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To cite this article: George Livadiotis and David J. McComas 2021 *EPL* **135** 49001

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Perspective

Black-body radiation in space plasmas

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received 27 September 2021; accepted in final form 8 October 2021

published online 25 October 2021

Abstract – We generalize Planck’s law for black-bodies, classically described by Maxwell-Boltzmann distributions, to include space and astrophysical plasmas described by kappa distributions. This provides the spectral intensity of electromagnetic radiation emitted by black bodies in thermal equilibrium with the surrounding and interacting plasmas. According to the generalized concept of thermal equilibrium, systems with correlations among their particles’ velocities/energies reside in stationary states described by kappa distributions associated with the thermodynamic parameters of temperature and kappa index. Using these distributions, we derive the generalized expressions of the i) mean energy of photon ensemble, ii) spectral intensity with respect to frequency and wavelength, and iii) Stefan-Boltzmann law, characterized by the well-known fourth power of temperature, but now multiplied by a new kappa-dependent factor. Finally, we discuss the implications of these new developments for space and astrophysical plasmas.

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Introduction. – A black body is an idealized opaque and non-reflective physical body, absorbing all incident electromagnetic radiation, regardless of frequency. The failure of classical physics to describe the black-body behavior helped establish the foundations of statistical and quantum mechanics [1]. Further understanding, modifications, and generalizations of the physical properties may lead to an improved black-body description. This paper develops the formulation of the black body radiation, using the generalized framework of statistical mechanics and thermodynamics that applies in space plasmas, as well as other systems that reside in stationary states out of the classical thermal equilibrium.

Black-body electromagnetic radiation is typically referred to as thermal, because it is emitted by a physical body in thermal equilibrium with its environment. Specifically, the relevant spectral intensity or radiance are denoted by $B_\nu(T)$ or $B_\lambda(T)$, respectively. These measure the power, per solid angle, area normal to propagation, and frequency ν or wavelength λ , of the black body in thermal equilibrium at temperature T with the surrounding, interacting medium. Therefore, the physical theory behind the study of the black body is encompassed in thermodynamics and statistical mechanics, also called

thermostatistics, which determines how a particle system behaves when it resides in thermal equilibrium [2]. However, space plasmas rarely reach thermal equilibrium; therefore, we ask: how would a black-body be properly described when it interacts with non-equilibrium plasmas?

Any initial state of a system eventually evolves to the stationary state of thermal equilibrium, where the entropy is maximized. Thermal equilibrium is a special stationary state of particle systems, in which any flow of heat (thermal conduction, thermal radiation) is in balance. It is associated with the definition of temperature, by means of the zeroth law of thermodynamics: two thermodynamic systems that are each in thermal equilibrium with a third are in thermal equilibrium with each other. Equivalently, two systems (*e.g.*, a system with its environment) are in thermal equilibrium if they are stationary while linked by a wall permeable only to heat exchange (*e.g.*, [3]). A classical system of particles, residing in the special stationary state of thermal equilibrium, is associated with a temperature, a procedure called “thermalization”, where particle velocities (or energies) are stabilized into a Maxwell-Boltzmann distribution.

In recent decades, however, space plasma observations and theoretical investigations have revealed that classical Maxwell-Boltzmann thermal equilibrium is not the only stationary state of a system. Systems such as space

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plasmas generally reside in stationary states where particles' velocities (or energies) are characterized by statistical correlations rather than being independent, and are temporarily stabilized into the kappa distributions instead of the Maxwell-Boltzmann distribution. For physical systems residing in these stationary states that constitute the generalized (also called nonthermal) thermal equilibrium, the temperature and zeroth law of thermodynamics are still meaningful, the kappa distributions formulate their statistics, and correlations further characterize the particle's velocities or energies. Starting from the zeroth law of thermodynamics it has been shown that kappa distributions constitute the most generalized form of particle distributions associated with a temperature [4,5].

A variety of mechanisms are responsible for generating kappa distributions, including superstatistics (*e.g.*, [6–10]), radiation in plasmas [11], shock waves (*e.g.*, [12]), turbulence (*e.g.*, [13–15]), effects of pickup ions [4,16], acceleration mechanisms (*e.g.*, [17]), polytropic behavior [18–20], and Debye shielding and magnetic coupling [21]. Such physical mechanisms connect particles through long-range interactions, producing statistical correlations among particles, and leading the system to be stabilized into specific kappa distributions.

Kappa distributions are common and observed across space and astrophysical plasmas, describing particles in the heliosphere, from solar wind and planetary magnetospheres to the distant heliospheric boundaries and beyond, and even to interstellar and intergalactic plasmas, and cosmology (see [22,23] and references therein). Half a century ago, kappa distributions were empirically derived as a suitable fitting model describing observations of space and other plasma particle populations [24–26]. A breakthrough in the field came with the connection of kappa distributions with the framework of nonextensive statistical mechanics [27], by maximizing entropy under the constraints of canonical ensemble [28–31]. Thereafter, numerous theoretical developments and applications in space science were published, emphasizing the importance of kappa distributions in describing space thermodynamics.

Processes and concepts in space plasmas related to statistical mechanics and/or thermodynamics need to be studied under the framework of space thermodynamics, namely, the theory of kappa distributions, the generalized thermal equilibrium implied by the stationary states in the presence of correlations, and their statistical framework of nonextensive statistical mechanics. The purpose of this paper is to examine the physics of black-body radiation within this framework of kappa distributions and generalized thermal equilibrium [5]. Next, we develop the intensity of the black-body radiation in space plasmas by following the formulation of the kappa distribution for discrete energy spectra and the statistics of the black-body discrete energy spectrum. Finally, we discuss implications of the new developments in space and astrophysical plasmas. The supplementary material

[Supplementarymaterial.pdf](#) (SM) shows the derivation of the near-Boltzmannian approximation.

Nonextensive statistical mechanics: brief overview. – Nonextensive statistical mechanics is the generalization of classical Boltzmann-Gibbs statistical mechanics based on a mono-parameterized entropic formulation and the concept of escort probabilities. This entropy generalizes the classical Boltzmann entropic formulation by means of a parameter q that recovers to Boltzmann entropy for $q = 1$. Following the Gibbs path, the Boltzmann entropy is maximized under the constraints of canonical ensemble, leading to the exponential function of the Boltzmann distribution [32]; similarly, the generalized entropy is maximized under the constraints of canonical ensemble leading to the distribution function

$$p(\varepsilon; T; q) \sim [1 - (q - 1) \cdot \varepsilon / (k_B T)]^{\frac{1}{q-1}}. \quad (1)$$

The above paragraph summarizes the pioneer paper on the topic published in 1988 by Tsallis [27]. The studies that followed, however, showed that there were several flaws with this formulation of the distribution. Briefly, it was understood that the generalization of the classical to the nonextensive statistical mechanics has to apply to both the entropic formulation and the averaging. Namely, the Boltzmann entropy is generalized to the Tsallis entropic form, but also the probability distribution is generalized under the concept of escort probability distribution $P \sim p^q$ [33]. The result of these considerations [34] led to the canonical distributions, that is, respectively, the ordinary and escort functions:

$$p(\varepsilon; T; q) \sim [1 + (q - 1) \cdot (\varepsilon - U) / (k_B T)]^{\frac{1}{1-q}}, \quad (2)$$

$$P(\varepsilon; T; q) \sim p(\varepsilon; T; q)^q \sim$$

$$[1 + (q - 1) \cdot (\varepsilon - U) / (k_B T)]^{-\frac{1}{q-1}-1}, \quad (3)$$

where the internal energy per particle U is the mean energy.

The modern framework of nonextensive statistical mechanics [34] is associated with eq. (3) and provides the same physical definitions as in classical statistical mechanics; namely, both the kinetic [35] and thermodynamic [5,36,37] definitions of temperature, and their equivalence [4,23,31,38].

The Hamiltonian function of a particle is $H(\vec{r}, \vec{u}) = \varepsilon_K(\vec{u}) + \Phi(\vec{r})$, where $\varepsilon_K(\vec{u}) = \frac{1}{2}m \cdot u^2$ is the kinetic energy and $\Phi(\vec{r})$ is the potential energy, which depends only on the position vector. The q -distribution of a Hamiltonian gives the probability distribution of a particle having its position and velocity in the infinitesimal intervals $[\vec{r}, \vec{r} + d\vec{r}]$ and $[\vec{u}, \vec{u} + d\vec{u}]$, respectively. The corresponding formulation of kappa distributions coincides exactly with that of q -exponential distribution, under the transformation of their indices $\kappa = 1/(q - 1)$ [31]. Under this transformation, the kappa distribution of velocities/energies that

follows from eq. (1) is

$$P(\vec{r}, \vec{u}) \sim \left[1 - (q-1) \cdot \frac{H(\vec{r}, \vec{u})}{k_B T} \right]^{\frac{1}{q-1}} \sim \left[1 + \frac{1}{\kappa} \cdot \frac{H(\vec{r}, \vec{u})}{k_B T} \right]^{-\kappa}. \quad (4)$$

In the modern version of nonextensive statistical mechanics, the kappa distribution is aligned to eq. (3) [23,39–41]:

$$\left[1 + (q-1) \cdot \frac{H(\vec{r}, \vec{u}) - \langle H \rangle}{k_B T} \right]^{-\frac{1}{q-1}} \sim \left[1 + \frac{1}{\kappa} \cdot \frac{H(\vec{r}, \vec{u}) - \langle H \rangle}{k_B T} \right]^{-\kappa-1}, \quad (5)$$

where $\langle H \rangle$ is the ensemble phase-space average of the Hamiltonian. For zero potential energy $\Phi \sim 0$, eq. (5) becomes

$$P(\vec{u}) \sim \left[1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_K(\vec{u}) - \langle \varepsilon_K \rangle}{k_B T} \right]^{-\kappa-1}. \quad (6)$$

Lastly, we refer to the invariant kappa index. The kappa index depends on the particle degrees of freedom d , $\kappa(d) = \kappa_0 + \frac{1}{2}d$ [40–43], where the 0-degrees kappa index κ_0 is a thermodynamic property invariant under variations of dimensionality. The degrees of freedom count the kinetic d_K and potential d_Φ degrees. The former is typically 3, while the latter depends on the form of potential energy Φ . For a central power potential, $\Phi(r) \sim \pm r^{\pm b}$, (any analytical form can be locally approximated by a power law), the potential degrees equal $(2/b)d_r$, where d_r is the dimensionality of the positional vector (typically, 3). Hence, $d = d_K + (2/b)d_r$; for a harmonic oscillator we have $b=2$, $d=6$, or $\kappa_0 = \kappa - 3$; the theory developed here represents photons in a box, akin to a harmonic oscillator.

The notion of the invariant kappa index is suitable for characterizing the thermodynamics of the system. In the case of the stationary state corresponding to the classical thermal equilibrium, the kappa index tends to $\kappa_0 \rightarrow \infty$. In the case of anti-equilibrium [22], the kappa index tends to the limit $\kappa_0 \rightarrow 0$. As the name suggests, this is the furthest possible stationary state from the classical thermal equilibrium. Each stationary state is described by kappa distributions of a certain kappa index, with the two limits i) at the classical thermal equilibrium for $\kappa_0 \rightarrow \infty$, described by the Maxwell-Boltzmann velocity distributions and zero correlation among particle velocities, and ii) at anti-equilibrium, where the velocity distribution tends to a simple power-law [4], and the correlation among particle velocities is maximized [22,42,44].

Throughout the paper, we will be using the standard kappa index formalism and symbol κ , but one can easily convert the results through the invariant kappa index $\kappa_0 = \kappa - 3$.

Formulation of discrete kappa distributions. –

We recall that kappa distributions are consistent with the concept of generalized thermal equilibrium, describing systems residing in stationary states characterized by the intensive thermodynamic parameters of temperature and kappa [5]. Here we show the formalism of discrete kappa distributions.

Let the discrete energy $\{\varepsilon_j\}$ of a particle system be associated with a discrete kappa distribution function $\{P_j\}$. If the system resides in stationary states and exhibits correlations among particles' velocities/energies, then it actually resides in the generalized thermal equilibrium [5], where its particles phase space is described by kappa distributions, *i.e.*,

$$P_j = \frac{1}{Z} \cdot \left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_j - \langle \varepsilon \rangle}{k_B T} \right)^{-\kappa-1}, \quad (7)$$

$$Z = \sum_{j=0}^{\infty} \left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_j - \langle \varepsilon \rangle}{k_B T} \right)^{-\kappa-1}.$$

The mean energy is determined by

$$\langle \varepsilon \rangle = \sum_{j=0}^{\infty} P_j(\langle \varepsilon \rangle) \cdot \varepsilon_j, \quad (8)$$

which can be given implicitly from

$$\sum_{j=0}^{\infty} \left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_j - \langle \varepsilon \rangle}{k_B T} \right)^{-\kappa-1} = \sum_{j=0}^{\infty} \left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon_j - \langle \varepsilon \rangle}{k_B T} \right)^{-\kappa}. \quad (9)$$

It is useful to express the internal energy $\langle \varepsilon \rangle$ for stationary states near classical equilibrium ($\kappa \rightarrow \infty$), that is, for $1/\kappa \ll 1$. We set $\beta(\varepsilon) \cong y_0 + \frac{1}{\kappa}y_1 + O(\frac{1}{\kappa})^2$ where $\beta \equiv (k_B T)^{-1}$ denotes the inverse temperature; in appendix A in the SM, we show that

$$y_0 = S_1/S_0, y_1 = \frac{1}{2}S_3/S_0 - \left(1 + \frac{3}{2}y_0 \right) S_2/S_0 + y_0^2 + y_0^3, \quad (10)$$

where the S_k coefficients can be defined and derived from

$$S_k \equiv \sum_{j=0}^{\infty} e^{-\beta \varepsilon_j} (\beta \varepsilon_j)^k, S_k = (-1)^k \beta^k \cdot \partial^{(k)} S_0 / \partial \beta^{(k)}. \quad (11)$$

The term y_0 denotes the result within the framework of the classical Boltzmann-Gibbs statistical mechanics, *i.e.*, $\langle \varepsilon \rangle = y_0 k_B T$ for $\kappa \rightarrow \infty$; the percentage deviation from the classical value is $\langle \varepsilon \rangle / (y_0 k_B T) - 1 = y_1 / (\kappa y_0)$ for $\kappa \rightarrow \infty$.

Black body in generalized thermal equilibrium.

– It is important to follow the classical path of derivation of the black-body radiation. This is developed by substituting the Boltzmannian formulation of the discrete distribution of energies with the one given by the discrete kappa distribution of energies shown in the previous section

and especially in eq. (7). In fact, this black-body radiation formulation is a natural thermodynamic generalization for systems out of thermal equilibrium. We note that there were several other attempts to approach the problem within a generalized statistical framework (*e.g.*, [45–47]). However, these were not based on the self-consistent framework of the modern adaptation of nonextensive statistical mechanics, which leads to eq. (3) (in terms of the entropic parameter q), or equivalently, to eq. (5) (in terms of the thermodynamic kappa index κ). (For details, see [22,23,31,43].)

The black-body radiation spectrum is characterized by a discrete energy and probability distribution and studied by the thermostatics of the generalized thermal equilibrium, which are described by the formulation of kappa distributions.

Let us consider a cubical black body of size L , filled with electromagnetic radiation residing in a stationary state (generalized thermal equilibrium at temperature T and kappa κ) with the surrounding space plasma. The ensemble of any number of photons, j , leads to the discrete energy of $\varepsilon_j = (j + \frac{1}{2}) \cdot h\nu$. The constant factor $\frac{1}{2}h\nu$ has no effect on the statistics, either for the classical Boltzmann exponential function (as an exponential function that can be factored out) or for kappa distributions (since the base of the distribution includes the difference between energy and its mean value, eq. (7)). Hence, we ignore the constant part of the energy levels, setting for simplicity that $\varepsilon_j = j \cdot h\nu$. The 3D wave number and wavelength components characterize the frequency, given by $2\pi \cdot \nu = ck = c(k_x^2 + k_y^2 + k_z^2)^{1/2}$ or $\nu = c(\lambda_x^{-2} + \lambda_y^{-2} + \lambda_z^{-2})^{1/2}$. According to the theory analogous to the wave function of a particle in a box, the electromagnetic field is expressed as superposition of periodic standing waves, where each wavelength component is determined by a positive integer, $\lambda_i = 2L/n_i$, with i : x , y , z , and the frequency is proportional to the magnitude of these numbers $n = (n_x^2 + n_y^2 + n_z^2)^{1/2}$, *i.e.*, $\nu = n \cdot [c/(2L)]$. Then, setting, $a \equiv \varepsilon_1/(k_B T) = h\nu/(k_B T)$,

$$\frac{\varepsilon_j}{k_B T} = j \cdot \frac{h\nu}{k_B T} \equiv j \cdot a, \quad \frac{\langle \varepsilon \rangle}{k_B T} = \langle j \rangle \cdot \frac{h\nu}{k_B T} = \langle j \rangle \cdot a, \quad (12)$$

where $\langle j \rangle$ is determined in terms of a using eqs. (8), (9), *i.e.*,

$$\langle j \rangle = \frac{\sum_{j=0}^{\infty} \left[1 + \frac{1}{\kappa} (j - \langle j \rangle) a \right]^{-\kappa-1} j}{\sum_{j=0}^{\infty} \left[1 + \frac{1}{\kappa} (j - \langle j \rangle) a \right]^{-\kappa-1}}, \quad (13)$$

$$\begin{aligned} \sum_{j=0}^{\infty} \left[1 + \frac{1}{\kappa} (j - \langle j \rangle) a \right]^{-\kappa-1} &= \\ \sum_{j=0}^{\infty} \left[1 + \frac{1}{\kappa} (j - \langle j \rangle) a \right]^{-\kappa} &. \end{aligned} \quad (14)$$

The above equations generalize the respective equation of the classical Boltzmannian case (*e.g.*, [48]):

$$\sum_{j=0}^{\infty} e^{-j \cdot a} (j - \langle j \rangle) = 0 \quad \text{or} \quad \langle j \rangle = \frac{\sum_{j=0}^{\infty} e^{-j \cdot a} j}{\sum_{j=0}^{\infty} e^{-j \cdot a}}. \quad (15)$$

The expressions in eqs. (13), (14) may be used to determine $\langle j \rangle$, which corresponds to the mean kinetic energy $\langle \varepsilon \rangle$ according to eq. (12). We avoid the summations in our calculations, using a known closed form, the Hurwitz zeta function [49],

$$\zeta(s, \lambda) \equiv \sum_{j=0}^{\infty} (\lambda + j)^{-s}. \quad (16)$$

Then, eq. (14) can be written without the summation as

$$(\kappa/a) \cdot \zeta(\kappa + 1, \kappa/a - \langle j \rangle) = \zeta(\kappa, \kappa/a - \langle j \rangle). \quad (17)$$

Still, the desired unknown, $\langle j \rangle$, is implicitly involved in the above equations. Then, we use the following mathematical substitution to have $\langle j \rangle$ simply expressed in terms of a . Setting $\Lambda \equiv \kappa/a - \langle j \rangle$, we derive the expression of $\langle j \rangle$ as a function of a , denoted by $\langle j \rangle(a)$, as a parametric equation, where Λ is the common parameter (see fig. 1(a)),

$$\begin{aligned} \langle j \rangle(\Lambda) &= \frac{\zeta(\kappa, \Lambda)}{\zeta(\kappa + 1, \Lambda)} - \Lambda, \\ a(\Lambda) &= \kappa \cdot \frac{\zeta(\kappa + 1, \Lambda)}{\zeta(\kappa, \Lambda)}. \end{aligned} \quad (18)$$

It is faster in coding to determine the Hurwitz zeta function just once; using an identity of this function, we rewrite (18):

$$\langle j \rangle(\Lambda) = -\kappa/\ln' \zeta(\kappa, \Lambda) - \Lambda, \quad a(\Lambda) = \ln' \zeta(\kappa, \Lambda). \quad (19)$$

Note that the above quantities depend on both Λ and kappa, $\langle j \rangle(\Lambda; \kappa)$ and $a(\Lambda; \kappa)$; for simplicity, we ignored showing the dependence on kappa; still, the desired equation $\langle j \rangle(a; \kappa)$ is derived from eliminating Λ from $\langle j \rangle(\Lambda; \kappa)$ and $a(\Lambda; \kappa)$.

Next, we calculate the spectral intensity (or radiance) at frequency ν , $B_\nu(T; \kappa)$, or at wavelength λ , $B_\lambda(T; \kappa)$, of the radiation emitted by the black body residing in nonthermal equilibrium with the surrounding space plasma characterized by temperature T and correlations with kappa κ (fig. 1(b), (c)).

Setting $\varepsilon_1 = ak_B T = h\nu = hc/\lambda$, the density of photon states is $g(n)dn = g(\varepsilon_1)d\varepsilon_1$, with $g(n)dn = \pi n^2 dn = V 8\pi h^{-3} c^{-3} \varepsilon_1^{-2} d\varepsilon_1 = V 8\pi c^{-3} \nu^2 d\nu = -V 8\pi \lambda^{-4} d\lambda$, where the black-body volume is $V = L^3$; we find $\langle j \rangle \nu^3 d\nu = -8\pi h c \langle j \rangle \lambda^{-5} d\lambda$. The spectral intensity $B_\nu(T; \kappa)$ or $B_\lambda(T; \kappa)$, is the energy $dU = \langle \varepsilon \rangle g(n)dn$ per volume V and frequency interval $d\nu$ or wavelength interval $d\lambda$:

$$\begin{aligned} B_\nu(\nu; T, \kappa) &\equiv (c/4\pi) \cdot |dU/(V d\nu)| = \\ &2hc^{-2} \cdot \langle j \rangle \left(\frac{h}{k_B T} \nu; \kappa \right) \cdot \nu^3, \end{aligned} \quad (20)$$

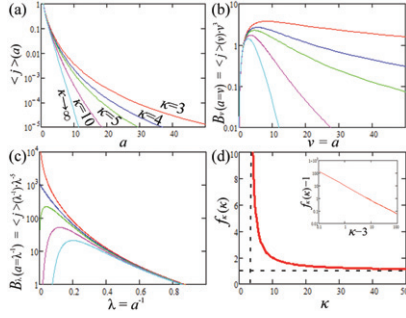


Fig. 1: (a) The average number of photons $\langle j \rangle$, which corresponds to the average energy $\langle \varepsilon \rangle = \langle j \rangle h\nu$ or $\langle \varepsilon \rangle / (k_B T) = \langle j \rangle \cdot a$, is plotted with respect to $a \equiv h\nu / (k_B T)$, and for kappa indices $\kappa = 3$ (red), $\kappa = 4$ (blue), $\kappa = 5$ (green), $\kappa = 10$ (pink), and $\kappa \rightarrow \infty$ (light-blue). (b) The spectral intensity $B_\nu(a = \nu; \kappa)$ in eq. (20) is plotted with respect to a , which coincides with the frequency with units $h/(k_B T) \equiv 1$, for the same values of kappa as in (a). (c) Similar to panel (b), the intensity $B_\lambda(a = \lambda^{-1}; \kappa)$ in eq. (21) is plotted with respect to $\lambda = a^{-1}$. (d) The kappa-dependent factor f_κ that modifies the Stefan-Boltzmann law diverges for $\kappa < 3$ and is plotted with respect to κ ; the input panel shows the approximately power-law relationship of $f_\kappa \approx 1 + 10 \cdot (\kappa - 3)^{-1}$. Note: $\kappa \rightarrow 3$ corresponds to the furthest state from thermal equilibrium, which is expected when the invariant kappa index $\kappa_0 = \kappa - 3$ reaches zero.

$$B_\lambda(\lambda; T, \kappa) \equiv (c/4\pi) \cdot |dU/(V d\lambda)| = 2hc^{-2} \cdot \langle j \rangle \left(\frac{h}{k_B T} \lambda^{-1}; \kappa \right) \cdot \lambda^{-5}. \quad (21)$$

The total energy emitted over the whole radiance spectrum U is $dU/V = \langle \varepsilon \rangle g(n) dn/V = \langle j \rangle a k_B T \cdot 8\pi h^{-3} c^{-3} \varepsilon_1^2 d\varepsilon_1$, that is, $dU/V = \langle j \rangle 8\pi h^{-3} c^{-3} (k_B T)^4 \cdot a^3 da$. Hence,

$$u \equiv \frac{1}{4} c \cdot U/V = 2\pi h^{-3} c^{-2} (k_B T)^4 \int_0^\infty \langle j \rangle(a; \kappa) a^3 da, \quad (22)$$

leading to the generalized Stefan-Boltzmann law (fig. 1(d)),

$$u = \frac{2}{15} \pi^5 h^{-3} c^{-2} (k_B T)^4 f_\kappa = \sigma \cdot T^4 \cdot f_\kappa, \quad (23)$$

which includes the known fourth power of temperature [50], but also an important new kappa-dependent factor

$$f_\kappa \equiv \frac{\int_0^\infty \langle j \rangle(a; \kappa) a^3 da}{\int_0^\infty \langle j \rangle(a; \kappa \rightarrow \infty) a^3 da} = 15\pi^{-4} \int_0^\infty \langle j \rangle(a; \kappa) a^3 da. \quad (24)$$

This recovers $f_\infty \rightarrow 1$ at $\kappa \rightarrow \infty$, because

$$\int_0^\infty \langle j \rangle(a; \kappa \rightarrow \infty) a^3 da = \int_0^\infty a^3 (e^a - 1)^{-1} da = \pi^4/15. \quad (25)$$

Next, we use the approximation in eqs. (9), (10). From eq. (14)

$$\beta \langle \varepsilon \rangle = \beta h\nu \langle j \rangle = a \langle j \rangle \cong y_0 + \frac{1}{\kappa} y_1, \quad \text{for } \kappa \gg 1, \quad (26)$$

where y_0 and y_1 are derived from eq. (10), with S_k coefficients:

$$S_k \equiv a^k \cdot \sum_{j=0}^{\infty} e^{-a \cdot j} j^k = (-a)^k \cdot \partial^{(k)} S_0 / \partial a^{(k)}, \quad (27)$$

leading to $S_k = a^k e^a (e^a - 1)^{-k-1} (1 + R_k)$, for $k=0,1,2,3$, with $R_k = \{0, 0, e^a, e^a(e^a + 4)\}$. Hence, we end up with $\langle j \rangle \cong a^{-1} (y_0 + \frac{1}{\kappa} y_1)$ (from eq. (26), where $y_0 = (1 - e^{-a})^{-1}$ and $y_1 = a^2 e^a (e^a - 1)^{-3} [(\frac{1}{2}a - 1)e^a + \frac{1}{2}a + 1]$, i.e.,

$$\langle j \rangle \cong (e^a - 1)^{-1} + \frac{1}{\kappa} \cdot a e^a (e^a - 1)^{-3} \left[\left(\frac{1}{2}a - 1 \right) e^a + \frac{1}{2}a + 1 \right]. \quad (28)$$

The spectral intensity per frequency in eq. (20) becomes

$$B_\nu(T; \kappa) \cong \frac{2hc^{-2} \nu^3}{e^{\frac{h}{k_B T} \nu} - 1} \cdot \left(1 + \frac{1}{\kappa} \cdot \delta \ln B_\nu \right), \quad (29)$$

with

$$\delta \ln B_\nu \equiv \frac{\frac{h}{k_B T} \nu e^{\frac{h}{k_B T} \nu}}{e^{\frac{h}{k_B T} \nu} - 1} \cdot \left(\frac{1}{2} \frac{h}{k_B T} \nu \frac{e^{\frac{h}{k_B T} \nu}}{e^{\frac{h}{k_B T} \nu} - 1} + 1 - 1 \right). \quad (30)$$

The near-Boltzmannian term of the order $(1/\kappa)$ can be interpreted as statistical fluctuations from the classical case of the Boltzmannian term. When a system resides near $(1/\kappa \ll 1)$ but not exactly at classical equilibrium $(1/\kappa \rightarrow 0)$, then there are deviations from the classical intensity (and its Boltzmannian term), which may be observable. Such deviations would have been interpreted as just statistical fluctuations, but instead, they could be physical and caused by the near-Boltzmannian term, corresponding to percentage deviation $\delta \ln B_\nu$. The respective approximation of the kappa-dependent correcting factor of eq. (24) that modifies the Stefan-Boltzmann law, eq. (23), is given by

$$f_\kappa \cong 1 + \frac{1}{\kappa} \cdot \frac{15}{\pi^4} \int_0^\infty \frac{a^4 e^a}{(e^a - 1)^2} \left(\frac{1}{2} a \frac{e^a + 1}{e^a - 1} - 1 \right) da = 1 + 6 \frac{1}{\kappa}. \quad (31)$$

Finally, we compare the exact spectral intensity per frequency, shown in eq. (20), and its approximation, shown in eq. (29). Figure 2 plots the percentage deviation of the spectral intensity from its classical value, expressed by $\delta \ln B_\nu$, for both the exact and approximate expressions, and for several kappa indices. We observe that the difference between the two graphs tends to zero as the kappa increases; for kappa ~ 100 or larger, the two expressions practically coincide.

Discussion. – In this study, we generalized Planck's law for black-body radiation for systems described by kappa distributions, such as in space plasmas, which reside in stationary states out of classical thermal equilibrium. This provides the spectral intensity of electromagnetic radiation, emitted by a black body residing in nonthermal or

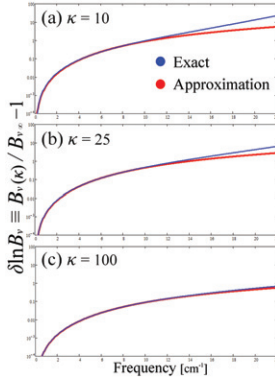


Fig. 2: Percentage deviation of the spectral intensity from its classical value, $\delta \ln B_\nu$, plotted with respect to frequency, and for kappa indices, (a) $\kappa = 10$, (b) $\kappa = 25$, and (c) $\kappa = 100$. We plot both the exact (blue) and approximate (red) expressions of the spectral intensity, using eqs. (20) and (29), respectively.

generalized thermal equilibrium, that is, in the stationary state described within the framework of kappa distributions. Then, statistical correlations may characterize the particle velocities/energies as generally observed in space and astrophysical plasmas.

In summary, in this paper, we: i) developed the equation of the mean energy of photon ensemble, and spectral intensity (radiance) with respect to frequency and wavelength for black bodies residing in the stationary states of the generalized thermal equilibrium with the surrounding and interacting material or particles (*e.g.*, space plasma), at temperature T and kappa index κ ; ii) updated the Stefan-Boltzmann law of the total energy emitted from the whole radiance spectrum for black bodies residing in generalized thermal equilibrium, which is characterized by the well-known fourth power of temperature, but now multiplied by a kappa-dependent factor; and iii) derived first-order approximation formulae for all the above developments, in the near-Boltzmannian case, where $0 < 1/\kappa \ll 1$, and the form of percentage deviation from the exact Boltzmannian case.

With the developments provided here, it is straightforward to compare the spectral intensities measured from various space physics observations with the modelled values using the statistical framework of kappa distributions. In this way, we minimize the chances for misinterpretation and error, such as from temperature measurements derived from radiation spectra, by also including the values of kappa that characterise these non-thermal black-body systems.

Black-body radiation is a common phenomenon; in fact, all objects are potential black-body radiators (anything at a higher temperature than its surroundings can radiate), with the amount of radiation and position in the spectrum depending on the emissivity, as well as the object's temperature and kappa values. There are every-day examples of black-body radiators, such as heaters, light bulbs, stoves, night-vision equipment, burglar alarms,

warm-blooded animals, etc. They can be used in numerous applications, such as lighting, heating, security, thermal imaging, metrology, etc. Black-body spectral radiation physics is also applicable in many space and astrophysical systems, even to cosmological scales: Solar spectroscopy; stellar spectroscopy, *e.g.*, characterization of stellar age, types, and Hertzsprung-Russell (HR) diagrams; planetary surfaces, atmospheres and clouds; thermal noise spectroscopy; gamma-ray bursts; Cosmic Microwave Background (CMB).

The kappa index that characterizes spectral radiation may lead to a new thermal-type parameterization of the source properties, such as its thermodynamic state, polytropic behavior, or other physical properties such as the connection with the magnetic field, or even conclusions about its age.

As an example, we may mention quasi-thermal noise spectroscopy, which has been applied to solar wind and Earth's plasmasphere [51]. This is based on the generalization of an antenna immersed in black-body radiation. Whereas analyses of electromagnetic fluctuations yield the temperature, an antenna in an equilibrium plasma is excited by thermal Langmuir waves, enabling the measurements of the plasma density and temperature. Improvements of this technique could also provide measurements of the kappa index.

In planetary science, clouds are also approximated as black bodies. The effect of clouds can be implemented by modifying the bulk emissivity depending on the cloud fraction and/or type. However, the most relevant parameter for the transmitted radiation is the temperature of its lower boundary, or cloud base (*e.g.*, [52]). Here the modified Stefan-Boltzmann law and involved kappa index should play a significant role in the modelling of atmospheric radiation.

The spectral intensity and Stephan-Boltzmann law are also used in astrophysics and cosmology, and its non-equilibrium modification may lead to important modifications of the existing physical concepts. For instance, the HR diagram plots stars' absolute magnitudes or luminosities *vs.* their stellar classifications or effective temperatures. The stellar lifetime is a sequence of luminosity- and temperature-dependent changes; thus, the HR diagram reflects the basics of these stellar lifetime stages. A new version of the HR diagram should include both the thermodynamic parameters of temperature and kappa.

Other phenomena involve types of gamma-ray bursts (GRBs) which are characterized by the presence of a significant thermal component following the prompt emission, as well as by the absence of a typical afterglow. These are black-body-dominated GRBs, where the thermal emission is attributed to the interaction of an ultra-relativistic jet with a hydrogen shell in the vicinity of the progenitor star that generates the GRBs [53]; their kappa indices characterize the thermodynamics of this interaction.

The CMB is sometimes considered as a system in thermal equilibrium; *e.g.*, [54] reviewed temperature

anisotropies of CMB radiation and noted that “the almost perfect blackbody shape of the energy spectrum could only be realized by the thermal equilibrium between photons, electrons, and protons in the early universe.” On the other hand, there is a merit to the theoretical view that the thermal equilibrium of CMB is a fossil, since it refers to an era when the photons were closely interacting with the cosmic plasma; very long wavelength modes, residing outside the horizon at this epoch, were never in thermal equilibrium with each other (e.g., [55]). The black-body formalism for nonthermal equilibrium described by kappa distributions might be just the right modification for describing the complex thermodynamics of the early universe.

In general, complicated systems under physical processes out of the classical thermal equilibrium are investigated within the framework of nonextensive statistical mechanics, which is associated with the concept of nonthermal or generalized thermal equilibrium [5]. The modifications developed in this paper apply across space plasmas, as well as any other areas where kappa distributions, rather than Maxwellian’s, characterize the physical systems.

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This work was funded in part by the Interstellar Boundary Explorer (IBEX) mission (80NSSC20K0719), a part of NASA’s Explorer Program.

Data availability statement: No new data were created or analysed in this study.

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