu}_{\nu}, \tag{1}$$

where the cosmological constant is equal to $\Lambda = -\dfrac{1}{l^2}$. The Newton constant, in the natural system of units, has units of length $[\bar{G}] = \ell$. For simplicity, we consider arbitrarily that this constant has a magnitude equal to 1. The energy momentum tensor describing a perfect fluid, is given by

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p_r, p_\theta). \tag{2}$$

We shall study the following spherically symmetric space-time:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta. \tag{3}$$

This form of the metric has the following consequence on the energy momentum tensor through the Einstein field equations:

$$\rho = -p_r. \tag{4}$$

So, for the line element (3), the (t,t) and (r,r) component of the Einstein field equations are

$$\frac{1}{2r}\frac{\mathrm{d}f}{\mathrm{d}r} + \Lambda = -8\pi\rho. \tag{5}$$

On the other hand, from the conservation law, $\nabla_\mu T^{\mu\nu} = 0$, one gets

$$p_\theta = r\frac{\mathrm{d}}{\mathrm{d}r}p_r + p_r. \tag{6}$$

We will chose the following ansatz:

$$f(r) = 1 - 8\bar{G}m(r) - \Lambda r^2. \tag{7}$$

Next, replacing the ansatz (7) into eq. (5), one arrives at

$$\frac{\mathrm{d}}{\mathrm{d}r}m(r) = 2\pi r\rho(r). \tag{8}$$

The new model. – The mass function for the $(2+1)$ case is defined as

$$m(r) = 2\pi\int_0^r x\rho(x)\mathrm{d}x. \tag{9}$$

As was mentioned above, in this work we will provide a new RBH solution based on the inclusion of matter field in the energy momentum tensor. For this, we will propose a new model of energy density. In order to have a well-posed physical situation, the energy density model that we will choose follows the requirements described in ref. [12]:

– The energy density must be positive and continuously differentiable to avoid singularities.

– The energy density must have a finite single maximum at the origin. The finiteness of $\rho(0)$ forbids the presence of a central singularity.

– In order to guarantee a well-defined asymptotic behavior, the energy density must be a decreasing radial function and must vanish at infinity. Furthermore, the mass function must reach its finite maximum value at infinity, *i.e.*,

$$\lim_{r\to\infty}\rho = 0, \tag{10}$$

$$\lim_{r\to\infty} m(r) = \mathrm{const.} \tag{11}$$

Thus, we propose the following model for energy density:

$$\rho = \frac{2LM^2}{\pi(2LM + r^2)^2}, \tag{12}$$

and replacing into eq. (9) we obtain the mass function

$$m(r) = \frac{Mr^2}{2LM + r^2}, \tag{13}$$

where L is a positive constant of units $[L] = \ell^3$. This constant must be positive to ensure that $\rho(0) = 1/(2\pi L) > 0$. The parameter M is the so-called mass parameter, in units of $[M] = \ell^{-1}$. This constant must also be positive to ensure the absence of singularities in the energy density.

Considering the previous analysis, the mass function has units $[m(r)] = \ell^{-1}$ and since $[\bar{G}] = \ell$, the factor $\bar{G}m(r)$ is dimensionless, where the magnitude of $\bar{G}$ is equal to the unity.

This model could be viewed as an extension in $(2+1)$ dimensions of the higher-dimensional model of ref. [12], which, for the $(3+1)$-dimensional case, coincides with the Hayward metric [9].

For the Hayward metric in $(3+1)$ dimensions, when the energy density is of the order of the Planck units near the origin, the formation of a de Sitter core is associated with quantum fluctuations. These models are known as Planck Stars [37,38]. In ref. [39], using radio astronomy data, it is conjectured that Planck Stars represent a speculative but realistic possibility to test quantum gravity effects. Thus, if our energy density were of the order of the Planck units near the origin, our model could serve as a toy model to study these ideas in a future work due to the simplicity of the equations of motion in $(2+1)$ dimensions. As we shall see below, the inclusion of a negative cosmological constant not only provides the formation of a dS core, but also provides the formation of a flat/AdS core. This latter feature could be analyzed for Planck Stars models elsewhere. It is worth mentioning that the formation of a flat or AdS core is a new feature for RBHs without non-linear electromagnetic sources.

The solution is:

$$f(r) = 1 - \Lambda r^2 - 8\bar{G}m(r) = 1 + \frac{r^2}{l^2} - \frac{8Mr^2}{2LM + r^2}. \tag{14}$$

For this solution, the Ricci and Kretschmann invariants are given by

$$R = -\frac{6}{l^2} + \frac{48M}{2LM + r^2} - \frac{112Mr^2}{(2LM + r^2)^2} + \frac{64Mr^4}{(2LM + r^2)^3}, \tag{15}$$

$$K = \left(\frac{2}{l^2} - \frac{16M}{2LM + r^2} + \frac{80Mr^2}{(2LM + r^2)^2} - \frac{64Mr^4}{(2LM + r^2)^3}\right)^2 + 2\left(\frac{2}{l^2} - \frac{16M}{2LM + r^2} + \frac{16Mr^2}{(2LM + r^2)^2}\right)^2. \tag{16}$$

We consider as regular when the curvature invariants are free of singularities. In our case both Ricci and Kretschmann do not diverge at $r = 0$ nor other values of r due to the fact that $L > 0$ and $M > 0$.

Behavior near the origin. From eq. (14) it is evident that this function behaves near the origin as

$$f\big|_{r\approx 0} \approx 1 + \left(\frac{1}{l^2} - \frac{4}{L}\right) r^2, \tag{17}$$

thus, we can define the effective radius ($\tilde{l}$):

$$\frac{1}{\tilde{l}} = \left(\frac{1}{l^2} - \frac{4}{L}\right). \tag{18}$$

– For $\frac{1}{l^2} = \frac{4}{L}$ the behavior near the origin corresponds to a flat space-time. Regular solutions with a flat core have been studied for $(3+1)$ dimensions with a non-linear electromagnetic source in refs. [6,7]. However, for the $(2+1)$ case, with a nonzero energy density, this behavior is a proper feature of our solution.

– For $\frac{1}{l^2} < \frac{4}{L}$ the behavior near the origin corresponds to a dS space-time. Several models of regular black holes share this feature [9,11–13,37,38]. In this case eq. (18) represents an effective dS radius.

– For $\frac{1}{l^2} > \frac{4}{L}$ the behavior near the origin corresponds to an AdS space-time. So, in our model this possibility could be valid, which differs from most of the RBHs models. Due to the simplicity of the $(2+1)$ models, the physical consequences of this particular feature could be tested elsewhere. In this case eq. (18) represents an effective AdS radius.

Thus, our model of energy density added to a negative cosmological constant allows that the solution can have both flat as de Sitter or Anti-de Sitter core, depending on the values of the l and L. This latter is a proper characteristic of our solution, because other models of regular black holes have only a single type of core.

Thermodynamic analysis. –

Conserved charges. Recently, in refs. [35,36] a new definition of conserved charges has appeared. In [35] the energy and momentum can be computed by integrating a covariantly conserved current $J^\mu = T^\mu_\nu \xi^\nu$ in a volume integral. It is worth to mention that the definition of ref. [35] is reduced to the definition of conserved energy of ref. [36] for a Killing vector $\xi^\mu = -\delta^\mu_0$. Following [35], the energy is defined as

$$E = \int \mathrm{d}^{d-1}x\sqrt{-g}J^0 = \int \mathrm{d}^{d-1}x\sqrt{-g}T^0_\nu \xi^\nu, \tag{19}$$

where ξ^ν is a Killing vector and d is the number of dimensions.

It is worth mentioning that this definition is diffeo-invariant. Furthermore the current is covariantly conserved due to $\nabla_\mu T^\mu_\nu = 0$ and $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$.

Following [35], we choose $\xi^\mu = -\delta^\mu_0$. Evaluating our solution into eq. (19) we have

$$E = -2\pi \int_0^\infty r\mathrm{d}r T^0_0 = 2\pi \int_0^\infty r\mathrm{d}r\rho(r) = \lim_{r\to\infty} m(r) = M. \tag{20}$$

Thus, the parameter M represents the total energy of the black hole. On the other hand, the obtained mass M coincides with the value of the vacuum BTZ solution computed in ref. [40] using another definition of conserved charges.

So, the mass of the black hole in $(2+1)$ dimensions is computed without adding boundary terms as in refs. [40,41]. The effectiveness of this formalism for computing the mass of our $(2+1)$ RBHs model could make us think about using this formalism for computing the mass of other models of regular black holes in $d \geq 4$ dimensions, where the energy density also fulfills the conditions mentioned before. Some examples of these models are [8,9,12,37,38].

The first law of thermodynamic in $(2+1)$ dimensions with matter fields. It is well known that the first law of thermodynamics is modified for RBHs due to the presence of matter fields in the energy momentum tensor. Following [42,43] we define the thermodynamics pressure as $p = -\frac{\Lambda}{8\pi}$. In order to propose a structure for the first law of thermodynamics for RBHs, we will use the conditions $f(r_+, M, p) = 0$ and $\delta f(r_+, M, p) = 0$, which can be viewed as constraints on the evolution along the space parameters [11,33]. Thus, from the condition

$$0 = \frac{\partial f}{\partial r_+}\mathrm{d}r_+ + \frac{\partial f}{\partial M}\mathrm{d}M + \frac{\partial f}{\partial p}\mathrm{d}p, \tag{21}$$

it is straightforward to check that for the present solution (14), the first law takes the form:

$$\frac{\partial m}{\partial M}\mathrm{d}M = \left(\frac{1}{4\pi}f'\big|_{r=r_+}\right)\mathrm{d}\left(\frac{\pi r_+}{2}\right) + (\pi r_+^2)\mathrm{d}p. \tag{22}$$

The above equation can be rewritten as

$$\mathrm{d}u = T\mathrm{d}S + V\mathrm{d}p, \tag{23}$$

where the temperature and entropy are easily computed as

$$T = \frac{1}{4\pi}f'\big|_{r=r_+}, \tag{24}$$

$$S = \frac{\pi r_+}{2}, \tag{25}$$

$$V = \pi r_+^2, \tag{26}$$

where our definition of volume coincides with [42].

One can notice that p, the pressure, has units of $[p] = \ell^{-2}$, eq. (14). Likewise, one can check that the factor $[\bar{G}m] = \ell^0$ is dimensionless and $[S] = \ell^1$. Following *Euler's theorem* [44], one can construct the *Smarr formula*:

$$TS - 2pV = 0, \tag{27}$$

which coincides with the vacuum (2+1) solution of ref. [42] for the non-rotating case. In a future work, the case where the constant L is a thermodynamics parameter could be studied.

It is direct to check that the temperature for our model (14) is

$$T = \frac{2}{r_+}m(r_+) - \frac{2}{r_+} - 8\frac{\mathrm{d}m}{\mathrm{d}r}\Big|_{r=r_+}. \tag{28}$$

On the other hand, the heat capacity at constant pressure is computed as

$$C_p = T\frac{\mathrm{d}S}{\mathrm{d}T}\Big|_{p=cte} = T\frac{\mathrm{d}S}{\mathrm{d}r_+}\left(\frac{\mathrm{d}T}{\mathrm{d}r_+}\right)^{-1}\Big|_{p=cte}. \tag{29}$$

In the vacuum case, the first law is of the form $\mathrm{d}M = T\mathrm{d}S$, where M is the energy of the system. However in our case the left side of the first law is modified in order to obtain the correct value of the entropy due to the presence of matter fields in the energy momentum tensor. So, in our case, the term on the left side, $\mathrm{d}u$, corresponds to a local definition of the variation of the energy.

The modification of the first law for regular black holes was studied in ref. [14], without our constraint in the space of parameters, by including a correction factor which corresponds to an integration of the radial coordinate up to infinity. Unlike this latter reference, in our case, both terms, $\mathrm{d}u$ and $\mathrm{d}S$, are local variables defined at the horizon.

It is easy to check that the factor $\mathrm{d}m/\mathrm{d}M$ in eq. (23) is always positive, and thus, the sign of the variation of $\mathrm{d}u$ always coincides with the sign of the variation of the total energy $\mathrm{d}M$. Furthermore, at infinity it is fulfilled that $\lim_{r\to\infty} \mathrm{d}m/\mathrm{d}M = 1$, therefore the variation of the local and total energy are similar at the asymptotical region. Thus, at the hypothetical limit when the horizon radius $r_+ \to \infty$, the usual first law $\mathrm{d}M = T\mathrm{d}S$ is recovered.

In a future work, following [45], we could test a possible relation between l and L to study the ensemble dependency problem.

Stability of our solutions. In fig. 1 the behavior of the mass parameter (top panel), temperature (middle panel) and heat capacity (bottom panel) are shown. It is direct to check that this behavior is generic for other values of the parameter l.

The equation (14) can take the form $f(r) = 0$, which taking arbitrarily fixed l and L, can have zero roots, two roots or one root, depending on the value of the mass parameter M.

In the top panel of fig. 1, following the analysis of refs. [11,46], we plot the mass parameter $M(R)$, where $r = R$ corresponds to the root of the equation $f(r) = 0$. R can take the values r_- and r_+, which correspond to the internal horizon and the black hole horizon, respectively. There is a critical value of the mass parameter $M_* = M(r_*)$, which corresponds to the minimum value on the curve, and where there is an extremal black hole. At this point the internal horizon r_- and the black hole horizon r_+ coincide, *i.e.*, $r_* = r_- = r_+$. For $M < M_*$ the solution does not have horizons and for $M > M_*$ the solution has both internal and black hole horizon.

In the middle panel of fig. 1, we note that the temperature is vanishing just in the extremal black hole. This does not have a linear dependence on r_+ as the spinless BTZ solution [47] and its dependence on r_+ also differ from the rotating BTZ solution [16].

In the bottom panel of fig. 1, the heat capacity is displayed, which is always positive, therefore the solution is always stable. Its behavior also differs from the BTZ solution whose heat capacity is constant for the static case [47].

Discussion and conclusions. – We have obtained a new regular black hole solution in $(2+1)$ dimensions with the presence of matter fields in the energy momentum tensor. For this, we have proposed a new model of energy density following the requirements described in ref. [12]. This energy density has a finite value at the center, avoiding the formation of a central singularity.

Our solution differs from the BTZ-like geometry of ref. [48], because in our solution the mass function is present, unlike in this reference, where only the constant mass parameter is present. Furthermore, in ref. [48] the BTZ horizon is characterized for small values of the mass and cosmological constant. In our model the energy

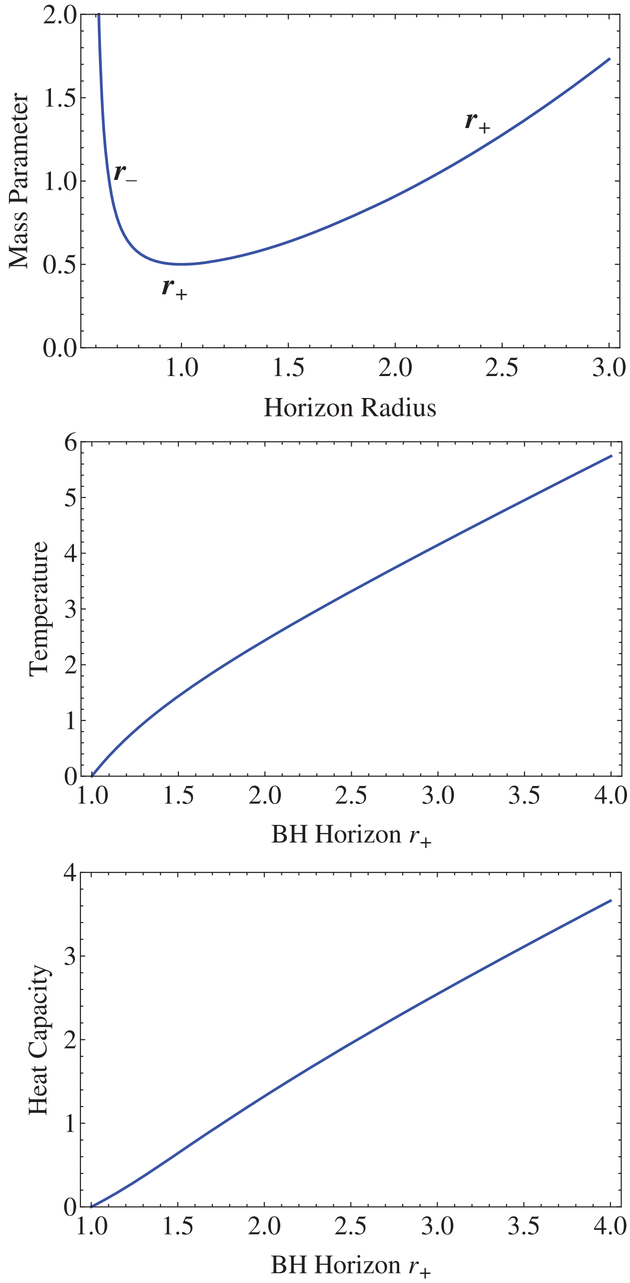

Fig. 1: Top panel: the mass parameter. Middle panel: the temperature. Bottom panel: the heat capacity. All these plots were obtained by considering $l = L = \bar{G} = 1$.

density tends to zero at infinity, eq. (10), this differs from the model of ref. [49], where the energy density diverges at infinity. Furthermore, our solution differs from ref. [50], which is based on the matching between the internal geometry obtained by means the Gravitational Decoupling algorithm and the external BTZ geometry.

The inclusion of our type of energy density plus a negative cosmological constant allows that the solution can have both flat as de Sitter or Anti-de Sitter core, depending on the values of the l and L. This latter is a proper characteristic of our solution, because other models of regular black holes have only a single type of core.

Regular solutions with a flat core have been studied in (3+1) dimensions with a non-linear electromagnetic source in refs. [6,7]. However, for the $(2+1)$ case, with a nonzero energy density, the flat core is a proper feature of our solution. In $(3+1)$ dimensions, regular black hole solutions with a dS core have been widely studied in the literature. Our model with a dS core could be viewed as a $(2+1)$ extension of the Hayward metric. Since the Hayward model has been used to test quantum gravity effects [39], our model could serve to study these effects elsewhere, due to the simplicity of the equations of motion in $(2+1)$ dimensions. Furthermore our solution admits a core with AdS structure, which is not a studied feature so far.

Using the recently definition of conserved charges of refs. [35,36] we have computed the total energy of our regular black hole solution, which is equal to the mass parameter M. The effectiveness of the definition used for computing the mass of our (2+1) regular black hole model suggests that this formalism could be used to compute the mass of other models of regular black holes in $d \geq 4$ dimensions with the presence of matter in the energy momentum tensor, as for example the models of refs. [8,9,12,37,38].

The structure of the first law of thermodynamic for regular black holes is modified by the presence of matter fields in the energy momentum tensor [14]. So, we have provided a new version of the first law of thermodynamic (23), where a local definition of the variation of the energy is defined, namely du, and, where the values of entropy and temperature are consistent with the previously known ones [16]. We have showed that the sign of the local variation of the energy du always coincides with the sign of variation of the total energy dM. Furthermore, at the hypothetical limit when the horizon radius $r_+ \to \infty$ the usual first law $\mathrm{d}M = T\mathrm{d}S$ is recovered.

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