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Perspective

Propagation of instability fronts in modulationally unstable systems

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Abstract – We study the evolution of pulses propagating through focusing nonlinear media. A small disturbance of a smooth initial non-uniform pulse amplitude leads to formation of a region of strong nonlinear oscillations. We develop here an asymptotic method for finding the law of motion of the front of this region. The method is based on the conjecture that instability fronts propagate with the minimal group velocity of linear waves. To support this conjecture, at first we review several physical situations where this statement was obtained as a result of direct calculations. Then we generalize it to situations with a non-uniform flow and apply it to the focusing nonlinear Schrödinger equation for the particular cases of Talanov and Akhmanov-Sukhorukov-Khokhlov initial distributions. The approximate analytical results agree very well with the exact numerical solutions for these two problems.

Introduction. – Modulation instability of initially smooth wide wave beams was first observed [1] in the form of filamentation of light beams propagating through nonlinear media. This phenomenon was explained in ref. [2] with the use of the focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0, \qquad (1)$$

written here in standard non-dimensional variables and in 1D geometry, ψ being the wave field variable. Bespalov and Talanov obtained the dispersion relation

$$\begin{aligned} \omega(k) &= k\sqrt{k^2/4-\rho}, \quad k > k_c = 2\sqrt{\rho}, \\ \gamma(k) &= k\sqrt{\rho - k^2/4}, \quad 0 < k < k_c = 2\sqrt{\rho}, \end{aligned} \qquad (2)$$

for linear waves $\propto \exp[i(kx-\omega t)]$ propagating along a uniform background $\psi = \psi_0 e^{i\rho_0 t} = \sqrt{\rho}\, e^{i\rho_0 t}$ and noticed that the frequency $\omega(k)$ becomes complex for $0 < k < 2\sqrt{\rho}$ where it transforms to the growth rate $\gamma(k)$ of unstable modes; see fig. 1. This means occurrence of instability of small perturbations of such a background, and the growth of these unstable modes leads eventually to filamentation of the beam at the nonlinear stage of evolution.

Discovered independently [3] modulation instability of Stokes water waves drew much attention and it was followed by intense study of different aspects of this phenomenon (see, *e.g.*, ref. [4] for the early history of these studies). This simple theory of modulation instability is applicable to uniform states only. However, in a number of applications, such as a filamentation of optical beams, the more natural formulation of the problem is related with propagation of instability fronts into the "still" region —how does the area of strong oscillations spread out to the region without oscillations? So far this question

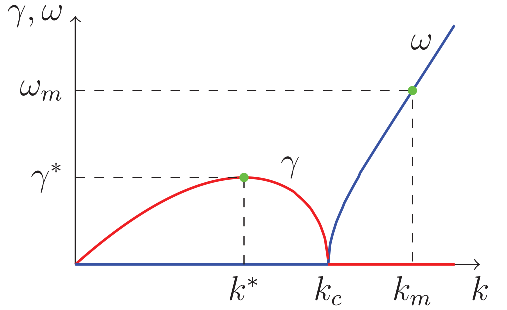

Fig. 1: Dispersion relation $\omega = \omega(k)$ of propagating modes (blue line) and the instability growth rate $\gamma = \gamma(k)$ (red line) as functions of the wave number k of harmonic waves. The dispersion relation $\omega = \omega(k)$ has an inflection point at $k = k_m = \sqrt{6\rho} > k_c$ where the group velocity takes its minimal value.

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was addressed in very few particular situations and for a uniform background only. To get a clue to a more general approach, at first we shall review several earlier theories of the front propagation in different physical situations. According to these particular cases, in modulationally unstable systems the instability front propagates with the minimal group velocity corresponding to the inflection point of the dispersion relation $\omega = \omega(k)$ for the stable branch of linear harmonic waves, as is shown in fig. 1. We take this statement as a conjecture applicable to slowly non-uniform and slowly time-dependent backgrounds and formulate the method of calculation of the path of the instability front in such situations. Our conjecture is confirmed by comparison with numerical solutions for two examples related with filamentation of intensive light beams propagating through nonlinear media.

Typical examples. –

Reaction-diffusion systems. A prototype example of the front propagation into an unstable state is given by the well-known Fisher-Kolmogorov-Petrovsky-Piskunov equation [5,6] which describes propagation of waves in biological systems of diffusion-reaction type. In many situations the main quantity of interest in this type of theories is the velocity of the front propagation, when a stable non-uniform state of a nonlinear dissipative system propagates into an initially unstable homogeneous region. As a result of numerous investigations (see, *e.g.*, the review article [7] and references therein) it was found that in typical situations the front's velocity quickly approaches some asymptotic value v^*, and, as was first suggested in ref. [8], the velocity v^* is the one at which the front is "marginally stable", that is the front solutions that move slower than v^* are unstable to perturbations while those that move faster are stable. If the dispersion relation of linearized equations is given by the complex function $\omega = \omega(k) = \omega'(k) + i\omega''(k)$, then the above condition yields the following equations for calculation of v^*:

$$v^* = \left.\frac{\mathrm{d}\omega'(k)}{\mathrm{d}k}\right|_{k=k^*}, \qquad \left.\frac{\mathrm{d}\omega''(k)}{\mathrm{d}k}\right|_{k=k^*} = 0. \qquad (3)$$

Thus, the front propagates asymptotically with the group velocity corresponding to the wave number k^* of the mode with maximal instability growth rate. However, this theory cannot be applied to non-dissipative systems. For instance, in the case of modulation instability described by the NLS equation (1) the maximal instability growth rate γ^* corresponds to the wave number k^* of non-propagating mode with vanishing group velocity (see fig. 1). The aim of this letter is to develop the theory of the instability front propagation in such situations.

Instability of a uniform plane wave propagating through a nonlinear self-focusing medium. Here we shall consider the problem of evolution of initially smooth wave packets (or wide light beams) in a focusing nonlinear medium. Numerical experiments (see, *e.g.*, [9–11]) showed that an

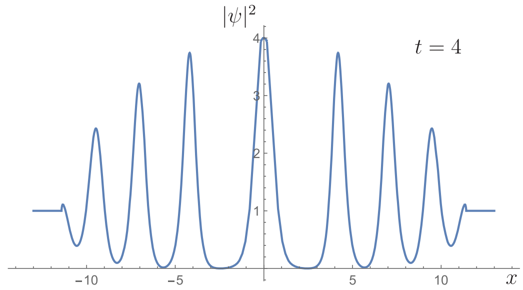

Fig. 2: Wave pattern evolving from an initially localized small disturbance introduced at the moment $t = 0$ in the vicinity of the point $x = 0$. In the case of the NLS equation (1) the edges of the wave pattern propagate to the undisturbed region with velocities $v_g = \pm 2\sqrt{2\rho_0}$.

initially smooth localized distribution focuses forming a spike at some moment of time, and after that a region of large amplitude nonlinear oscillations appears at the center of the distribution and this region spreads with time outwards over the smooth evolving background distribution. Formation of the region of large nonlinear oscillations can be triggered by a localized disturbance which violates smoothness of the initial distribution [12,13]. For example, let a small initial disturbance be located at $x = 0$ and let it evolve along a constant background $|\psi_0| = \sqrt{\rho}$ according to the NLS equation (1). Then we find that the region of strong nonlinear oscillations appears around the origin [12,13] which spreads outwards with time; see fig. 2. For not too large time of evolution, the region of oscillations can be represented as a modulated periodic solution of eq. (1) and, according to the Gurevich-Pitaevskii approach [14] to the theory of dispersive shock waves, the evolution of the modulation parameters obeys the Whitham modulation equations [15,16]. In the case of the periodic solution of the NLS equation (1) the intensity of light $\rho = |\psi|^2$ is given by the expression (see, *e.g.*, [17])

$$\begin{aligned}\rho(x,t) &= (\gamma+\delta)^2 - 4\gamma\delta \\ &\times sn^2\left(\sqrt{(\alpha-\beta)^2+(\gamma+\delta)^2}(x-Vt), m\right),\end{aligned} \qquad (4)$$

where m and V are equal, respectively, to

$$m = \frac{4\gamma\delta}{(\alpha-\beta)^2+(\gamma+\delta)^2}, \qquad V = -(\alpha+\beta). \qquad (5)$$

In a strictly periodic solution the parameters $\alpha, \beta, \gamma, \delta$ are constant but in a modulated wave they become slow functions of x and t changing little in one wavelength or one period of oscillations in the periodic wave. In the Whitham theory [15,16], averaging over these fast oscillations yields the modulation equations for these parameters. Solution of the Whitham equations for the problem posed in refs. [12,13] with a small disturbance located in the vicinity of the point $x = 0$ at the moment $t = 0$ was obtained in refs. [18,19] for the NLS equation case and a similar problem with identical Whitham equations was solved in ref. [20] for the one-dimensional Heisenberg ferromagnet. It is clear that it must be self-similar, so the parameters can depend on x and t only through the variable $\zeta = x/t$. Then one can find that either (α, γ) or (β, δ) must be constant and if for definitness we take (α, γ) constant, then dependence of (β, δ) on ζ is determined by the equations

$$\begin{aligned} \frac{E(m)}{K(m)} &= \frac{(\alpha-\beta)^2 + (\gamma-\delta)^2}{(\alpha-\beta)^2 + \gamma^2 - \delta^2}, \\ -2\beta - \frac{\gamma^2-\delta^2}{\beta-\alpha} &= \frac{x}{t}, \end{aligned} \tag{6}$$

$K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively. Substitution of these $\beta = \beta(x/t)$, $\gamma = \gamma(x/t)$ for the particular choice $\alpha = 0$, $\gamma = 1$ into expression (4) gives the dependence of $\rho = \rho(x,t)$ on x at the fixed moment of time $t = 4$ shown in fig. 2.

The small-amplitude edge of the oscillatory region corresponds to the limit $m \to 0$. If we take for simplicity $\alpha = 0$, then we obtain $\beta = -\gamma/\sqrt{2}$, $\delta = 0$, and then eq. (6) yields $v = x/t = 2\sqrt{2}\gamma$. On the other hand, we find from eq. (4) the background intensity $\rho_0 = \gamma^2$ and the wave number $k = \sqrt{6}\,\gamma = \sqrt{6\rho_0}$ of the wave at the small-amplitude edge. Thus, the wave number corresponds exactly to the inflection point of the dispersion relation (2) and the small-amplitude edge propagates along a uniform background with the group velocity

$$v_g = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_m} = 2\sqrt{2\rho_0} \tag{7}$$

equal to its minimal value at $k = k_m = \sqrt{6\rho_0}$ (see fig. 1).

Snake instability of dark solitons. Now we shall consider the two-dimensional NLS equation

$$i\psi_t + \frac{1}{2}\Delta\psi - (|\psi|^2 - 1)\psi = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{8}$$

with defocusing nonlinearity and background density $\rho_0 = 1$ in standard non-dimensional variables. For convenience of notation, we have excluded the trivial time dependence $\psi_0 = \sqrt{\rho_0}e^{-i\rho_0 t}$ from the wave function by adding the term $\rho_0\psi$ to the equation. It is well known that this equation has the dark soliton solution [21] that can be written for the wave function $\psi(x,y,t) = \sqrt{\rho(x,y,t)}\,\exp(i\phi(x,y,t))$ in the form

$$\begin{aligned} \rho &= \rho_s(x - Vt) = 1 - \frac{1-V^2}{\cosh^2[\sqrt{1-V^2}(x-Vt)]}, \\ u &= \frac{\partial\phi}{\partial x} = V\left(1 - \frac{1}{\rho}\right), \qquad v = \frac{\partial\phi}{\partial y} = 0. \end{aligned} \tag{9}$$

This solution manifests itself as a dip in the uniform background density parallel to the y-axis and this dip propagates in the direction of the x-axis with velocity V. If we impose a small "snake-like" bending disturbance, then we find that this solution is unstable: amplitude of the disturbance grows up leading eventually to breaking of the soliton to a sequence of vortex–anti-vortex pairs. The undisturbed soliton is located along a straight line, so propagation of the small-amplitude disturbance waves has effectively the one-dimensional form and can be characterized by the wave number and the frequency of harmonic waves propagating along the soliton. In the limit of shallow solitons and long wavelength disturbances the instability increment was found in ref. [22] and for arbitrary wavelength in ref. [23]. It has a form similar to eq. (2) (see fig. 1):

$$\omega(k) = \frac{\sqrt{2}}{3^{3/4}}k\sqrt{k-k_c}, \quad k > k_c = \sqrt{3}(1-V). \tag{10}$$

This formula implies small depth of the soliton, *i.e.*, $1 - V \ll 1$, and for arbitrary depth or for soliton's velocity in the interval $0 < V < 1$ the stability of solitons was studied in ref. [24]. Qualitatively the spectrum of harmonic disturbances of dark solitons is the same for any depth: the bending waves are unstable for small enough wave numbers $0 < k < k_c$ and they correspond to propagating stable waves for $k > k_c$. Instability of dark solitons was observed experimentally [25,26] and studied in numerous papers (see the review article [27] and references therein).

A different situation occurs for dark solitons generated by a supersonic flow of atomic or polariton condensate past an obstacle (they were predicted in ref. [28]): numerical and real [29,30] experiments showed that such a soliton is stable for large enough supersonic flow velocity. This phenomenon was explained in [31] as a transition from an absolute instability of dark solitons to their convective instability. Simple theory of such a transition was developed in ref. [32]. Let a dark soliton be disturbed in the vicinity of the point $y = 0$. Numerical solution of the 2D NLS equation (8) demonstrates formation of a "wave of destruction" —in its center vortex pairs are formed and its edges propagate along the soliton with some velocity v; see fig. 3. This velocity can be estimated in the following way. Let the initial disturbance have the Fourier representation $\phi_0(y) = \int f(k)e^{iky}\mathrm{d}k$ with a smooth spectrum $f(k)$ so that the disturbance is localized around the point $y = 0$. The disturbance leads to waves propagating along the soliton and, as long as their amplitude is small, they are represented by the Fourier integral

$$\phi(y,t) = \int f(k)e^{i(ky-\omega(k)t)}\mathrm{d}k \tag{11}$$

with the known dispersion relation $\omega = \omega(k)$ (for example, in the case of shallow solitons it is given by eq. (10)). For large time t the integral can be evaluated by the steepest descent method and the main contribution to it is given by vicinity of the saddle points k_s defined as solutions of the equation

$$\frac{y}{t} = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}. \tag{12}$$

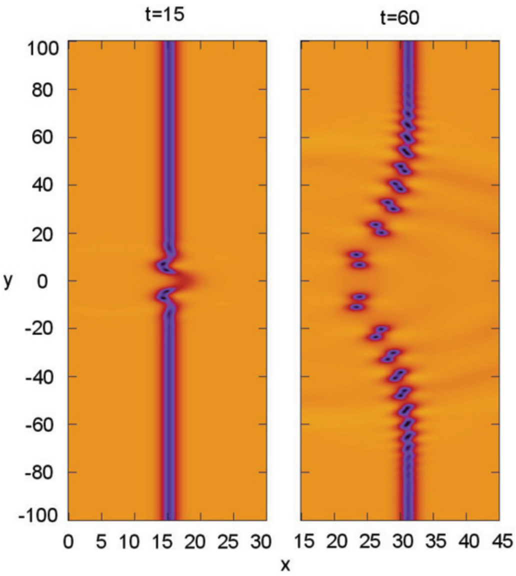

Fig. 3: Evolution of a localized disturbance of the dark soliton with time [32]. A plane soliton slightly disturbed in vicinity of the point $y = 0$ starts its motion at $x = 10$ with velocity $V = 0.355$. The disturbance leads to breaking the soliton into a sequence of vortex–anti-vortex pairs in the central part and to propagation of the edges of the instability region with velocity v_g equal to the minimal group velocity of propagating along the soliton "snake"-like bending waves.

The values of k_s depend on y/t and the integrand function includes the fast oscillating factor $e^{ik_s y}$. Consequently, the resulting integral corresponds to a usual dispersive wave packet slowly decaying with time as $t^{-1/2}$. However, if with decrease of y the roots of eq. (12) move along the real k-axis and collide with each other at some value of $y = y_m$ bifurcating here into two complex roots k_s and k_s^*, then the saddle points move into the complex plane and the integral (11) acquires the factor $\exp[-\mathrm{Im}(k_m(y))y]$ which depends exponentially on y. Hence, the bifurcation point $y = y_m$ corresponds to the edge of a large amplitude pulse. At the point of bifurcation of two real roots into two complex ones eq. (12) has a double root, that is we have here $\mathrm{d}^2\omega/\mathrm{d}k^2 = 0$, and that means that the edge of the pulse corresponds to the extremum of the group velocity. For example, the amplitude of disturbance propagating along a shallow soliton is proportional to

$$\phi(y,t) \propto \exp\left[-\frac{y^2}{t}\sqrt{2\sqrt{3}k_m - 3\frac{y^2}{t^2}}\right], \qquad (13)$$

near the edge of the large amplitude region, and this edge propagates with the velocity $y_m/t = \sqrt{2k_m}/3^{1/4} = \sqrt{2(1-V)}$ equal to the minimum of the group velocity at $k = k_m = 4k_c/3$,

$$v_f = \min\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}\right) = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_m}, \qquad (14)$$

corresponding to the dispersion law (10).

The last two examples suggest generalization to propagation of instability fronts in modulationally unstable systems along non-uniform and time-dependent backgrounds.

Propagation of instability fronts in modulationally unstable systems. –

General theory. We assume that the system under consideration is described by two wave variables $\rho(x,t)$ and $u(x,t)$ which we shall call the "density" and "flow velocity", correspondingly. As was shown in ref. [33], in many physical situations the equations of wave dynamics can be cast in the form

$$\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (\rho u^2 + P)_x = 0, \qquad (15)$$

where $P = P(\rho, u, \rho_x, \rho_t, u_x, u_t, \rho_{xx}, \ldots)$ and the higher-order derivatives in P correspond to dispersive (or dissipative) effects. In dispersionless limit, these higher-order derivatives are disregarded, $P = P(\rho, u)$, and eqs. (15) take the form of equations of compressible fluid dynamics. The system is modulationally unstable in the long wavelength limit if $\partial P(\rho,u)/\partial\rho < 0$. For example, the substitution $\psi = \sqrt{\rho}\exp\left(i\int^x u(x',t)\mathrm{d}x'\right)$ casts the NLS equation (1) to the system

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &u_t + uu_x - \rho_x + \left(\frac{\rho_x^2}{8\rho^2} - \frac{\rho_{xx}}{4\rho}\right)_x = 0, \end{aligned} \qquad (16)$$

and in dispersionless limit $P = -\rho^2/2$, $\mathrm{d}P/\mathrm{d}\rho = -\rho < 0$.

If we omit the terms in eq. (15) with higher-order derivatives, then we arrive at dispersionless equations which describe the self-focusing of initial distributions in the geometric optics approximation. Although, strictly speaking, in this case the geometric optics problem is ill-posed, nevertheless, if the initial data are represented by analytic functions, then solutions of dispersionless equations provide a very good approximation (see, *e.g.*, [9,34,35]) up to the focusing moment and they often can be obtained by classical methods of the compressible fluid dynamics (see, *e.g.*, [36]). We assume that such a dispersionless solution is known and it is convenient for what follows to write this solution in an implicit form

$$x = x(\rho, u), \qquad t = t(\rho, u), \qquad (17)$$

so that solving this system with respect to ρ and u yields the time-dependent distributions $\rho = \rho(x,t)$ and $u = u(x,t)$. We shall assume that the instability is triggered by a small perturbation located in the vicinity of the point $x = 0$ in the initial distribution. Then for $t > 0$ the region of strong nonlinear oscillations develops and, following the ideas of the Gurevich-Pitaevskii theory of dispersive shock waves [14], we assume that at the asymptotic stage of evolution, when the typical wavelength of these nonlinear waves is much smaller than the length of the whole oscillatory region, the wave dynamics outside this region can be described by the above-considered dispersionless limit of the equations under consideration with disregarded dispersion (diffraction) effects. Near the edges of the oscillatory region the amplitude of oscillations is small and these waves propagate along the smooth background (17) according to the linearized equations (15) which yield some dispersion law

$$\omega = \omega(k, \rho, u), \tag{18}$$

where the wavelength $2\pi/k$ is assumed much smaller than the characteristic size of the background flow distributions. Then, according to the above-considered examples, the edges of the oscillatory region propagate with the minimal group velocity

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_g(\rho, u) = \left.\frac{\partial\omega}{\partial k}\right|_{k=k_m}, \quad \left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=k_m} = 0. \tag{19}$$

These equations replace eqs. (3) known from the theory of pattern formation in reaction-diffusion systems. Naturally, the use of the notion of the group velocity means that our theory is asymptotic and is applicable only to the stage of evolution later enough after the moment of formation of the region of nonlinear oscillations.

We assume for simplicity that the variable u can be excluded from system (17) and from the function $v_g = v_g(\rho, u)$, so we get

$$x = x(\rho, t), \qquad v_g = v_g(\rho, t), \tag{20}$$

where ρ plays the role of the parameter along the path of the small amplitude wave packet. Substitution of these expressions into eq. (19) gives the differential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{v_g - \partial x/\partial t}{\partial x/\partial \rho}, \tag{21}$$

for variation of the background density along the packet's path. In the case of instability triggered by a small initial disturbance localized in vicinity of the point $x = 0$, this equation should be solved with the initial condition

$$\rho = \rho_0(0), \qquad \text{at} \qquad t = 0, \tag{22}$$

where $\rho = \rho_0(x)$ is the initial distribution of the density. The resulting solution $\rho = \rho(t)$ of eq. (21) after substitution into eq. (20) yields the asymptotic expression for the path of the edge. Now we illustrate this approach by two particular examples.

Talanov's self-focusing solution. We take the initial distribution in the form of a parabolic hump

$$\rho_0(x) = \begin{cases} a^2(1 - x^2/l^2), & |x| \le l, \\ 0, & |x| > l, \end{cases} \tag{23}$$

and the initial phase is equal to zero everywhere, hence $u_0(x) = 0$. System (16) reduces in the dispersionless limit to the "inverted" shallow water equations

$$\rho_t + (\rho u)_x = 0, \qquad u_t + uu_x - \rho_x = 0. \tag{24}$$

As was shown by Talanov [37], this system with the initial conditions (23) has the exact solution in the form

$$\begin{aligned} \rho_b(x,t) &= \frac{a^2}{f(t)}\left(1 - \frac{x^2}{l^2 f^2(t)}\right), \\ u_b(x,t) &= \frac{f'(t)}{f(t)}x = -\frac{2a}{l}\frac{\sqrt{1-f}}{f^{3/2}}x, \end{aligned} \tag{25}$$

where the function $f(t)$ is defined implicitly by the expression

$$(2a/l)t = \sqrt{f(1-f)} + \arccos\sqrt{f}. \tag{26}$$

This solution determines profiles of the background density and the flow velocity as functions of time.

Now we suppose that the initial distribution (23) is disturbed and for definiteness we take this disturbance in the form of a tiny hillock over the density distribution at $x = 0$. (In fact, the asymptotic evolution of the instability region does not depend on the form of the disturbance). As a result of such a disturbance, the region of strong oscillations is formed, and our aim is to find the low of motion of its edges. The edge wave packet propagates upstream or downstream the background flow (25) that has the local flow velocity $u_b(x, t)$, so the Doppler-shifted local dispersion relation reads

$$\omega(k) = k\left(u_b \pm \sqrt{k^2/4 - \rho_b}\right), \tag{27}$$

where the sign depends on the direction of the propagation of the packet. The function $\omega = \omega(k)$ has inflection points at $k = \pm\sqrt{6\rho_b}$ and the corresponding group velocity is equal to

$$v_g = u_b \pm 2\sqrt{2\rho_b}. \tag{28}$$

For definiteness, we shall consider the left instability front at $x < 0$ for which $u_b > 0$ and $v_g < 0$, that is we get with account of eqs. (25)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2a}{l}\frac{\sqrt{1-f}}{f^{3/2}} - 2\sqrt{2}a \cdot \sqrt{\frac{1}{f}\left(1 - \frac{x^2}{l^2 f^2}\right)}. \tag{29}$$

With the use of eq. (26) we transform eq. (29) to

$$\frac{\mathrm{d}x}{\mathrm{d}f} = \frac{x}{f} + \sqrt{2}l \cdot \sqrt{\frac{1 - x^2/(lf)^2}{1-f}}. \tag{30}$$

Due to the initial disturbance, the instability front starts its propagation at the point $x = 0$ at the moment $t = 0$, when $f = 1$, so eq. (30) should be solved with the initial condition $x(f = 1) = 0$ and the solution reads

$$x(f) = -lf\sin\left(\sqrt{2}\ln\frac{1+\sqrt{1-f}}{1-\sqrt{1-f}}\right). \tag{31}$$

Together with eq. (26), this equation determines the path of the instability front in a parametric form $(x(f), t(f))$ with f playing the role of the parameter. The front's coordinate $x(f)$ touches the edge of the background distribution (25) when the argument of the sine-function in

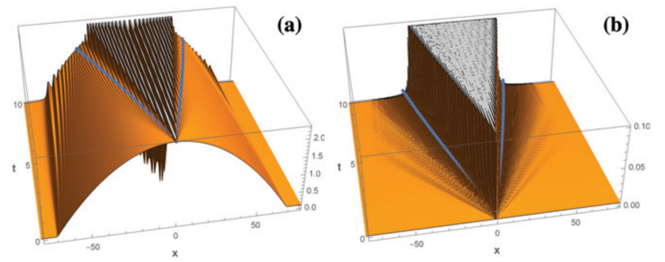

Fig. 4: (a) Plot of the numerical solution of the NLS equation (1) for the initial condition (23) with $a = 1.5$, $l = 70$, and a tiny disturbance of density at $x = 0$. (b) Plot of $|\rho(x,t) - \rho_b(x,t)|$ with subtracted background evolution (25) from the exact solution $\rho(x,t)$. Analytical paths of the instability fronts are shown by blue lines.

eq. (31) equals $\pi/2$, that is at $f = f_c = 1/\cosh^2(\pi/4\sqrt{2}) \approx 0.745$. The background density vanishes at this point and our approach becomes inapplicable. Thus, the parameter f in eq. (31) changes in the interval $1 \geq f > f_c$.

We compare our analytical theory with the exact numerical solution of the NLS equation with the disturbed parabolic initial distribution (23). In fig. 4(a) the surface plot of the function $\rho(x,t)$ is shown where for clarity the large amplitude peaks are cut at the level $\rho = 2.3$. Theoretical paths of the instability fronts given parametrically by the formulas $(\pm x(f), t(f), \rho_b(\pm x(f), t(f)))$ are shown by blue lines. To show the evolution of the region of large amplitude oscillations only, we subtracted the background distribution $\rho_b(x,t)$ given by eq. (25) from the exact numerical solution $\rho(x,t)$ of the NLS equation and the corresponding surface plot of $|\rho(x,t) - \rho_b(x,t)|$ is shown in fig. 4(b). Again the large amplitude oscillations are cut for clarity at the level $|\rho(x,t) - \rho_b(x,t)| = 0.1$ and the fronts' paths are shown by blue lines. The theory does not contain any fitting parameters and its agreement with the numerical solution seems quite good.

Akhmanov-Sukhorukov-Khokhlov self-focusing solution. As another example, let us consider evolution of the background with the initial conditions

$$\rho_0(x) = \frac{a^2}{\cosh^2(x/l)}, \quad u_0(x) = 0 \quad \text{at} \quad t = 0. \tag{32}$$

Solution of this problem was obtained in ref. [38] and it can be written in the following parametric form:

$$\rho(\xi,\eta) = a^2(1+\xi^2)(1-\eta^2), \quad u(\xi,\eta) = 2a\xi\eta, \tag{33}$$

$$\begin{aligned} t &= \frac{l}{a}\cdot\frac{\xi}{(1+\xi^2)(1-\eta^2)}, \\ x &= l\left[\frac{2\xi^2\eta}{(1+\xi^2)(1-\eta^2)} - \frac{1}{2}\ln\frac{1+\eta}{1-\eta}\right], \end{aligned} \tag{34}$$

where ξ and η are parameters ($\xi \geq 0, -1 \leq \eta \leq 1$). Their elimination yields the original implicit form of the solution

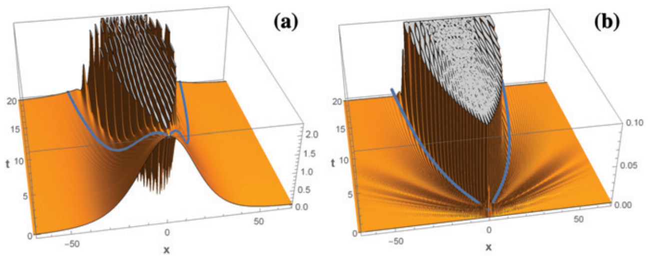

Fig. 5: (a) Plot of the numerical solution of the NLS equation (1) for the initial condition (32) with $a = 1.5$, $l = 20$, and a tiny disturbance of density at $x = 0$. (b) Plot of $|\rho(x,t) - \rho_b(x,t)|$ with subtracted background evolution (35), (36) from the exact solution $\rho(x,t)$. Analytical paths of the instability fronts are shown by blue lines.

found in ref. [38]. For us it is more convenient to exclude only the parameter η to obtain

$$\rho(\xi,t) = al\cdot\frac{\xi}{t}, \quad u(\xi,t) = 2a\xi\sqrt{1 - \frac{l\xi}{at(1+\xi^2)}} \tag{35}$$

and

$$x(\xi,t) = 2at\xi\sqrt{1-\frac{l\xi}{at(1+\xi^2)}} - \frac{l}{2}\ln\frac{1+\sqrt{1-\frac{l\xi}{at(1+\xi^2)}}}{1-\sqrt{1-\frac{l\xi}{at(1+\xi^2)}}}. \tag{36}$$

If we fix the moment of time t, then these formulas give us parametric forms of distributions of the density $(\rho(\xi,t), x(\xi,t))$ and the flow velocity $(u(\xi,t), x(\xi,t))$ as functions of x. This solution becomes singular in the limit $t \to t_f = l/(2a)$ with space derivatives of the distributions $\rho(x,t), u(x,t)$ tending to infinity. To avoid complications related with dispersion effects near singularity point, we shall consider formation of the instability region due to small disturbance in the initial condition (32).

Formulas (35), (36) show that it is convenient to choose ξ as a parameter along the front's path. The group velocity of linear waves propagating in the neighborhood of the point $x(\xi,t)$ with the carrier wave number $k_m = \sqrt{6\rho}$ at the moment t can also be expressed as a function of ξ and t:

$$v_g(\xi,t) = 2a\xi\sqrt{1-\frac{l}{at}\frac{\xi}{1+\xi^2}} - 2\sqrt{\frac{2al\xi}{t}}. \tag{37}$$

If we denote by $x = \bar{x}(\xi(t),t)$ the front's path, then we have $\bar{x}_\xi(\mathrm{d}\xi/\mathrm{d}t) + \bar{x}_t = v_g(\xi,t)$ (see eq. (21) where the parameter ρ is replaced by ξ), and taking into account eq. (36) this equation reduces to the differential equation

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\xi(1+\xi^2)}{t[4a\xi t(1+\xi^2)^2 + l(1-6\xi^2-3\xi^4)]}\Bigg[& l(1-\xi^2) \\ & -4(1+\xi^2)\sqrt{2al\xi t\left(1-\frac{l}{at}\frac{\xi}{1+\xi^2}\right)}\Bigg], \end{aligned} \tag{38}$$

for the dependence of the parameter ξ along the path. This equation must be solved with the initial condition $\xi = 0$ at $t = 0$ which means instant formation of the oscillatory region due to the disturbance. This equation can be easily solved numerically and substitution of the solution into eq. (36) gives us the path of the edge (the interval $1 > \xi > 0$ corresponds to the left edge of the oscillatory region). An example of the resulting plots is shown in fig. 5. Again we see quite good agreement of our analytical theory with numerical solution.

Conclusion. – In this letter, we have developed the analytical approach to the problem of propagation of instability fronts in modulationally unstable systems. The theory is based on the conjecture that such a front propagates with minimal group velocity of linear waves propagating in such systems. This conjecture is based on a few earlier studied examples where it was derived in the framework of either Whitham's theory of modulations or asymptotic theory of propagation of linear wave packets. Taking this conjecture as a general principle, we formulate the theory of instability front propagation in non-uniform and time-dependent situations. As a result, eqs. (3) which determine the instability front velocity in dissipative systems should be replaced by eqs. (19) for such a velocity in modulationally unstable non-dissipative systems. The theory is applied to the systems whose evolution is described by the focusing NLS equation and it is confirmed by comparison of the results with numerical solutions.

* * *

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