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Faithful entanglement distribution using quantum multiplexing in noisy channel

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Abstract – Quantum multiplexing provides us with a powerful approach that can distribute entanglement using only a single photonic pulse to interact with remote quantum memories. In common entanglement distribution protocols, the entanglement purification is often exploited if the distributed entanglement is degraded by the channel noise. In this protocol, we propose a faithful entanglement distribution protocol using quantum multiplexing of photon in a collective noise channel. By designing the encoding and decoding setup, our protocol can resist the collective noise. When the protocol is successful, the parties can obtain two pairs of maximally entangled states. When the protocol fails, the parties can still obtain one pair of maximally entangled state. Our protocol does not require the entanglement purification, so that it is more efficient. The above advantages enable our protocol to have important application potential in the future quantum communication field.

Introduction. – Quantum communication has attracted widespread interest for its high security during the past few decades. There are many branches of quantum communication, such as quantum key distribution (QKD) [1–5], quantum teleportation (QT) [6–8], quantum secure direct communication (QSDC) [9–14] and some other interesting protocols [15–21]. QKD provides an unconditionally secure method of distributing secret keys between two spatially separated parties. QT can transmit the unknown quantum states of an arbitrary particle without transmitting the particle itself. QSDC enables the secret information to be transmitted directly through a quantum channel without keys or ciphertexts. These branches of quantum communication were widely developed in recent years, among which numerous schemes require remote communication parties to share the entanglement. As a result, long-distance entanglement distribution is an important issue in quantum communication field.

Photons are often used to transmit information due to their rapidity of transmission. For a single photon, the polarization degree of freedom (DOF) is usually used to encode information for the polarization is easy to manipulate [1–3,22–24]. Actually, in the quantum communication, there is not only a requirement for confidentiality, but also a high channel capacity. The usual 2-dimensional polarization coding can only transmit one bit of information per single photon. High-dimensional encoding in single DOF such as spatial-mode [25,26], frequency [27], and orbital angular momentum (OAM) is an effective method to increase the channel capacity. Furthermore, multiple DOFs of a single photon can also be employed simultaneously to encode information, such as spatial-mode and polarization DOFs [28], time-bin and polarization DOFs [29], OAM and polarization DOFs [30]. Using multiple DOFs to increase the quantum channel capacity has been widely investigated in QT [31,32], QKD [28,29,33], and QSDC [34–37] and some other interesting protocols [38–40].

As we all know, photons are inevitably affected by noise during the transmission in practical quantum channels, which may cause the entanglement decoherence or photon loss. The influence from channel noise may decrease communication efficiency, cause communication errors or even threaten communication security. Quantum repeater is a powerful tool to realize the long-distance quantum

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communication in the noisy environments [41–45]. In quantum repeaters, entanglement purification is exploited to distill the high-quality entanglement states from low-quality entanglement states [46–57]. Quantum repeaters have been developed for three generations [43]. Recently, Lo Piparo *et al.* proposed a quantum multiplexing scheme based on the nitrogen-vacancy (NV) center which uses a single photon encoded in polarization and time-bin DOFs to create multiple entangled pairs between two distant parties Alice and Bob. Their protocol can reduce the number of resources and increase the capacity [58]. In their protocol, a novel entanglement purification protocol is used to distill the polarization entanglement using time-bin entanglement. After purification, the time-bin entanglement is consumed and the fidelity of polarization entanglement can be increased with some probability. On the other hand, before transmission, if the noise is known, we can adopt the quantum error correction approach to resist the noise and realize the faithful entanglement distribution. For example, the collective noise model [59] is widely studied and also discussed in single DOF quantum repeaters [60] and blind quantum computation [61,62]. In collective noise condition, the effect of noise on photon polarization can be expressed as a unitary transformation like $|H\rangle \to \delta|H\rangle + \eta|V\rangle$ ($|\delta|^2 + |\eta|^2 = 1$). Here, $|H\rangle$ and $|V\rangle$ denote the horizontal and vertical polarization of a photon, respectively.

In this paper, based on the previous works in quantum multiplexing and collective noise mode [58,63], we propose a high-capacity long-distance entanglement distribution protocol by using the quantum multiplexing in a collective noise channel. By designing the encoding and decoding setup, the protocol can resist the decoherence and the parties can obtain two pairs of maximally entangled states with some success probability. Even when the protocol fails, the parties can still obtain one pair of maximally entangled state. As this protocol adopts the quantum multiplexing and does not require the entanglement purification, it is more efficient than common long-distance entanglement distribution protocols. This paper is organized as follows. We first explain our high-capacity entanglement distribution protocol using the quantum multiplexing. Next, we present a discussion and conclusion.

Entanglement distribution protocol using the quantum multiplexing in a collective noise channel. – Our protocol requires the NV center in a superposition of electronic spin states $|\psi\rangle = |g\rangle + |e\rangle$, where we have omitted the normalization constant for simplicity [58]. Here, $|g\rangle$ and $|e\rangle$ are ground state and excited state, respectively. The NV center actually acts the role of quantum memory (QM). In our protocol, we will use $|\psi_i\rangle = |g_i\rangle + |e_i\rangle$ to represent the state of QMi, where $i = 1, 2, 3, 4$. When a diagonally polarized photon ($|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$) enters the cavity, the photon will interact with the NV center and entangle with it. Notice that if the NV center is in $|e\rangle$, the interaction between a photon and the NV center will make the vertically polarized photon ($|V\rangle$) have a phase shift of π and keep the horizontally polarized photon ($|H\rangle$) unchanged. In this way, we can obtain $|g\rangle|D\rangle \to |g\rangle|D\rangle$ and $|e\rangle|D\rangle \to |e\rangle|A\rangle$, where $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$.

In our protocol, Alice makes a $|D\rangle$ polarized photon interact with the first NV center (QM1), giving

$$|D\rangle|\psi_1\rangle \to |g_1\rangle|D_{a_1}\rangle + |e_1\rangle|A_{a_1}\rangle. \tag{1}$$

The a_i ($i = 1, 2$) here, b_j ($j = 1, 2, 3$), c_k ($k = 1, 2$), d_u ($u = 1, 2, 3$), e_v ($v = 1, 2, 3, 4$), f_w ($w = 1, 2$), g_x ($x = 1, 2$), h_y ($y = 1, 2$), i_z ($z = 1, 2$), and m below represent different spatial modes as shown in fig. 1. The CPBS is the circular polarization beam splitter which can transmit the diagonally polarized photon ($|D\rangle$) and reflect the anti-diagonally polarized photon ($|A\rangle$). The OS is the optical switch, allowing photons of different spatial modes to merge into the same spatial mode a_2, and can lead a single photon to different paths. HWP is the half-wave plate which can convert $|D\rangle$ to $|A\rangle$ and $|A\rangle$ to $|D\rangle$. Alice makes the photon in a_1 mode successively pass through the CPBS1, HWP1, and OS1. The state in eq. (1) becomes

$$|g_1\rangle|D_{a_2}\rangle_s + |e_1\rangle|D_{a_2}\rangle_l, \tag{2}$$

where the subscripts s and l indicate the short and long time-bin, respectively. The photon then interacts with the second NV center of Alice (QM3) and the whole state becomes

$$\begin{aligned}&|g_1\rangle|g_3\rangle|D_{b_1}\rangle_s + |g\rangle_1|e_3\rangle|A_{b_1}\rangle_s \\ &+ |e_1\rangle|g_3\rangle|D_{b_1}\rangle_l + |e\rangle_1|e_3\rangle|A_{b_1}\rangle_l.\end{aligned} \tag{3}$$

In order to describe this protocol clearly, we let

$$\begin{aligned}|a\rangle &\equiv |g_1\rangle|g_3\rangle, \quad |b\rangle \equiv |g_1\rangle|e_3\rangle, \\ |c\rangle &\equiv |e_1\rangle|g_3\rangle, \quad |d\rangle \equiv |e_1\rangle|e_3\rangle.\end{aligned} \tag{4}$$

Before Alice sends the photon to Bob through a noisy channel, she encodes the qubit in another two time-bins L and S like in ref. [63]. We take $|g_1\rangle|g_3\rangle|D_{b1}\rangle_s$ as an example, which can be written as

$$\begin{aligned}&|g_1\rangle|g_3\rangle|D_{b_1}\rangle_s = |a\rangle|D_{b_1}\rangle_s = \frac{1}{\sqrt{2}}(|a\rangle|H_{b_1}\rangle_s + |a\rangle|V_{b_1}\rangle_s) \\ &\xrightarrow{PBS1} \frac{1}{\sqrt{2}}(|a\rangle|H\rangle_{sS} + |a\rangle|V\rangle_{sL}) \\ &\xrightarrow{HWP2} \frac{1}{\sqrt{2}}(|a\rangle|H\rangle_{sS} + |a\rangle|H\rangle_{sL}) \\ &\xrightarrow{BS1} \frac{1}{2}[(|a\rangle|H\rangle_{sS} + i|a\rangle|H\rangle_{sL})_{b_2} \\ &\qquad + i(|a\rangle|H\rangle_{sS} - i|a\rangle|H\rangle_{sL})_{b_3}].\end{aligned} \tag{5}$$

As shown in fig. 1, the subscripts b_2 and b_3 represent two outputs of Alice's 50:50 beam splitter (BS1). Here,

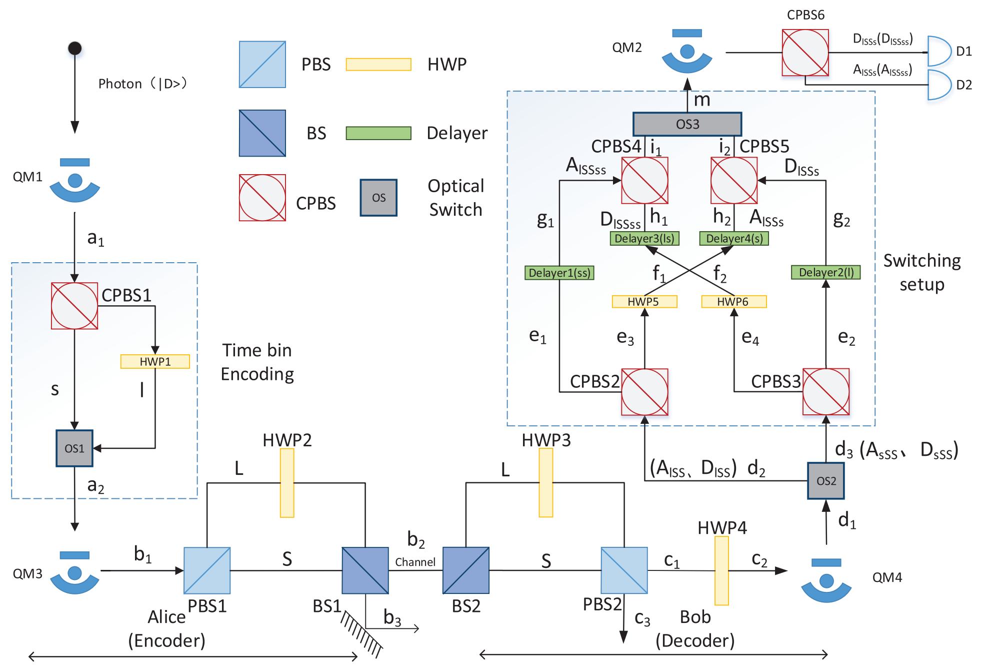

Fig. 1: Schematic diagram of our entanglement distribution protocol with noisy channels. BS represents the 50:50 beam splitter. PBS represents the polarization beam splitter. CPBS represents the circular polarization beam splitter. HWP means the half wave plate. OS means the optical switch. The parties can delay the photon by a certain amount of time by precisely adjusting the length of optical fiber (Delayer). D1 and D2 represent the single photon detectors.

the subscripts L and S also correspond to long arm and short arm, respectively, but they are different with time-bins l and s. Since the states in two output ports can be interchangeable by a unitary operation, we only consider the channel b_2 in detail below. Alice sends the photon in b_2 mode to Bob through the noise channel. The effect of noise on the photon's polarization can be expressed as $|H\rangle \to \delta|H\rangle + \eta|V\rangle$ under the collective noise model [59,63]. Then, the evolution of the state in channel b_2 can be described as

$$\begin{aligned}
&\frac{1}{2}(|a\rangle|H\rangle_{sS} + i|a\rangle|H\rangle_{sL})_{b_2} \to \\
&\frac{1}{2}(\delta|a\rangle|H\rangle_{sS} + \eta|a\rangle|V\rangle_{sS} + i\delta|a\rangle|H\rangle_{sL} \\
&+i\eta|a\rangle|V\rangle_{sL})_{b_2} \equiv |\phi\rangle_A. \qquad (6)
\end{aligned}$$

After the photon arriving at Bob's location, Bob needs to decode the photon state. He lets the photon pass through the BS2, HWP3, and PBS2, successively, and the state $|\phi\rangle_A$ evolves to

$$\begin{aligned}
|\phi\rangle_A \xrightarrow{BS2} &\frac{1}{2\sqrt{2}}(|a\rangle\delta|H\rangle_{sSS} + |a\rangle\eta|V\rangle_{sSS} \\
&+ i|a\rangle\delta|H\rangle_{sLS} + i|a\rangle\eta|V\rangle_{sLS} \\
&+ i|a\rangle\delta|H\rangle_{sSL} + i|a\rangle\eta|V\rangle_{sSL} \\
&- |a\rangle\delta|H\rangle_{sLL} - |a\rangle\eta|V\rangle_{sLL}) \\
\xrightarrow{HWP3} &\frac{1}{2\sqrt{2}}(|a\rangle\delta|H\rangle_{sSS} + |a\rangle\eta|V\rangle_{sSS} \\
&+ i|a\rangle\delta|H\rangle_{sLS} + i|a\rangle\eta|V\rangle_{sLS} \\
&+ i|a\rangle\delta|V\rangle_{sSL} + i|a\rangle\eta|H\rangle_{sSL} \\
&- |a\rangle\delta|V\rangle_{sLL} - |a\rangle\eta|H\rangle_{sLL}) \\
\xrightarrow{PBS2} &\frac{\delta}{2\sqrt{2}}[|a\rangle|H\rangle_{sSS} - |a\rangle|V\rangle_{sLL} \\
&+ i(\underline{|a\rangle|H\rangle_{sLS} + |a\rangle|V\rangle_{sSL}})]_{c_1} \\
&+ \frac{\eta}{2\sqrt{2}}[|a\rangle|V\rangle_{sSS} - |a\rangle|H\rangle_{sLL} \\
&+ i(\underline{|a\rangle|V\rangle_{sLS} + |a\rangle|H\rangle_{sSL}})]_{c_3}. \qquad (7)
\end{aligned}$$

Similarly, other items $|b\rangle|A\rangle_s$, $|c\rangle|D\rangle_l$ and $|d\rangle|A\rangle_l$ in eq. (3) will become

$$\begin{aligned}
|b\rangle|A\rangle_s \ \to \ &\frac{\delta}{2\sqrt{2}}[|b\rangle|H\rangle_{sSS} + |b\rangle|V\rangle_{sLL} \\
&+i(\underline{|b\rangle|V\rangle_{sSL} - |b\rangle|H\rangle_{sLS}})]_{c_1} \\
&+\frac{\eta}{2\sqrt{2}}[|b\rangle|V\rangle_{sSS} + |b\rangle|H\rangle_{sLL} \\
&+i(\underline{|b\rangle|H\rangle_{sSL} - |b\rangle|V\rangle_{sLS}})]_{c_3}, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
|c\rangle|D\rangle_l \ \to \ &\frac{\delta}{2\sqrt{2}}[|c\rangle|H\rangle_{lSS} - |c\rangle|V\rangle_{lLL} \\
&+i(\underline{|c\rangle|H\rangle_{lLS} + |c\rangle|V\rangle_{lSL}})]_{c_1} \\
&+\frac{\eta}{2\sqrt{2}}[|c\rangle|V\rangle_{lSS} - |c\rangle|H\rangle_{lLL} \\
&+i(\underline{|c\rangle|V\rangle_{lLS} + |c\rangle|H\rangle_{lSL}})]_{c_3}, \qquad (9)
\end{aligned}$$

$$
\begin{aligned}
|d\rangle|A\rangle_l \ \rightarrow & \frac{\delta}{2\sqrt{2}}[|d\rangle|H\rangle_{lSS} + |d\rangle|V\rangle_{lLL} \\
& + i(\underline{|d\rangle|V\rangle_{lSL} - |d\rangle|H\rangle_{lLS}})]_{c_1} \\
& + \frac{\eta}{2\sqrt{2}}[|d\rangle|V\rangle_{lSS} + |d\rangle|H\rangle_{lLL} \\
& + i(\underline{|d\rangle|H\rangle_{lSL} - |d\rangle|V\rangle_{lLS}})]_{c_3}.
\end{aligned} \tag{10}
$$

As shown in fig. 1, the subscripts c_1 and c_3 represent the two outputs of the PBS2. We can see that the output state of channel b_2 is divided into paths c_1 and c_3 by the PBS2. The underlined items are going to arrive at the two outputs c_1 and c_3 at a definite time (sSL in eq. (7) and eq. (8), lSL in eq. (9) and eq. (10)). Here, we only calculate the results in the path c_1 for simplicity. In addition, we use the HWP4 to perform a bit-flip in the path c_1 and the state in eq. (3) can be transformed to

$$
\begin{aligned}
& \left[\frac{|a\rangle}{\sqrt{2}}(|D\rangle_{sSS} - |A\rangle_{sSS}) - \frac{|a\rangle}{\sqrt{2}}(|D\rangle_{sLL} + |A\rangle_{sLL})\right]_{c_2} \\
& + \left[\frac{|b\rangle}{\sqrt{2}}(|D\rangle_{sSS} - |A\rangle_{sSS}) + \frac{|b\rangle}{\sqrt{2}}(|D\rangle_{sLL} + |A\rangle_{sLL})\right]_{c_2} \\
& + \left[\frac{|c\rangle}{\sqrt{2}}(|D\rangle_{lSS} - |A\rangle_{lSS}) - \frac{|c\rangle}{\sqrt{2}}(|D\rangle_{lLL} + |A\rangle_{lLL})\right]_{c_2} \\
& + \left[\frac{|d\rangle}{\sqrt{2}}(|D\rangle_{lSS} - |A\rangle_{lSS}) + \frac{|d\rangle}{\sqrt{2}}(|D\rangle_{lLL} + |A\rangle_{lLL})\right]_{c_2}. \\
& + i(|a\rangle|D\rangle_{sSL} + |b\rangle|A\rangle_{sSL} + |c\rangle|D\rangle_{lLS} + |d\rangle|A\rangle_{lLS})_{c_2}.
\end{aligned} \tag{11}
$$

Note that the coefficient $\frac{\delta}{2\sqrt{2}}$ caused by the noisy channel can be regarded as a common factor and it only decides the total probability of the state in spatial mode c_2 in eq. (11). In this way, we omit this coefficient in eq. (11). We rearrange the state in eq. (11) as

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}(|a\rangle|D\rangle_{sSS} - |b\rangle|A\rangle_{sSS} + |c\rangle|D\rangle_{lSS} - |d\rangle|A\rangle_{lSS})_{c_2} \\
& + \frac{1}{\sqrt{2}}(-|a\rangle|A\rangle_{sSS} + |b\rangle|D\rangle_{sSS} - |c\rangle|A\rangle_{lSS} + |d\rangle|D\rangle_{lSS})_{c_2} \\
& + \frac{1}{\sqrt{2}}(-|a\rangle|D\rangle_{sLL} + |b\rangle|A\rangle_{sLL} - |c\rangle|D\rangle_{lLL} + |d\rangle|A\rangle_{lLL})_{c_2} \\
& + \frac{1}{\sqrt{2}}(-|a\rangle|A\rangle_{sLL} + |b\rangle|D\rangle_{sLL} - |c\rangle|A\rangle_{lLL} + |d\rangle|D\rangle_{lLL})_{c_2} \\
& + i(|a\rangle|D\rangle_{sSL} + |b\rangle|A\rangle_{sSL} + |c\rangle|D\rangle_{lLS} + |d\rangle|A\rangle_{lLS})_{c_2}.
\end{aligned} \tag{12}
$$

Here, we explain the derivation of the items in the first line ($|\phi_1\rangle_{c_2} = |a\rangle|D\rangle_{sSS} - |b\rangle|A\rangle_{sSS} + |c\rangle|D\rangle_{lSS} - |d\rangle|A\rangle_{lSS}$) and those in the last line ($|\phi_2\rangle_{c_2} = |a\rangle|D\rangle_{sSL} + |b\rangle|A\rangle_{sSL} + |c\rangle|D\rangle_{lLS} + |d\rangle|A\rangle_{lLS}$) in detail. The remaining items can be obtained with the same principle. When the photon in eq. (12) interacts with Bob's first NV center (QM4), $|\phi_1\rangle_{c_2}$ combined with the state of QM4 becomes

$$
\begin{aligned}
|\phi_{d_1}\rangle_B = & \frac{1}{\sqrt{2}}(|a\rangle|D\rangle_{sSS} - |b\rangle|A\rangle_{sSS} \\
& + |c\rangle|D\rangle_{lSS} - |d\rangle|A\rangle_{lSS}) \otimes QM_4 \\
= & \frac{1}{\sqrt{2}}(|a\rangle|D\rangle_{sSS} - |b\rangle|A\rangle_{sSS} \\
& + |c\rangle|D\rangle_{lSS} - |d\rangle|A\rangle_{lSS}) \otimes (|g_4\rangle + |e_4\rangle) \\
= & \frac{1}{\sqrt{2}}(|g_1\rangle|g_3\rangle|g_4\rangle|D\rangle_{sSS} - |g_1\rangle|e_3\rangle|g_4\rangle|A\rangle_{sSS} \\
& + |e_1\rangle|g_3\rangle|g_4\rangle|D\rangle_{lSS} - |e_1\rangle|e_3\rangle|g_4\rangle|A\rangle_{lSS} \\
& + |g_1\rangle|g_3\rangle|e_4\rangle|A\rangle_{sSS} - |g_1\rangle|e_3\rangle|e_4\rangle|D\rangle_{sSS} \\
& + |e_1\rangle|g_3\rangle|e_4\rangle|A\rangle_{lSS} - |e_1\rangle|e_3\rangle|e_4\rangle|D\rangle_{lSS})_{d_1}.
\end{aligned} \tag{13}
$$

Next, Bob makes the photon in d_1 mode pass through OS2. With the help of OS2, the photon with the time-bin of lSS will be in the path d_2 and that with the time-bin of sSS will be in the path d_3. Then, the photon's polarization and time-bin DOFs will be swapped after it passes through switching setup [58]. In the switching setup, the "Delayers" represent the optical fibers. By precisely adjusting the length of the optical fibers, Bob can delay the time-bin of photon in e_1, e_2, f_1 and f_2 pathes by ss, l, s, and ls, respectively. Then, the OS3 can lead the photon in i_1 and i_2 to be in the same output mode m. For explaining the function of the switching setup, we take the photon in the state of D_{lSS} for example. After passing through the switching setup, the photon in D_{lSS} will evolve to

$$
\begin{aligned}
|D_{d_2}\rangle_{lSS} & \xrightarrow{CPBS2} |D_{e_3}\rangle_{lSS} \xrightarrow{HWP5} |A_{f_1}\rangle_{lSS} \\
& \xrightarrow{Delayer4} |A_{h_2}\rangle_{lSSs} \xrightarrow{CPBS5} |A_{i_2}\rangle_{lSSs} \\
& \xrightarrow{OS3} |A_m\rangle_{lSSs}.
\end{aligned} \tag{14}
$$

Similarly, we can obtain the state of the rest items in eq. (13) after the switching setup. As a result, $|\phi_{d_1}\rangle_B$ will finally evolve to

$$
\begin{aligned}
|\phi_{d_1}\rangle_B \ \xrightarrow{Switching} & \frac{1}{\sqrt{2}}(|g_1\rangle|g_3\rangle|g_4\rangle|D_m\rangle_{sSSl} \\
& -|g_1\rangle|e_3\rangle|g_4\rangle|D_m\rangle_{sSSls} + |e_1\rangle|g_3\rangle|g_4\rangle|A_m\rangle_{SSls} \\
& -|e_1\rangle|e_3\rangle|g_4\rangle|A_m\rangle_{lSSss} + |g_1\rangle|g_3\rangle|e_4\rangle|D_m\rangle_{sSSls} \\
& -|g_1\rangle|e_3\rangle|e_4\rangle|D_m\rangle_{sSSl} + |e_1\rangle|g_3\rangle|e_4\rangle|A_m\rangle_{lSSss} \\
& -|e_1\rangle|e_3\rangle|e_4\rangle|A_m\rangle_{lSSs}).
\end{aligned} \tag{15}
$$

Afterwards, the photon interacts with the last NV center (QM2) at Bob's side, resulting in the state

$$
\begin{aligned}
& \frac{1}{\sqrt{2}}(|g_1\rangle|g_2\rangle|g_3\rangle|g_4\rangle|D\rangle_{sSSl} - |g_1\rangle|g_2\rangle|e_3\rangle|g_4\rangle|D\rangle_{sSSls} \\
& +|e_1\rangle|g_2\rangle|g_3\rangle|g_4\rangle|A\rangle_{sSSl} - |e_1\rangle|g_2\rangle|e_3\rangle|g_4\rangle|A\rangle_{sSSls}
\end{aligned}
$$

$$
\begin{aligned}
&+|g_1\rangle|g_2\rangle|g_3\rangle|e_4\rangle|D\rangle_{sSSls} - |g_1\rangle|g_2\rangle|e_3\rangle|e_4\rangle|D\rangle_{sSSl}\\
&+|e_1\rangle|g_2\rangle|g_3\rangle|e_4\rangle|A\rangle_{sSSls} - |e_1\rangle|g_2\rangle|e_3\rangle|e_4\rangle|A\rangle_{sSSl}\\
&+|g_1\rangle|g_2\rangle|g_3\rangle|g_4\rangle|A\rangle_{sSSl} - |g_1\rangle|g_2\rangle|e_3\rangle|g_4\rangle|A\rangle_{sSSls}\\
&+|e_1\rangle|e_2\rangle|g_3\rangle|g_4\rangle|D\rangle_{sSSl} - |e_1\rangle|e_2\rangle|e_3\rangle|g_4\rangle|D\rangle_{sSSls}\\
&+|g_1\rangle|e_2\rangle|g_3\rangle|e_4\rangle|A\rangle_{sSSls} - |g_1\rangle|e_2\rangle|e_3\rangle|e_4\rangle|A\rangle_{sSSl}\\
&+|e_1\rangle|e_2\rangle|g_3\rangle|e_4\rangle|D\rangle_{sSSls} - |e_1\rangle|e_2\rangle|e_3\rangle|e_4\rangle|D\rangle_{sSSl}).
\end{aligned}
\tag{16}
$$

The above equation can be simplified as

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}[(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|g_3\rangle|g_4\rangle|D\rangle_{sSSl}\\
&-(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|e_3\rangle|e_4\rangle|D\rangle_{sSSl}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|g_3\rangle|g_4\rangle|A\rangle_{sSSl}\\
&-(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|e_3\rangle|e_4\rangle|A\rangle_{sSSl}\\
&-(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|e_3\rangle|g_4\rangle|D\rangle_{sSSls}\\
&+(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|g_3\rangle|e_4\rangle|D\rangle_{sSSls}\\
&-(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|e_3\rangle|g_4\rangle|A\rangle_{sSSls}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|g_3\rangle|e_4\rangle|A\rangle_{sSSls}].
\end{aligned}
\tag{17}
$$

Taking the items in the first two lines of eq. (17) as an example, we can simplify these items to obtain

$$
\frac{1}{\sqrt{2}}[(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)(|g_3\rangle|g_4\rangle - |e_3\rangle|e_4\rangle)]|D\rangle_{sSSl} = |\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sSSl}. \tag{18}
$$

The items in the other lines of eq. (17) can be simplified with the same principle. In this way, we can rewrite eq. (17) as

$$
\begin{aligned}
&|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sSSl} + |\psi^+_{12}\rangle|\phi^-_{34}\rangle|A\rangle_{sSSl}\\
&-|\phi^+_{12}\rangle|\psi^-_{34}\rangle|D\rangle_{sSSls} - |\psi^+_{12}\rangle|\psi^-_{34}\rangle|A\rangle_{sSSls},
\end{aligned}
\tag{19}
$$

where $|\phi^+_{ij}\rangle = \frac{1}{\sqrt{2}}(|g_i\rangle|g_j\rangle + |e_i\rangle|e_j\rangle)$ and $|\psi^+_{ij}\rangle = \frac{1}{\sqrt{2}}(|g_i\rangle|e_j\rangle + |e_i\rangle|g_j\rangle)$.

In the above description, we make a detailed derivation and calculation for the items in the first line of eq. (12). Then, we simply calculate the items in the fifth line of eq. (12), which are for inference calculation in the original work [58]. Firstly, after swapping the DOFs, the above items will be transformed to

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}(|g_1\rangle|g_3\rangle|g_4\rangle|D\rangle_{sSLl} + |g_1\rangle|e_3\rangle|g_4\rangle|D\rangle_{sSLls}\\
&+|e_1\rangle|g_3\rangle|g_4\rangle|A\rangle_{lLSs} - |e_1\rangle|e_3\rangle|g_4\rangle|A\rangle_{lLSss}\\
&+|g_1\rangle|g_3\rangle|e_4\rangle|D\rangle_{sSLls} - |g_1\rangle|e_3\rangle|e_4\rangle|D\rangle_{sSLl}\\
&+|e_1\rangle|g_3\rangle|e_4\rangle|A\rangle_{lLSss} - |e_1\rangle|e_3\rangle|e_4\rangle|A\rangle_{lLSs}).
\end{aligned}
\tag{20}
$$

Table 1: The entangled states between Alice and Bob corresponding to the measurement results in time-bin and polarization DOFs. The measurement results in the first two lines correspond to the protocol being successful, while those in the last four lines correspond to the protocol failing.

Time bin \ Polarization	D	A
$sSLl$	$\|\phi^+_{12}\rangle\|\phi^+_{34}\rangle$	$\|\psi^+_{12}\rangle\|\phi^+_{34}\rangle$
$lLSss$	$\|\phi^+_{12}\rangle\|\psi^+_{34}\rangle$	$\|\psi^+_{12}\rangle\|\psi^+_{34}\rangle$
$sSSl$	$\|\phi^+_{12}\rangle$	$\|\psi^+_{12}\rangle$
$sSSls$	$\|\phi^+_{12}\rangle$	$\|\psi^+_{12}\rangle$
$sLLl$	$\|\phi^+_{12}\rangle$	$\|\psi^+_{12}\rangle$
$lLLss$	$\|\phi^+_{12}\rangle$	$\|\psi^+_{12}\rangle$

Secondly, after the photon interacting with QM2 at Bob's side, we can obtain

$$
\begin{aligned}
&\tfrac{1}{\sqrt{2}}[(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|g_3\rangle|g_4\rangle|D\rangle_{sSLl}\\
&+(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|e_3\rangle|e_4\rangle|D\rangle_{sSLl}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|g_3\rangle|g_4\rangle|A\rangle_{sSLl}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|e_3\rangle|e_4\rangle|A\rangle_{sSLl}\\
&+(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|e_3\rangle|g_4\rangle|D\rangle_{lLSss}\\
&+(|g_1\rangle|g_2\rangle + |e_1\rangle|e_2\rangle)|g_3\rangle|e_4\rangle|D\rangle_{lLSss}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|e_3\rangle|g_4\rangle|A\rangle_{lLSss}\\
&+(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)|g_3\rangle|e_4\rangle|A\rangle_{lLSss}].
\end{aligned}
\tag{21}
$$

Similarly, we also rewrite eq. (21) with the form of Bell state as

$$
\begin{aligned}
&|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sSLl} + |\psi^+_{12}\rangle|\phi^+_{34}\rangle|A\rangle_{sSLl}\\
&+|\phi^+_{12}\rangle|\psi^+_{34}\rangle|D\rangle_{lLSss} + |\psi^+_{12}\rangle|\psi^+_{34}\rangle|A\rangle_{lLSss}.
\end{aligned}
\tag{22}
$$

We perform similar operations on the remaining items of eq. (12), and can obtain the whole output state at Bob's side as

$$
\begin{aligned}
&(|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sSSl} + |\psi^+_{12}\rangle|\phi^-_{34}\rangle|A\rangle_{sSSl}\\
&-|\phi^+_{12}\rangle|\psi^-_{34}\rangle|D\rangle_{sSSls} - |\psi^+_{12}\rangle|\psi^-_{34}\rangle|A\rangle_{sSSls})\\
&+(-|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sSSls} - |\psi^+_{12}\rangle|\phi^-_{34}\rangle|A\rangle_{sSSls}\\
&+|\phi^+_{12}\rangle|\psi^-_{34}\rangle|D\rangle_{sSSl} + |\psi^+_{12}\rangle|\psi^-_{34}\rangle|A\rangle_{sSSl})\\
&+(-|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sLLl} - |\psi^+_{12}\rangle|\phi^-_{34}\rangle|A\rangle_{sLLl}\\
&+|\phi^+_{12}\rangle|\psi^-_{34}\rangle|D\rangle_{lLLss} + |\psi^+_{12}\rangle|\psi^-_{34}\rangle|A\rangle_{lLLss})\\
&+(|\phi^+_{12}\rangle|\phi^-_{34}\rangle|D\rangle_{sLLls} + |\psi^+_{12}\rangle|\phi^-_{34}\rangle|A\rangle_{sLLls}\\
&-|\phi^+_{12}\rangle|\psi^-_{34}\rangle|D\rangle_{lLLs} - |\psi^+_{12}\rangle|\psi^-_{34}\rangle|A\rangle_{lLLs})\\
&+(\underline{|\phi^+_{12}\rangle|\phi^+_{34}\rangle|D\rangle_{sSLl} + |\psi^+_{12}\rangle|\phi^+_{34}\rangle|A\rangle_{sSLl}}\\
&+\underline{|\phi^+_{12}\rangle|\psi^+_{34}\rangle|D\rangle_{lLSss} + |\psi^+_{12}\rangle|\psi^+_{34}\rangle|A\rangle_{lLSss}}).
\end{aligned}
\tag{23}
$$

Finally, we measure the polarization and time-bin features of the output photon. As shown in table 1, if the output photon is in $|D\rangle_{sSLl}$, Alice and Bob will share two pairs of maximally entangled states $|\phi^+_{12}\rangle|\phi^+_{34}\rangle$. On the other hand, if the measurement results are $|A\rangle_{sSLl}$, they will share $|\psi_{12}^{+}\rangle|\phi_{34}^{+}\rangle$. They will also obtain the state $|\phi_{12}^{+}\rangle|\psi_{34}^{+}\rangle$ and $|\psi_{12}^{+}\rangle|\psi_{34}^{+}\rangle$, corresponding to the measurement results of $|D\rangle_{lLSss}$ and $|A\rangle_{lLSss}$, respectively. We define that when Bob obtains one of the four above measurement results, our protocol is successful. From eqs. (7)–(11), the success probability of our protocol corresponding to the state in path b_2 is 0.25, which is independent of the noise parameters δ and η, so that the success probability of generating two pairs of maximally entangled states is 0.25. Meanwhile, by considering the total contribution of paths b_2 and b_3, the probability of our protocol will be 0.5. Interestingly, it can be found that if our protocol fails, say, Bob obtains one of the measurement results in the last four lines of table 1, Alice and Bob can also share one pair of maximally entangled state. For example, if the measurement result is $|D\rangle_{sSSl}$, they will obtain $|\phi_{12}^{+}\rangle|\phi_{34}^{-}\rangle + |\phi_{12}^{+}\rangle|\psi_{34}^{-}\rangle$, which allows them to obtain one pair of maximally entangled state $|\phi_{12}^{+}\rangle$. Therefore, after the entanglement distribution, Alice and Bob can obtain two pairs of maximally entangled states or one pair of maximally entangled state, according to Bob's measurement result.

Discussion and conclusion. – So far, we have completely described this protocol based on the protocols in refs. [58,63]. In ref. [58], they described the quantum multiplexing protocol which can distribute two pairs of maximally entangled states simultaneously. In their protocol, if the entanglement is degraded by the channel noise, they have to exploit the entanglement purification to distill the high-quality entangled states from low-quality entangled states. Their entanglement purification is quite different from existing entanglement purification protocols, in which they use the time-bin entanglement to purify the polarization entanglement. When the purification is successful, the parties can obtain a high fidelity polarization entanglement by consuming the time-bin entanglement. When the entanglement purification fails, both the entanglement in polarization and time-bin DOFs have to be discarded. In our protocol, we consider the collective noise mode. We show that by designing the encoding and decoding setup, the parties can faithfully obtain two pairs of maximally entangled states with some success probability. On the other hand, if the protocol fails, Alice and Bob can still obtain one pair of maximally entangled state. The main reason is that the collective noise may cause error in the polarization DOF but does not affect the entanglement in the time-bin DOF. In practical transmission, the time-bin entanglement is robust which has been verified in experiments [64–66]. Moreover, our protocol also has application potential in the high-dimensional entanglement distribution, like the effort of Ecker *et al.* [67].

In conclusion, we propose a faithful entanglement distribution protocol using quantum multiplexing in a collective noise channel. By designing the encoding and decoding setup, our protocol can efficiently resist the collective noise. When the protocol is successful, the parties can obtain two pairs of maximally entangled states. Even if the protocol fails, the parties can still obtain one pair of maximally entangled state. This protocol does not require the entanglement purification to consume the low-quality entangled states to obtain the high-quality entangled states, so that it has higher entanglement distribution efficiency. Based on the above features, our entanglement distribution protocol has potential application in future quantum communication field.

* * *

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