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On gravity and quantum interactions

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Abstract – We introduce the metric of general relativity into a description of baryon mass spectra which otherwise has been founded entirely on the concept of an intrinsic configuration space, the Lie group $U(3)$. We find that the general relativistic metric influences the mass eigenstates in gravitational fields. We discuss parts per million effects that may be observed in space missions close to the Sun or the planet Jupiter, for instance by accurate Cavendish experiments or energy shifts in gamma decays of metastable nuclei like Ba-137m. We review how the particle and gauge fields are generated by momentum forms on the intrinsic wave functions to form the quantum field bases for instance of quantum chromodynamics. Our strategy to combine quantum interactions and general relativity is that of geometrising quantum mechanics rather than quantising gravity.

Introduction. – It was a great step forward in our understanding of fundamental interactions when electromagnetic interactions and weak interactions were put into a common setting as gauge theories based on gauge groups $U(1)$ and $SU(2)$, respectively [1]. This was seen as a first step in unifying the quantum interactions and once $SU(3)$ was found to be the relevant gauge group for the strong interactions, ambitions rose to quantise also gravity which so far had been left alone as a classical field theory, namely the general theory of relativity [2] —that, however, still stands to test in its classical form [3,4].

The present work does *not* try to quantise gravity. We try to describe baryon mass eigenstates in gravitational fields and distinguish between rest masses and eigenvalues of mass eigenstates.

Our idea is to use a description of elementary particles that starts out *completely* detached from spacetime, see fig. 1. We shall use intrinsic configuration variables to describe baryon mass spectra [5,6]. From the intrinsic wave functions we then generate —by exterior derivatives also called momentum forms— the quantum fields that make up the basic fields of gauge theories. The exterior derivatives are "just" the group generators acting as derivatives on the wave functions and these generators are very close analogues of the kinematical operators that led to the creation of the intrinsic (particle) states in the first place.

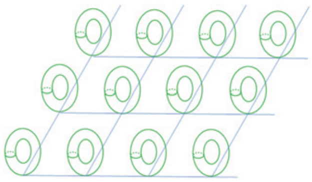

Fig. 1: The intrinsic space —shown as a torus— can be reached from any point in laboratory space —shown as a patterned floor. Letting the floor pattern be visible through the intrinsic toroidal space is to stress that the intrinsic space should not be considered as just "extra" spatial dimensions like in string theory. Rather the intrinsic space should be likened to a generalised spin space, *i.e.*, with no physical dimensional units. Drawing inspired by Maldacena [7]. Figure and caption edited from [8].

Mass eigenstates as intrinsic configurations. – We derived the local gauge invariance for the quark and gluon fields of quantum chromodynamics from an intrinsic description of baryons as mass eigenstates on the intrinsic configuration space, the Lie group $U(3)$, see the appendices for *résumés*. The stationary mass eigenstates have eigenvalues $\mathcal{E} = mc^2$ determined by the following *mass Hamiltonian* operating on an intrinsic wave function Ψ [5]:

$$\frac{\hbar c}{a}\left[-\frac{1}{2}\Delta + \frac{1}{2}d^2(e,u)\right]\Psi(u) = \mathcal{E}\Psi(u), \quad u = e^{i\chi} \in U(3). \tag{1}$$

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The generator χ of the configuration variable u can be expanded on nine group generators $T_j, S_j, M_j, j = 1, 2, 3$,

$$\chi = \theta_j T_j + (\alpha_j S_j + \beta_j M_j)/\hbar, \quad \theta_j, \alpha_j, \beta_j \in \mathbb{R}. \tag{2}$$

We use dynamical angles

$$\theta_j = x_j/a \tag{3}$$

to map from laboratory space to intrinsic space by introducing a length scale a which is set by the classical electron radius [5,9]

$$\pi a = r_e. \tag{4}$$

In that way we can interpret all nine generators to have analogues in nine kinematical operators from laboratory space. The toroidal generators iT_j are equivalent to momentum operators in laboratory space

$$iT_j = \frac{\partial}{\partial \theta_j}, \quad p_j = -i\hbar \frac{1}{a} \frac{\partial}{\partial \theta_j} = \frac{\hbar}{a} T_j. \tag{5}$$

The operators S_j in coordinate representation [10] are analogues of angular momentum operators and take care of spin, *e.g.*,

$$S_3 = a\theta_1 p_2 - a\theta_2 p_1 \sim \hbar\lambda_2 \tag{6}$$

and the mixing operators M_j are analogues of the less well-known Laplace-Runge-Lenz vector operator which is conserved in the hydrogen atom [10] and in Kepler orbits [11], *e.g.*,

$$M_3/\hbar = \theta_1\theta_2 + \frac{a^2}{\hbar^2} p_1 p_2 \sim \lambda_1, \tag{7}$$

with matrix representations corresponding to Gell-Mann's lambda matrices [10]. The M_j's mix spin and flavour degrees of freedom. The spin operators commute like intrinsic spin operators in body fixed coordinate systems in nuclear physics (note the minus sign in (8)) and the mixing operators close the algebra by commuting into the subspace of the spin algebra

$$[M_i, M_j] = [S_i, S_j] = -i\hbar\varepsilon_{ijk} S_k. \tag{8}$$

The generators act as derivatives in the Laplacian [12]

$$\Delta = \sum_{j=1}^{3} \frac{1}{J} \frac{\partial^2}{\partial \theta_j^2} J + 2 - \sum_{1 \le i < j, k \ne i,j}^{3} \frac{(S_k^2 + M_k^2)/\hbar^2}{8 \sin^2 \frac{1}{2}(\theta_i - \theta_j)}, \tag{9}$$

where

$$J = \prod_{1 \le i < j}^{3} 2 \sin\left(\frac{1}{2}(\theta_i - \theta_j)\right). \tag{10}$$

The potential $\frac{1}{2}d^2$ is half the square of the shortest geodesic from the *origo* e in the configuration space to the configuration variable u and may be thought of as an intrinsic harmonic oscillator potential. It maps out in the angular space as a periodic potential manifesting the compactness of the configuration space. It can be expressed as

$$\frac{1}{2} d^2(e, u) = \frac{1}{2} \mathrm{Tr}\ \chi^2 \tag{11}$$

and thus only depends on the *eigenangles* θ_j in the eigenvalues $e^{i\theta_j}$ of u. This follows from the fact that the trace is invariant under similarity transformations $u \to v^{-1}uv, v \in U(3)$, in particular transformations to diagonalise u. The potential may be seen as the Euclidean measure folded into the group manifold [13]. It is through this latter conception of the potential that we find an opening for the inclusion of effects of the generally curved metric in the general theory of relativity. But first we must consider the time-dependent case

$$\frac{\hbar c}{a} \left[-\frac{1}{2}\Delta + \frac{1}{2} d^2(e, u) \right] \Psi(u, t) = i\hbar \frac{\partial}{\partial t} \Psi(u, t). \tag{12}$$

We use the same length scale a as for the spatial coordinates and introduce an imaginary "time angle" θ_0 by

$$a\theta_0 = ict, \tag{13}$$

where t is the time parameter in spacetime and c is the speed of light in empty space. Further, the time derivative corresponds to the time coordinate field generator

$$\frac{\partial}{\partial \theta_0} = \frac{a}{ic} \frac{\partial}{\partial t} = -H/\Lambda \equiv iT_0, \tag{14}$$

where $\Lambda = \hbar c/a$ is the energy scale in (1) amounting to approximately 214 MeV at nucleonic values of the fine structure coupling inherent in the classical electron radius in a from (4). We then have for

$$\tilde{u} \equiv e^{i\theta_0 T_0} u \quad \in U(1) \times U(3) \tag{15}$$

a left invariant time coordinate field

$$\partial_0|_{\tilde{u}} = \frac{\mathrm{d}}{\mathrm{d}\theta} (\tilde{u} \exp \theta i T_0)|_{\theta=0} = \tilde{u} i T_0 \tag{16}$$

with the generator iT_0 and corresponding time form $\mathrm{d}\theta_0$ fulfilling $\mathrm{d}\theta_0(\partial_0) = 1$.

We factorize the time-independent wave function Ψ into a toroidal part τ and an off-toroidal part Υ analogous of the radial part and the spherical harmonics introduced in solving the hydrogen atom [14], *i.e.*, we write

$$\Psi(u) = \tau \Upsilon. \tag{17}$$

In that way the measure-scaled, time-dependent wave function $\Phi(u, t) = J\Psi(u, t)$ for a time-independent, toroidally symmetric potential $V(u, t) = V(\theta_1, \theta_2, \theta_3)$ becomes

$$\Phi(u, t) = e^{-i\mathcal{E}t/\hbar} R(\theta_1, \theta_2, \theta_3) \Upsilon \equiv \mathcal{R}(\theta_0, \theta_1, \theta_2, \theta_3) \Upsilon \tag{18}$$

with *measure-scaled* toroidal wave function

$$R(\theta_1, \theta_2, \theta_3) = J(\theta_1, \theta_2, \theta_3) \tau(\theta_1, \theta_2, \theta_3). \tag{19}$$

For the mapping between spacetime and the torus, we have in particular the following corresponding bases:

$$\partial_\mu|_e = iT_\mu = e_\mu \tag{20}$$

and can write at each event x

$$x = x^\mu e_\mu, \qquad x^0 = ct \tag{21}$$

with contravariant spacetime coordinates $x^\mu, \mu = 0,1,2,3$ and covariant base e_μ and with Einstein's summation convention as throughout. For the induced base $\{iT_\mu\}$ at the *origo* $e = I$ in the $4D$ torus we may choose a representation with

$$\begin{aligned} iT_0 &= \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \qquad iT_1 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \\ iT_2 &= \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \qquad iT_3 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{Bmatrix}. \end{aligned} \tag{22}$$

In special relativity we have

$$x^\mu x_\mu = x^\mu \eta_{\mu\nu} x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{23}$$

from a metric tensor with non-zero components $\eta_{00} = 1, \eta_{11} = \eta_{22} = \eta_{33} = -1$. We get accordingly with trace taking

$$\mathrm{Tr}\,(\theta_\mu iT_\mu)\,(\theta_\nu iT_\nu) = (\theta_0)^2 - (\theta_1)^2 - (\theta_2)^2 - (\theta_3)^2. \tag{24}$$

We see that the generators $\{iT_\mu\}$ carry the Minkowski metric intrinsically. Thus as

$$e_\mu \cdot e_\nu = \eta_{\mu\nu} \tag{25}$$

we likewise have

$$\mathrm{Tr}\,(iT_\mu iT_\nu) = \eta_{\mu\nu}. \tag{26}$$

Gravitational term in the intrinsic dynamics. – We shift to the metric sign convention of Weinberg [2] and write the general metric as a sum of two parts

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1). \tag{27}$$

This gives the proper time invariant under Lorentz transformations in flat spacetime, p. 26 in [2]

$$\mathrm{d}\tau^2 \equiv \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{28}$$

As mentioned, the intrinsic potential in (1)

$$\begin{aligned} &\frac{1}{2}d^2(e,u) = \frac{1}{2}\mathrm{Tr}\ \chi^2 = \sum_{j=1}^{3} w(\theta_j), \\ &w(\theta) = \frac{1}{2}(\theta - n\cdot 2\pi)^2, \\ &\theta \in [(2n-1)\pi, (2n+1)\pi)], \quad n \in \mathbb{Z} \end{aligned} \tag{29}$$

may be seen as the Euclidean metric folded into the group manifold and may therefore be seen to represent the flat spacetime part resulting from the $\eta_{\mu\nu}$ part in (27). In order to take into account the full metric $g_{\mu\nu}$ we therefore add a gravitational term to our theory for baryon mass eigenstates

$$\begin{aligned} &\frac{\hbar c}{a}\left[-\frac{1}{2}\Delta + \frac{1}{2}d^2(e,u) + \frac{1}{2}\varepsilon\ \bar\theta^*_\mu h_{\mu\nu}\bar\theta_\nu\right]\Psi(u,t) = \\ &i\hbar\frac{\partial}{\partial t}\Psi(u,t), \quad \text{where}\ \ \bar\theta_j, \bar\theta_0/i \in [-\pi,\pi]. \end{aligned} \tag{30}$$

We consider three cases for the parameter ε

$$\varepsilon = \begin{cases} 1, & \text{generic}, \\ l^2_{Pl}/a^2, & \text{Planck scale}, \\ \varepsilon, & \text{empirical}. \end{cases} \tag{31}$$

For positive ε we get an increase in the eigenvalues of the mass eigenstates of the baryons from (30) when compared to (1). For the generic case $\varepsilon = 1$ the effect is at the 5.5 ppm level at the surface of the Sun and at the 51 ppb level if considered in the gravitational field at the surface of the planet Jupiter. To see this, we need to know the metric components and estimate the integral over the time-dependent gravitational term

$$\frac{1}{2}\bar\theta^*_\mu h_{\mu\nu}\bar\theta_\nu. \tag{32}$$

Disregarding the off-diagonal terms which are vanishing in an asymptotic weak field approximation (37) we have

$$\frac{1}{2}\bar\theta^*_\mu h_{\mu\nu}\bar\theta_\nu \approx \sum_{\mu=0}^{3} h_{\mu\mu} w(\theta_\mu). \tag{33}$$

For the measure-scaled wave function R we use a Slater determinant

$$R_{0\frac{1}{2}1} = \frac{1}{N}\begin{vmatrix} 1 & 1 & 1 \\ \sin\dfrac{\theta_1}{2} & \sin\dfrac{\theta_2}{2} & \sin\dfrac{\theta_3}{2} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{vmatrix}, \quad \mathcal{R} = e^{-i\mathcal{E}t/\hbar}R, \tag{34}$$

with N a normalisation constant over $[-2\pi,2\pi]^3$, $N^2 = 96\pi^3$. This R approximates the toroidal part of the stationary protonic state in (1). It has yielded a quite reasonable proton spin structure function [6] and reasonable parton distribution functions for the up and down valence quarks in the proton [5].

Weinberg derives the terms $h_{\mu\nu}$ in a weak field approximation and finds (p. 78 in [2]) for the time component

$$h_{00} = -2\phi, \tag{35}$$

where ϕ is the Newtonian gravitational potential from a mass M at distance r from its center in relativistic units

$$\phi = -\frac{1}{c^2}\frac{G_N M}{r}. \tag{36}$$

Newton's gravitational constant is $G_N = 6.67430(15) \cdot 10^{-11}\ \mathrm{Nm^2kg^{-2}}$ [15]. From the asymptotic part of the spatial metric Weinberg finds (p. 182 in [2]) as $r \to \infty$

$$h_{ij} \to \frac{2G_N M}{c^2 r} n_i n_j + O\left(\frac{1}{r^2}\right), \quad n_j \equiv x^j/r, \quad j = 1,2,3. \tag{37}$$

It is reasuring in a general relativistic setting to see that the time and space components are equal in size $h_{00} = h_{jj}$. With solar mass $M_{Sun} = 1.9885 \cdot 10^{30}$ kg and radius $r_{Sun} = 6.9551 \cdot 10^8$ m [15] we get at the surface

$$h_{\mu\mu} \approx \frac{2G_N M_{Sun}}{c^2 r_{Sun}} = 4.24635(24) \cdot 10^{-6} \approx 4 \text{ ppm}. \tag{38}$$

Generic case. To estimate the effect on the eigenvalues in (30) we integrate $\mathcal{R}$ over the gravitational term. For the time component we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\mathcal{E}t/\hbar}\frac{1}{2}\theta_0^*\theta_0 e^{-i\mathcal{E}t/\hbar}\mathrm{d}(\theta_0/i) = \frac{\pi^2}{6}. \tag{39}$$

The wave function R for the spatial part has doubling in the periods[1] and we get

$$\int\int\int_{(-2\pi)^3}^{(2\pi)^3} R^* \sum_{j=1}^{3} w(\theta_j)\ R\ \mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3 = \frac{\pi^2}{2} + \frac{5}{4}. \tag{40}$$

Hovering in the spherical, static field ϕ we have for the length scale a that $a^2 \to a'^2 = g_{jj}a^2 = (1-2\phi)a^2 = (1+h_{jj})a^2$ [16]. Thus with $\hbar c/a = 214\,\mathrm{MeV}$ and $\varepsilon = 1$, we get the approximate proton eigenvalue shift from (1) to (30)

$$\begin{aligned}&\mathcal{E}' - \mathcal{E} = \\ &\frac{\mathcal{E}}{\sqrt{1+h_{jj}|_{Sun}}} + \left(\frac{\hbar c}{a}\left(\frac{\pi^2}{6}+\frac{\pi^2}{2}+\frac{5}{4}\right)\frac{h_{\mu\mu}|_{Sun}}{\sqrt{1+h_{jj}|_{Sun}}}\right) \\ &-\mathcal{E} = \mathcal{E}\cdot(-2.1+7.6)\text{ ppm} = \mathcal{E}\cdot 5.5\text{ ppm}.\end{aligned} \tag{41}$$

This contribution is larger than the uncertainty with which certain nuclear masses are known. To test it in a far future one may imagine a robotic mission in close orbit around the Sun deploying a space shuttle to hover in the Sun's almost static field at a fixed radius vector relative to the stars —*i.e.*, disregarding the Sun's differential rotation— doing an analogue of Cavendish's experiment with two balls in mutual mass attraction. The mass eigenstate eigenvalues in the shuttle might be compared to the unaffected *rest* masses determined in the mothership in a freely falling frame of reference when in orbit around the Sun. *Rest masses should be measured in freely falling frames of reference, i.e., locally Minkowskian spaces while the eigenvalues of quantum mechanical mass eigenstates according to* (30) *might be influenced by the presence of gravitational fields when the particles are* not *freely falling.* The predicted increases in mass eigenstate eigenvalues do not imply composition-dependent rest masses because in free fall the extra gravitational term in (30) vanishes. Thus the Eötvös/Eöt-Wash experiments [3,4] on the weak equivalence principle WEP do not question the predictions. Instead something like a Cavendish experiment of mutually attracting test masses is needed. The maneuvering stability of the MICROSCOPE satellite [4] in either "spin mode" or "inertial pointing" is encouraging for such undertakings. Extraterrestrial measurements of *"hovering masses"* may also be compared with masses determined at the surface of the Earth. Apart from the huge technical challenges a Cavendish experiment asks for an increased accuracy on the value of Newton's gravitational constant G_N which so far is known "only" at the 23 ppm level. A less futuristic, yet still quite challenging experiment would be on a mission going close to the surface of Jupiter. With mass $M_{Jup} = 1.89815(4) \cdot 10^{27}$ kg and radius $r_{Jup} = 7.1492 \cdot 10^7$ m [17] we get correspondingly an effect at the 51 ppb level. Here one could measure the gamma decay energy of the metastable barium-137 isotope whose energy level $J^P = \frac{11}{2}^-$ is known almost at the ppm level [18],

$$E_\gamma = 661.659(3)\ \mathrm{keV}. \tag{42}$$

The half-life of this state is 2.552 minutes [18] corresponding to a level width of

$$\Gamma(\text{Ba-137m}) = \frac{h}{(2.552/\ln 2\ \text{min})} = 1.872\cdot 10^{-17}\ \mathrm{eV}. \tag{43}$$

Since the effect of the gravitational term is more or less relative[2] it will influence the nuclear binding energies to the same degree and to measure a shift in the gamma energy at the ppm-level should be reachable in a not too far future, in a robotic mission close to Jupiter.

Planck case. We define the Planck length l_{Pl} as

$$\frac{l_{Pl}^2}{\kappa} \equiv hc \quad \to \quad l_{Pl} = 2.0 \cdot 10^{-19}\ \mathrm{fm}, \tag{44}$$

where $\kappa = \frac{8\pi}{c^4}G_N$ is the strength of mutual influence between the energy-momentum tensor $\mathrm{T}_{\mu\nu}$ and the metric $g_{\mu\nu}$ in Einstein's general theory of relativity [2]

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{R} = -\kappa \mathrm{T}_{\mu\nu}. \tag{45}$$

Here $\mathrm{R}_{\mu\nu}$ is the curvature tensor derived from the metric and R its contraction over spacetime indices $\mu,\nu = 0,1,2,3$. One may think of (44) as expressing an exchange of one unit of *space action hc* between the gravitational sector and a certain quantum sector. In the present connection that would be the strong interaction sector and the choice

$$\varepsilon = \frac{l_{Pl}^2}{a^2} \tag{46}$$

[1] We interpret the period doubling as the topological origin of the non-zero charge of the proton [5].

[2] For a neutronic Slater determinant with no period doubling in the sines, eq. (40) yields $4.9\cdots$ instead of $6.18\cdots$ so some differences are expected in the effects on neutrons as compared to protons.

would express a ratio between Gaussian curvatures

$$\frac{K_a}{K_{Pl}} = \frac{1/a^2}{1/l_{Pl}^2}. \tag{47}$$

We thus interpret the choice (46) as the case for the quantisation of gravity. From the vanishingly small value $\approx 4 \cdot 10^{-38}$ it follows that the effect of the gravitational term in (30) on the eigenvalues of baryon mass eigentates would not be detectable in any foreseeable future. Any consequences of the gravitational term would have to be topological to be detectable and would otherwise remain (only) conceptually unifying. Note that (47) signifies a second-order effect in κ since κ is already making its influence on the metric in (45).

Empirical case. We stated a plus sign for the gravitational term in (30) in analogy with the phenomenon of gravitational redshift of photons, for instance emitted from the surface of the Sun at r_2 and received on the Earth at r_1,

$$\frac{\Delta\nu}{\nu} = \phi(r_2) - \phi(r_1) < 0. \tag{48}$$

It amounts to 2.1 ppm and was confirmed already in 1959 by measurements [2,19]. Thus, the photons have a higher energy near the surface of the Sun than at a distance away from it. Analogously we chose the plus sign for the gravitational effect on massive particles. But there might be surprises and ε would have to be determined experimentally. In case other consequences of the gravitational term can be derived, it would make the opportunity to find out if ε does have a unique value.

Conclusion. – We developed concepts on metrics and quantum fields to introduce a gravitational term in a theory for baryon mass eigenstates. The theory is based on an intrinsic configuration space, the Lie group $U(3)$. The effect is an increase in eigenvalues of mass eigenstates at the 51 parts per billion level at the "surface" of the planet Jupiter and we discussed possible experiments to test it. A confirmation would imply the entrance of general relativity into the realm of elementary particles.

Appendix A: first quantisation. – The conjugacy of the coordinate forms $\mathrm{d}x_j$ to the coordinate fields $\frac{\partial}{\partial x_j}$ *geometrises the commutation relations between conjugate variables in quantum mechanics.* Thus

$$\mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij} \sim [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{A.1}$$

both express the basic relation of first quantisation and generalises to the *global* relations on the configuration space

$$\mathrm{d}\theta_i(\partial_j) = \delta_{ij} \tag{A.2}$$

If the expressions in (A.2) and (B.1) seem like completely new territory, the reader is advised to consult [20] or appendix A10 in [8]; otherwise the present appendices will be incomprehensible.

Appendix B: spacetime fields as exterior derivatives. – Imagine a scalar state φ with intrinsic configuration variable u. We read off intrinsic momenta $\pi_j(u)$ by applying $\mathrm{d}\varphi$ to a basis $\frac{\partial}{\partial x_j}$ induced from laboratory space,

$$\pi_j(u) = -i\hbar\, d\varphi_u\left(\frac{\partial}{\partial x_j}\right) = \frac{-i\hbar}{a}\partial_j|_u[\varphi]. \tag{B.1}$$

The dynamics inherent in the time-dependent Schrödinger equation (12) can be embedded in $U(1)\otimes U(3)$ based on four-dimensional representations like

$$T_0 = \begin{Bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \quad T_3 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix}. \tag{B.2}$$

and

$$S_3 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}, \quad M_3 = \begin{Bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}. \tag{B.3}$$

Embedding in a product $U(1)\otimes U(3)$ with the time dimension separated from the intrinsic configuration space $U(3)$ allows for time not to be a dynamical quantum variable [21] and at the same time to have a four-dimensional formulation of the fields in spacetime projection. We now consider projections along the torus $U(1)\otimes U_0(3)$ given by

$$\tilde{u} = e^{i\theta_0 T_0 + i\boldsymbol{\theta}\cdot\mathbf{T}}, \quad \mathbf{T} = q_1T_1 + q_2T_2 + q_3T_3 = \frac{a}{\hbar}\mathbf{p}, \tag{B.4}$$

with the three iT_j as the toroidal generators of $U(3)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. The space projection $\mathrm{d}R$ for R —which would correspond to (B.1) —is then replaced by the spacetime projection

$$\mathrm{d}\mathcal{R} \equiv \frac{a}{-i\hbar}\pi_\mu \mathrm{d}\theta_\mu, \quad \mu = 0,1,2,3. \tag{B.5}$$

If we want to project the structure inherent in the solution $\mathcal{R}$ on a given basis at a particular event in the Minkowski spacetime we must consider the directional derivative at a fixed basis $\{iT_\mu\}$, *i.e.*,

$$(iT_\mu)_{\tilde{u}}[\mathcal{R}] = \mathrm{d}\mathcal{R}_{\tilde{u}}(iT_\mu). \tag{B.6}$$

From the left invariance

$$\partial_\mu|_{\tilde{u}} = \tilde{u}\partial_\mu|_e \tag{B.7}$$

of the coordinate fields ∂_μ, we have

$$\mathrm{d}\mathcal{R}_{\tilde{u}}(iT_\mu) = \mathrm{d}\mathcal{R}_{\tilde{u}}(\tilde{u}^{-1}\partial_\mu) = \tilde{u}^{-1}\mathrm{d}\mathcal{R}_{\tilde{u}}(\partial_\mu) = \tilde{u}^{-1}\frac{a}{-i\hbar}\pi_\mu(\tilde{u}), \tag{B.8}$$

where the latter expression uses (B.5) and

$$\mathrm{d}\theta_\mu(\partial_\nu) = \delta_{\mu\nu}. \tag{B.9}$$

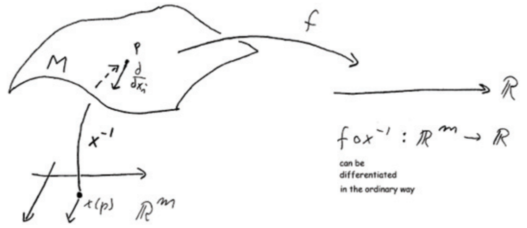

Fig. 2: Derivation of a function f at point p in a manifold M is defined by using a local smooth map $x: M \to R^m$ to pull back the problem to an ordinary derivation on R^m using the pullback function $f \circ x^{-1}$. Figure from [8,22].

From the pull-back $\mathcal{R}^*$ of $\mathcal{R}$ from $U(1) \otimes U_0(3)$ to $\mathbb{R} \otimes \mathbb{R}^3$ given by (see fig. 2)

$$\mathcal{R}^* = \mathcal{R} \circ \exp\colon\ \mathbb{R} \oplus \mathbb{R}^3 \to \quad U(1) \otimes U_0(3) \to \mathbb{C} \qquad \text{(B.10)}$$

we also have the directional derivative using (B.8)

$$\mathrm{d}\mathcal{R}_{\tilde{u}}(iT_\mu) = \tilde{u}^{-1} \frac{\partial \mathcal{R}^*}{\partial \theta_\mu} \left(\frac{\mathcal{E}t}{\hbar}, \frac{p_1 x^1}{\hbar}, \frac{p_2 x^2}{\hbar}, \frac{p_3 x^3}{\hbar} \right). \qquad \text{(B.11)}$$

We use $\theta_0 = ict/a$ from (13), introduce $q_0 = \mathcal{E}/\Lambda = \frac{a}{\hbar}\frac{\mathcal{E}}{c}$ and get for the phase factor

$$\begin{aligned} \tilde{u}^{-1} &= \exp(q_0\theta_0 - i\mathbf{q}\cdot\boldsymbol{\theta}) = \exp\left(i\frac{\mathcal{E}t}{\hbar} - i\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} \right) \\ &\equiv \exp(i\omega t - i\mathbf{k}\cdot\mathbf{x}) \equiv e^{ik\cdot x} \end{aligned} \qquad \text{(B.12)}$$

with $\hbar\omega = \mathcal{E}$ and $\hbar\mathbf{k} = \mathbf{p}$.

In the pull-back (B.10) we have used the coordinate fields as induced base

$$\partial_\mu|_{\tilde{u}} = \left.\frac{\partial}{\partial \theta_\mu}\right|_{\tilde{u}} = \mathrm{d}(\exp)_{\exp^{-1}(\tilde{u})}(e_\mu), \qquad \text{(B.13)}$$

where $\{e_\mu\}$ is a basis in the parameter space, *i.e.*, a basis at the event in Minkowski space, see fig. 2 where the manifold M in the present case could be $U(1) \otimes U(3)$, the inverse map $x^{-1} = \exp$, and the complex-valued function f would be the measure-scaled wave function $\mathcal{R}$ introduced in (18).

Appendix C: second quantisation. – Reading off intrinsic momenta in (B.1) at different laboratory space points $\mathbf{x}$ and $\mathbf{x}'$ corresponds to generating conjugate fields $\pi(\mathbf{x})$ and $\pi(\mathbf{x}')$. We take this as the origin of second quantization: Read offs of intrinsic variables are independent when done at different laboratory space points $\mathbf{x}$. Below we unfold some details of this conception [8].

From the commutators

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \qquad \text{(C.1)}$$

in (A.1), we introduce raising and lowering operators in a coordinate representation (cf. p. 182 in [10], cf. also appendix B in [22])

$$\hat{a}_j^\dagger = \frac{1}{\sqrt{2}}\left(\theta_j - i\frac{a}{\hbar}\hat{p}_j \right), \quad \hat{a}_j = \frac{1}{\sqrt{2}}\left(\theta_j + i\frac{a}{\hbar}\hat{p}_j \right) \qquad \text{(C.2)}$$

to rewrite the momentum component operators as

$$\hat{p}_j = \frac{-i\hbar}{a}\frac{1}{\sqrt{2}}\left(\hat{a}_j - \hat{a}_j^\dagger \right). \qquad \text{(C.3)}$$

Note that a without a hat is the length scale introduced in (4) for the projection from the intrinsic, toroidal coordinates to laboratory space.

We now return to the *interpretation of momentum components as directional derivatives* in (B.11). By comparison with (B.8), we infer intrinsic momenta

$$p_\mu = \frac{-i\hbar}{a}\frac{\partial}{\partial \theta_\mu}|_e \mathcal{R}^*. \qquad \text{(C.4)}$$

Using (C.3) and (B.11) we have for a fixed basis projection to space ($j = 1, 2, 3$)

$$\mathrm{d}\mathcal{R}_{\tilde{u}}(iT_j) = e^{ik\cdot x}\frac{1}{\sqrt{2}}\left(\hat{a}_j(\mathbf{k}) - \hat{a}_j^\dagger(\mathbf{k}) \right)\mathcal{R}^*|_{(\omega t, k_1x^1, k_2x^2, k_3x^3)}. \qquad \text{(C.5)}$$

We interpret (C.5) as Fourier components of a conjugate momentum field

$$\pi(x) = \int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}}\sqrt{\frac{\mathcal{E}(\mathbf{k})}{2}}\left(\hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k}) \right)e^{ik\cdot x} \qquad \text{(C.6)}$$

to be excited at the spacetime coordinate x where the intrinsic momenta are read off according to (C.4). We incorporated the $1/\sqrt{2}$ prefactor on the annihilation and creation operators into the standard normalization of the momentum field and omitted a factor $-i\hbar/a$.

If we uphold the canonical relation

$$\dot{\phi}(x) = \pi^\dagger(x), \qquad \text{(C.7)}$$

where "dot" represents derivation with respect to time t, we get for the field ϕ conjugate to the momentum field π, that

$$\phi(x) = -i\int \frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3}}\frac{1}{\sqrt{2\mathcal{E}(\mathbf{k})}}\left(\hat{a}(\mathbf{k}) - \hat{a}^\dagger(\mathbf{k}) \right)e^{-ik\cdot x}. \qquad \text{(C.8)}$$

It can be shown [8] that (C.8) yields a standard propagator.

Appendix D: local gauge transformation equates left translation in intrinsic space. – We require local gauge invariance of a field Hamiltonian [23] constructed from colour quark fields generated as[3]

$$\psi_j(u) = \mathrm{d}R_u(\partial_j) \qquad \text{(D.1)}$$

for use in

$$H = \int \psi^\dagger\left(-i\hbar c\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2 \right)\psi\,\mathrm{d}x^3,\ \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*). \qquad \text{(D.2)}$$

Here we suppressed spinor indices which are mixed by the 4×4 Dirac matrices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β. The spinor

[3]This section is edited from [6,24].

indices commute with the colour indices. Using left invariance of the coordinate fields

$$\partial_j|_u = uiT_j = u\partial_j|_e, \tag{D.3}$$

we get in the mass term of (D.2)

$$\begin{aligned}\psi'(u')^\dagger\psi'(u') &= (u'iT_j[R])^\dagger\,(u'iT_j[R]) = \\ (iT_j[R])^\dagger\,(u')^\dagger u'\,(iT_j[R]) &= \psi(u)^\dagger\psi(u)\end{aligned} \tag{D.4}$$

provided the configuration variables are unitary, i.e., $(u')^\dagger u' = u^\dagger u = \mathbf{1}$. Next we impose the local gauge transformation

$$\psi \to \psi' = g(x)\psi, \quad g(x) \in SU(3), \quad \partial_\mu \to D_\mu = \partial_\mu + \mathcal{G}_\mu \tag{D.5}$$

with colour gauge field, $\mathcal{G} = ig_s\mathbf{G}$ [15] containing a strong coupling g_s

$$\begin{aligned}\mathcal{G}_\mu &= ig_sG^k_\mu\frac{\lambda_k}{2}, \quad k = 1, \cdots, 8, \\ G^k &\sim G^{(k)}(e) = \mathrm{d}\Phi_e\left(i\frac{\lambda_k}{2}\right)\end{aligned} \tag{D.6}$$

and transforming (when $e \to g(x)$ in $G^{(k)}$) like in [25]

$$\begin{aligned}\mathcal{G}'_\mu &= g(x)\mathcal{G}_\mu g(x)^{-1} - \partial_\mu(g(x))g(x)^{-1} \to \\ \left(D'_\mu\psi'\right)^2 &= \left(D_\mu\psi\right)^2.\end{aligned} \tag{D.7}$$

We thus have the basic ingredient colour fields for setting up QCD. Choosing $u = g(x)$ in (D.5) and in (D.3) equates local gauge transformation in laboratory space to left translation of the intrinsic coordinate fields,

$$\psi_j(u) = \partial_j|_u[\Phi] = u\partial_j|_e[\Phi] = u\psi_j(e). \tag{D.8}$$

* * *

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