



LETTER

# Factorized Hilbert-space metrics and non-commutative quasi-Hermitian observables

To cite this article: Miloslav Znojil 2022 *EPL* **139** 32001

View the [article online](#) for updates and enhancements.

## You may also like

- [Dissipative instabilities in beam-plasma systems](#)  
Eduard V. Rostomyan
- [Relativistic quantum oscillator model under the effects of the violation of Lorentz symmetry by an arbitrary fixed vector field](#)  
Faizuddin Ahmed
- [Interplay of 3d and 4f electrons in ZrCuSiAs-type oxypnictide CeCrAsO](#)  
J. Chen, M. M. Wang, L. Qiao et al.

# Factorized Hilbert-space metrics and non-commutative quasi-Hermitian observables

MILOSLAV ZNOJIL<sup>(a)</sup>

*The Czech Academy of Sciences, Nuclear Physics Institute - Hlavní 130, 250 68 Řež, Czech Republic,  
Department of Physics, Faculty of Science, University of Hradec Králové - Rokitanského 62,  
50003 Hradec Králové, Czech Republic and  
Institute of System Science, Durban University of Technology - Durban, South Africa*

received 5 April 2022; accepted in final form 5 July 2022  
published online 1 August 2022

**Abstract** – In 1992, Scholtz *et al.* (*Ann. Phys.*, **213** (1992) 74) showed that a set of non-Hermitian operators can represent observables of a closed unitary quantum system, provided only that its elements are quasi-Hermitian (*i.e.*, roughly speaking, Hermitian with respect to an *ad hoc* inner-product metric). We show that such a version of quantum mechanics admits a simultaneous closed-form representation of the metric  $\Theta_N$  and of the observables  $\Lambda_k$ ,  $k = 0, 1, \dots, N + 1$  in terms of auxiliary operators  $Z_k$  with  $k = 0, 1, \dots, N$ . At  $N = 2$  the formalism degenerates to the well-known  $\mathcal{PT}$ -symmetric quantum mechanics using factorized metric  $\Theta_2 = Z_2 Z_1$ , where  $Z_2 = \mathcal{P}$  is parity and where  $Z_1 = \mathcal{C}$  is charge.

Copyright © 2022 EPLA

**Introduction.** –  $\mathcal{PT}$ -symmetric quantum mechanics of review [1] offers one of the best known examples of technical and conceptual merits of the use of a nontrivial Hilbert-space inner-product metric  $\Theta$ , which is assumed to be factorized

$$\Theta_{\mathcal{PT}} = \mathcal{P}\mathcal{C} \neq I. \quad (1)$$

The underlying abstract formulations of quantum mechanics using any nontrivial inner-product metric  $\Theta \neq I$  are known as quasi-Hermitian quantum mechanics [2], or as pseudo-Hermitian quantum mechanics [3]. Once one adds the specific factorization ansatz (1), operators  $\mathcal{P}$  and  $\mathcal{C}$  are interpreted, most often, as parity and charge, respectively. Still, also certain less specific forms of the factors  $\mathcal{P}$  and  $\mathcal{C}$  forming the metric can be found discussed in the literature [4–7]. In our present letter we intend to reveal and describe an unexpected new connection between the abstract mathematics of the Hilbert-space-metric factorization as sampled by eq. (1) and the realistic requirements in physics where one often has to demand the generic consistent coexistence of several non-commutative non-Hermitian quantum observables (say,  $\Lambda_j$ ).

*The challenge of non-commutativity.* The challenging nature of all of the mutually more or less equivalent approaches to quantum dynamics based on the nontriviality of the physical Hilbert-space metric (1) can be illustrated

by the Bagchi's and Fring's conjecture [8] that the use of such a formalism could enrich even the study of quantized gravity. The latter authors argued that the use of Hilbert spaces with  $\Theta \neq I$  could lead to a new and consistent description of quantum systems which admit the existence of an observable minimal length and/or of some innovative forms of experimentally verifiable uncertainty relations.

In the latter, most ambitious physical project the main technical obstacle has later been found to lie in the necessity of reflecting the non-commutative nature of the sets of the relevant observables [9]. In fact, this was bad news which still belong among the key motivating forces in our present letter. The more so because earlier, the obstacle had already been encountered as serious in the framework of the abstract theory [2] as well as in several very concrete calculations and pragmatic studies, say, in condensed matter physics [10] or in nuclear physics [11].

At present, the technical subtlety of the non-Hermitian non-commutativity still belongs, in spite of an enormous recent progress in the field, among the main limiting factors and open questions in the otherwise highly promising and fairly rapid developments in the field of the non-Hermitian coupled-cluster method (see, *e.g.*, the recent progress reports in [7,12]), etc.

*The challenge of technical feasibility.* In general, it is really tempting to expect that using the quasi-Hermitian (QH) quantum theory one could really achieve a

<sup>(a)</sup>E-mail: znojil@ujf.cas.cz (corresponding author)

conceptual compatibility between a manifestly non-Hermitian operator of momentum (say,  $\Lambda_0$ ) with another, still non-Hermitian operator representing position (say,  $\Lambda_1$ ), etc. The more explicit technical realization of such an idea and project appeared, unfortunately, to be more difficult than expected. In [9], in particular, we demonstrated that “whenever we are given more than one candidate for an observable (*i.e.*, say, two operators  $\Lambda_0$  and  $\Lambda_1$ ) in advance”, the consistent QH theory “need not exist in general”. The methodical as well as phenomenological scepticism implied by such a discouraging result is to be weakened in our present letter.

Our present encouragement of a return to optimism will rely upon letter [13] in which we proposed a specific reformulation of the  $\mathcal{PT}$ -symmetric quantum mechanics of ref. [1]. Moreover, we will also incorporate the results of paper [14] where we generalized ansatz (1) and where we considered a more flexible version of the QH quantum theory which admits any number  $N$  of factors forming the metric

$$\Theta_N = Z_N Z_{N-1} \dots Z_2 Z_1. \quad (2)$$

After the transition from the specific ansatz (1) to its general form (2), and after a brief outline of our basic idea in the following section, our present main message will concern the metric-factorization-related systematic construction of the sets

$$\Lambda_0, \Lambda_1, \Lambda_2, \dots, \quad (3)$$

of the admissible non-Hermitian (*i.e.*, QH) operators representing the observables. The details of the construction will be formulated in the third to sixth sections and summarized in the last section.

**The concept of physical inner-product metric.** – The origin of the idea of the possible usefulness and consistency of quantum mechanics using a nontrivial operator metric  $\Theta \neq I$  can be traced back to the year 1992 when the authors of the review paper [2] declared the conventional use of trivial metrics  $\Theta_{\text{textbook}} = I$  “somewhat restrictive”. They showed that under suitable mathematical conditions a broader class of “consistent quantum mechanical systems” can be described using a certain “set of non-Hermitian operators” representing observable quantities. In other words, these operators (see eq. (3) above) had to be compatible with the metric-dependent observability *alias* quasi-Hermiticity condition

$$\Lambda_k^\dagger \Theta = \Theta \Lambda_k, \quad k = 0, 1, \dots \quad (4)$$

In this light, the  $\mathcal{PT}$ -symmetric quantum mechanics can be perceived as one of the most successful realizations of the amended formalism with the formal and trivial observable  $\Lambda_0 = I$ , with the observable charge  $\mathcal{C} = \Lambda_1$  and metric  $\mathcal{PC} = \Lambda_2$  of eq. (1), and, for unitary systems, also with the observable QH Hamiltonian,  $\Lambda_3 = H$ .

*Inspiration: antilinear symmetries.* The inspiration of our present letter may be traced back to the Bender’s and Boettcher’s conjecture of an upgrade of quantum

theory [15] which made the abstract QH formalism user-friendlier. The goal has been achieved via a rather unconventional assumption that a given non-Hermitian Hamiltonian  $H$  can be made acceptable when required to exhibit certain auxiliary antilinear symmetries.

These symmetries were of two forms. One was the parity-time-reversal antilinear symmetry *alias*  $\mathcal{PT}$ -symmetry

$$H\mathcal{PT} = \mathcal{P}TH. \quad (5)$$

It combined, successfully, a phenomenological appeal of the linear parity  $\mathcal{P}$  with the antilinear time reversal  $\mathcal{T}$ . In comparison with the abstract QH approach of Scholtz *et al.* [2], the updated theory based on the *ad hoc* technical assumption (5) proved much more intuitive and more friendly in applications (see, *e.g.*, the collection of reviews in [6] for details).

Another, less well-known but much more fundamental antilinear symmetry of the Hamiltonian has been found in its parity-charge-time *alias*  $\mathcal{PCT}$  symmetry [16],

$$H\mathcal{PCT} = \mathcal{PCT}H. \quad (6)$$

In a way explained in [3] the action of the time-reversal operator  $\mathcal{T}$  in both of these antilinear symmetries has precisely the same meaning as Hermitian conjugation. This means that eq. (6) can be given the following equivalent form:

$$H^\dagger \mathcal{PC} = \mathcal{PC}H. \quad (7)$$

In what follows we will prefer the latter notation convention.

*Physics: quasi-Hermitian observables.* Before we proceed to the description of details let us point out that the originally purely formal nature of ansatz (2) as introduced in [14] will be complemented here by the turn of attention to physics. The QH quantum dynamics will be assumed based, in the spirit of ref. [2], not only on the specification of the Hamiltonian  $H$  but also of a multiplet of candidates for the other, phenomenologically potentially relevant observables.

Our attention will be paid to a constructive guarantee that all of the latter operators (3) admit the standard probabilistic experimental interpretation. According to assumptions as formulated in [2], our knowledge of  $H$  and of the set (3) has to represent a dynamical input information about the system in question. All of these operators will have to satisfy the quasi-Hermiticity relations

$$\Lambda_\ell^\dagger \Theta_N = \Theta_N \Lambda_\ell, \quad \ell = 0, 1, 2, \dots, \quad (8)$$

of course. In paper [2] we read the reason: “for the given set of non-Hermitian observables”, the essence of the QH model-building recipe lies in the “construction of a metric (if it exists)”. In this context, the additional, more specific recommendation of the construction of the metric as inspired by eq. (1) and as generalized in paper [14] can be read as a requirement that the metric entering the

hidden-Hermiticity condition (4) should have a factorized form. In our present letter we are going to turn attention to physics behind such a factorization.

**Framework: QH quantum mechanics.** – In the two most elementary special cases with  $N \leq 1$  there is in fact no factorization. At  $N = 0$ , in particular, the degenerate factorization ansatz (2) yields trivial  $\Theta_0 = I$ . This choice leads to the conventional quantum mechanics of textbooks. Relation (8) then means that all of the admissible and eligible observables (3) must be self-adjoint. Their selection (hinted, typically, by the hypothetical quantum-classical correspondence [17]) remains formally unconstrained.

*Bra-ket notation.* At any  $N > 0$  the acceptability of operators (3) is only constrained by their QH property (8). One might recall paper [14] and, using the terminology of functional analysis, notice that the traditional  $N = 0$  Hilbert space of textbooks becomes split into the doublet of the two formally non-equivalent Hilbert spaces  $\mathcal{H}_{phys}$  and  $\mathcal{H}_{math}$  (in [14] we recommended the abbreviations  $\mathcal{H}_{phys} = \mathcal{R}_0$  and, at any preselected  $N$ ,  $\mathcal{H}_{math} = \mathcal{R}_N$ ). The latter space is to be interpreted as auxiliary, strongly preferred in calculations but manifestly unphysical. Only the former Hilbert space  $\mathcal{R}_0$  provides the correct probabilistic picture of the quantum system in question.

The less friendly space  $\mathcal{R}_0$  differs from its partner  $\mathcal{R}_N = \mathcal{H}_{math}$  by the less elementary inner product,

$$(\psi_a, \psi_b)_{phys} = (\psi_a, \Theta \psi_b)_{math}. \quad (9)$$

This means that the Dirac's bra-ket notation must be used here with due care. In what follows, it will still be used in the preferred manipulation space  $\mathcal{R}_N = \mathcal{H}_{math}$ ,

$$(\psi_a, \psi_b)_{math} = \langle \psi_a | \psi_b \rangle. \quad (10)$$

The transition to  $\mathcal{R}_0 = \mathcal{H}_{phys}$  can be then represented, at any  $N$ , by the formula

$$(\psi_a, \psi_b)_{phys} = \langle \psi_a | \Theta_N | \psi_b \rangle. \quad (11)$$

This means that the measurable predictions (*i.e.*, the relevant physical matrix elements) can always be evaluated without ever leaving the user-friendlier representation space  $\mathcal{R}_N = \mathcal{H}_{math}$ .

*The choice of  $N = 1$  and the QH formalism without factorization.* One returns to the standard QH formulation of quantum mechanics when choosing  $N = 1$ . In its framework one deals with a nontrivial Hermitian and positive definite metric  $\Theta_1$  *alias*  $Z_1$  which is self-adjoint (*i.e.*,  $Z_1 = Z_1^\dagger$ ) and, in principle, eligible as an observable (so that we can set  $\Lambda_1 = Z_1$  in (3)). Such an operator becomes nontrivial (*i.e.*,  $Z_1 \neq I$ ) whenever our preselected observable quantum Hamiltonian  $H$  *alias*  $Z_0$  *alias*  $\Lambda_2$  is chosen non-Hermitian,  $Z_0 \neq Z_0^\dagger$ . Naturally, as long as the Hamiltonian carries information about the closed-system dynamics, one must know or prove that its spectrum is real, *i.e.*, compatible with the unitarity of the evolution.

Once such a spectrum is shown real (and also discrete, see the reasons discussed in [18]), one arrives at the fundamental (one could say Dieudonné's [19]) Hamiltonian hidden-Hermiticity condition (8),

$$Z_0^\dagger Z_1 = Z_1 Z_0, \quad N = 1. \quad (12)$$

This condition becomes tractable as an equation to be solved. All of its Hamiltonian-dependent solutions  $\Theta_1 = Z_1(Z_0)$  are the formally eligible metric operators (see the list of their necessary mathematical properties as listed in eq. (2.1) in [2]). Any one of these Hilbert-space metrics defines a different, conceptually consistent quantum system. All of them would lead to the experimental predictions in a way which would vary with the different choices of the sets of the operators (3) representing measurable quantities. In this setting it makes sense to prove the following simple but important result.

**Lemma 1** *The knowledge of Hilbert-space metric  $\Theta = \Theta^\dagger$  enables us to define all of the eligible observables as the operator products*

$$\Lambda = M \Theta, \quad (13)$$

*with arbitrary  $M = M^\dagger$ .*

**Proof.** Referring to the list of the necessary mathematical properties of the Hilbert-space metrics (see, *e.g.*, [2]), and having in mind, for the sake of the simplicity of the proof, just the special models living in finite-dimensional Hilbert spaces, we leave, to the readers, all of the necessary care to be paid, in the general case, to the domains of the operators, etc. (see, *e.g.*, the dedicated reviews in [20]). Then, once we insert expression (13) in the criterion (8) of observability

$$\Lambda^\dagger \Theta = \Theta \Lambda,$$

we obtain the relation

$$\Theta^\dagger M^\dagger \Theta = \Theta M \Theta,$$

and recall the Hermiticity and invertibility of the metric.  $\square$

**Quantum models with parity-times-charge metrics.** – At  $N = 2$ , the QH-based quantum mechanics is described and discussed in [13]. In factorization formula (2) the two new operators can be interpreted as traditional charge ( $Z_1 = \mathcal{C}$ ) and parity ( $Z_2 = \mathcal{P}$ ), both, possibly, generalized [5]. In the language of physics the theory relies upon a pre-selection of the Hamiltonian  $H$  denoted here, alternatively, by the zero-subscripted symbol  $Z_0$  and determining the quantum evolution dynamics in the QH Schrödinger picture,

$$i \frac{d}{dt} |\psi(t)\rangle = Z_0 |\psi(t)\rangle, \quad |\psi(t)\rangle \in \mathcal{H}_{math}. \quad (14)$$

In [13] the corresponding quantum theory with  $N = 2$  was dubbed “intermediate-space Schrödinger picture” (ISP).

From the point of view of mathematics its three operators  $Z_k$  were shown constrained, solely, by the triplet of relations

$$Z_0^\dagger (Z_2 Z_1) = (Z_2 Z_1) Z_0, \quad (15)$$

$$Z_1^\dagger Z_2 = Z_2 Z_1, \quad (16)$$

$$Z_2^\dagger = Z_2 \quad (17)$$

(see table 1 in [13]).

These equations guarantee the unitarity of the evolution. Directly, this feature of the system is only controlled by the Hamiltonian-containing equation (15) where one can immediately identify  $Z_2 Z_1 \equiv \Theta_2(H)$ . Still, the role of the other two relations is also related and nontrivial. For explanation it makes sense to return to the  $\mathcal{PT}$ -symmetric quantum mechanics of review [1] in which a very specific parity-times-charge factorization of the metric

$$\Theta_2 = \mathcal{P}\mathcal{C} = Z_2 Z_1 = \Theta_2^\dagger \quad (18)$$

has been introduced and shown useful in applications. On this background an understanding of the amendments provided by the  $N = 2$  formalism of ref. [13] lies in an enhancement of the economy of assumptions. Indeed, among relations (15)–(17) one does not find the popular  $\mathcal{PT}$ -symmetry assumption (5) even when, in our present notation, re-written as the  $\mathcal{P}$ -pseudo-Hermiticity condition  $H^\dagger \mathcal{P} = \mathcal{P} H$ . A minor but interesting generalization of the conventional  $\mathcal{PT}$ -symmetric quantum mechanics is obtained. Even though the relation between  $H$  and  $\mathcal{P}$  remains unspecified, the ISP theory is consistent. The operator  $\mathcal{P}$  itself is purely auxiliary, not carrying any immediate physical meaning at all.

The latter non-observability paradox is caused by the fact that in the framework of ISP we have, in general,  $\mathcal{P}^\dagger \Theta_2 \neq \Theta_2 \mathcal{P}$ . Such a statement looks counter-intuitive. Still, its intuitive acceptability can be supported by the well-known observation that even the coordinate itself is not an observable quantity in general [9]. Although the coordinate is often assumed to be measurable, the construction of its operator representation appeared to be a difficult task even in the most elementary QH square-well models (see, *e.g.*, an example of the construction in [21]). As a consequence, the generalized parity operator constrained by Hermiticity (17) is just a freely variable “parameter” specifying the quantum model via its observables [2].

Once we accept the ISP Hamiltonian  $Z_0$  as an observable which is given in advance, we are prepared to specify the necessary Hermitian metric (18) as one of the eligible solutions of eq. (15). This step being completed, we are left with the single mathematical constraint (16). Nevertheless, it is trivial to see that this is just an identity, *i.e.*, just an equation which is equivalent to the Hermiticity of the solution  $\Theta_2(H)$  of eq. (15). This means that the acceptable generalized charge  $Z_1 = \mathcal{C}$  is not arbitrary, being

fully defined in terms of the metric and of the preselected parity,

$$\mathcal{C} = \mathcal{P}^{-1} \Theta_2, \quad N = 2. \quad (19)$$

The multiplication of the identity (16) by  $Z_1$  from the right yields the new, equivalent formula

$$\mathcal{C}^\dagger \Theta_2 = \Theta_2 \mathcal{C}, \quad N = 2. \quad (20)$$

It says that operator (19) called generalized charge is not only  $\mathcal{PT}$  symmetric (see eq. (16)) but also quasi-Hermitian. It *can* consistently be used as representing a measurable physical quantity. In eq. (3) we may identify, in such a case,  $\Lambda_1 = Z_1 = \mathcal{C}$ . Formula (19) can be read as a sample of eq. (13) with  $M \equiv \mathcal{P}^{-1}$ . Finally, an entirely analogous argument implies that also the metric itself could play an analogous role of another observable,  $\Lambda_2 = Z_2 Z_1 = \Theta_2$ . Once more, it is also possible to add  $\Lambda_3 = Z_0 = H$ .

We may conclude that at  $N = 2$ , a fully consistent unitary quantum system is obtained. In terms of the input information represented by the “suitable physical” operator  $H$  and its “arbitrary mathematical” operator-valued parameter complement  $\mathcal{P}$ , we are able to define the metric  $\Lambda_2 = \Theta_2(H)$  (as any solution of eq. (15)) and via eq. (19), the charge,  $\Lambda_1 = \mathcal{C}$ . Such an explicit constructive specification of the operators of observables converts all of the three obligatory consistency constraints (15), (16) and (17) into mathematical identities.

### Non-standard quantum models with $N = 3$ .

Table 2 in [14] samples the one-to-one correspondence between the factorization of metric and the unitarity of evolution of a quantum system in the generalized Schrödinger picture (GSP) using a preselected  $N$ . At  $N = 3$ , with the physical Hilbert-space metric  $\Theta_3 = Z_3 Z_2 Z_1$ , the internal consistency of the GSP quantum theory has been shown guaranteed by the quadruplet of constraints

$$Z_3^\dagger = Z_3, \quad (21)$$

$$Z_2^\dagger Z_3 = Z_3 Z_2, \quad (22)$$

$$Z_1^\dagger (Z_3 Z_2) = (Z_3 Z_2) Z_1, \quad (23)$$

$$Z_0^\dagger \Theta_3 = \Theta_3 Z_0. \quad (24)$$

The physical meaning of the last item (24) is obvious because it guarantees the hidden Hermiticity of our preselected Hamiltonian  $H = Z_0$ . Using this equation one can determine and select one of the eligible solutions  $\Theta_3 = \Theta_3(Z_0)$  of this equation and assign it the role of the correct physical Hilbert-space metric.

The first, simplest constraint (21) is imposed upon the “generalized parity”  $Z_3 = \mathcal{P}$  which remains, in a parallel to the previous  $N = 2$  scenario, unobservable. This enables us to treat such a (necessarily, self-adjoint) operator, as before, as a carrier of a “dynamical input information”. Its unconstrained variability can be



used to characterize the differences between various phenomenologically non-equivalent quantum systems which are sharing the same Hamiltonian (*i.e.*, the same form of the operator of energy).

The interpretation of the next constraint (22) remained purely formal in [14]. The product  $Z_3 Z_2$  has been identified there with a new operator  $Y_3$  which happens to be self-adjoint,  $Y_3 = Y_3^\dagger$ . This operator cannot be treated as an observable because  $Y_3^\dagger \Theta_3 \neq \Theta_3 Y_3$  in general. It enters the GSP theory, therefore, simply as another, freely variable operator parameter of the model. From this point of view, eq. (22) degenerates to an identity in which, in terms of the already available operators  $Z_3$  and  $Y_3$ , the unique “unobservable quasiparity”  $Z_2 = \mathcal{Q}$  is defined as follows:

$$\mathcal{Q} = Z_3^{-1} Y_3.$$

In the last step we are left with constraint (23). After its pre-multiplication by (presumably, non-singular factor)  $Z_1$  from the right it acquires the equivalent form

$$Z_1^\dagger \Theta_3 = \Theta_3 Z_1.$$

This confirms that  $Z_1$  can represent an observable. For the sake of definiteness let us speak about a renormalized charge,  $Z_1 = \mathcal{R}$ . As long as it is uniquely defined,

$$\mathcal{R} = Y_3^{-1} \Theta_3, \quad (25)$$

we may recall the list of the potentially eligible observables (3) and identify  $\Lambda_1 = \mathcal{R}$ .

In such an upgraded notation our  $N = 3$  eq. (24) yields the solution  $\Theta_3 = \mathcal{P}\mathcal{Q}\mathcal{R} = \Theta_3(H)$  which is required to be self-adjoint, but this constraint appears precisely equivalent to our last relation (23). In other words, the whole set of constraints (24), (23) becomes converted into identities. In their light it is then easy to identify the other two eligible observables, *viz.*,  $\Lambda_2 = \mathcal{Q}\mathcal{R}$  and  $\Lambda_3 = \mathcal{P}\mathcal{Q}\mathcal{R}$  and, finally,  $\Lambda_4 = H$  —see [14].

In the latter paper, unfortunately, the scenarios with  $N > 3$  remained unexplored. Here, we will describe the explicit constructive identification of the potential operators of the phenomenologically relevant observables at any preselected number  $N$  of factors in the general factorized metric of eq. (2).

**Non-standard quantum models with arbitrary  $N$ .** — In paper [14] the quasi-Hermiticity property

$$H^\dagger \Theta_N = \Theta_N H \quad (26)$$

of a preselected Hamiltonian  $H = Z_0$  was assumed to be satisfied by the factorized Hilbert-space metric  $\Theta_N = Z_N Z_{N-1}, \dots, Z_2 Z_1$  at an arbitrary  $N$ . The separate factors  $Z_j$  were required to obey the set of the GSP theoretical consistency requirements

$$Z_1^\dagger (Z_N \dots Z_3 Z_2) = (Z_N \dots Z_3 Z_2) Z_1 \quad (= A_N \equiv \Theta_N), \quad (27)$$

$$Z_2^\dagger (Z_N \dots Z_4 Z_3) = (Z_N \dots Z_4 Z_3) Z_2 \quad (= B_N), \quad (28)$$

...

$$Z_{N-2}^\dagger (Z_N Z_{N-1}) = (Z_N Z_{N-1}) Z_{N-2} \quad (= X_N), \quad (29)$$

$$Z_{N-1}^\dagger Z_N = Z_N Z_{N-1} \quad (= Y_N), \quad (30)$$

$$Z_N^\dagger = Z_N. \quad (31)$$

Recalling the last item (31) we deduce that  $Y_N = Y_N^\dagger$  and, subsequently, that  $X_N = X_N^\dagger$  and so on, until  $B_N = B_N^\dagger$  and  $A_N = A_N^\dagger$ . One of the most important subtle consequences of this observation is the following result.

**Theorem 2** The set (8) of conditions of the quantum-theoretical observability is satisfied by the set of the operator products  $\Lambda_k = Z_k Z_{k-1}, \dots, Z_2 Z_1$ , with  $k = 1, 2, \dots, N$ .

**Proof.** First we notice that the right-hand side of eq. (27) is equal to the metric,  $A_N = \Theta_N$ . What is less obvious is that after the multiplication of the next relation (28) by  $Z_1^\dagger \equiv \Lambda_1^\dagger$  from the left we may recall the previous identity and obtain the metric as well,  $\Lambda_1^\dagger B_N = \Theta_N$ . Similarly, we have  $\Lambda_2^\dagger C_N = \Theta_N$ , etc., until  $\Lambda_{N-1}^\dagger Z_N = \Theta_N$ . Every such a relation has the form

$$\Lambda_k^\dagger M_k = \Theta_N, \quad k = 0, 1, \dots, N-1,$$

where  $M_0 = A_N$ ,  $M_1 = B_N$ , etc., and where we use  $\Lambda_0 = I$  at  $k = 0$ . After we multiply each of these equations by  $\Lambda_{k+1}$  from the right, we reveal that  $\Lambda_0^\dagger M_0 \Lambda_1 = \Lambda_1^\dagger \Theta_N$ ,  $\Lambda_1^\dagger M_1 \Lambda_2 = \Lambda_2^\dagger \Theta_N$  and, in general,  $\Lambda_k^\dagger M_k \Lambda_{k+1} = \Lambda_{k+1}^\dagger \Theta_N$ . In other words, we reveal that the resulting sequence of equations coincides with the QH conditions (8).  $\square$

From this result we may immediately deduce that in the GSP formulation of quantum mechanics at a fixed  $N$ , the system under consideration is characterized, first of all, by its Hamiltonian  $H$  (which is assumed observable) and by the Hilbert-space metric which is, due to constraint (26) (*i.e.*, due to eq. (4) at  $k = 0$ ), Hamiltonian dependent,  $\Theta = \Theta_N(H)$ .

In the light of theorem 2 the subscript  $N$  characterizes not only the number of factors in the metric-operator ansatz (2) but also the size of the multiplet of the invertible and freely variable self-adjoint operator-parameters  $Z_N, Y_N, X_N, \dots, B_N$  of the model as well the size of the multiplet of the observables define in closed form. This is our final result: their first few  $N$ -dependent lists are sampled here in the form of the first few columns of table 1.

**Discussion.** — A decisive technical merit of all of the existing QH quantum models is that their correct physical Hilbert space  $\mathcal{H}_{phys}$  is so easily represented in another, extremely user-friendly Hilbert space  $\mathcal{H}_{math}$ . After the mere amendment  $\langle \psi_a | \psi_b \rangle \rightarrow \langle \psi_a | \Theta | \psi_b \rangle$  of the inner product the experimental predictions of the theory acquire the standard probabilistic form which is, naturally, strictly

Table 1: Observables  $\Lambda_k$  available for a given QH Hamiltonian  $H$ , integer  $N$ , Hilbert-space metric  $\Theta_N(H)$  and for an  $(N - 1)$ -plet of invertible self-adjoint operator xparameters  $Z_N, Y_N, X_N, \dots, B_N$ . The first-line items  $\Lambda_0$  are trivial since  $Y_2 \equiv \Theta_2(H)$  at  $N = 2$ , etc. At any  $N$ , we also recalled the observability status of the “input” Hamiltonian and added the symbol  $\Lambda_{N+1} \equiv H$ .

$N$	2	3	4	5	...
$\Lambda_0$	$I$	$I$	$I$	$I$	...
$\Lambda_1$	$Z_2^{-1}\Theta_2(H)$	$Y_3^{-1}\Theta_3(H)$	$X_4^{-1}\Theta_4(H)$	$W_5^{-1}\Theta_5(H)$	...
$\Lambda_2$	$\Theta_2(H)$	$Z_3^{-1}\Theta_3(H)$	$Y_4^{-1}\Theta_4(H)$	$X_5^{-1}\Theta_5(H)$	...
$\Lambda_3$	$H$	$\Theta_3(H)$	$Z_4^{-1}\Theta_4(H)$	$Y_5^{-1}\Theta_5(H)$	...
$\Lambda_4$	–	$H$	$\Theta_4(H)$	$Z_5^{-1}\Theta_5(H)$	...
$\Lambda_5$	–	–	$H$	$\Theta_5(H)$	...
$\vdots$				$\ddots$	$\ddots$

equivalent to its conventional (but, by assumption, prohibitively complicated) textbook Schrödinger-picture alternative [2,3].

Technically, this means that the unitary evolution of a quantum system in question has to be described not only by the kets  $|\psi(t)\rangle$  and by the corresponding conventional time-dependent Schrödinger equation,

$$i \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle, \quad |\psi(t)\rangle \in \mathcal{H}_{math}, \quad (32)$$

but also, in parallel, by the metric-multiplied kets  $|\psi(t)\rangle\rangle = \Theta_N |\psi(t)\rangle$  with the evolution controlled by another Schrödinger equation in which the Hamiltonian is replaced by its Hermitian conjugate,

$$i \frac{d}{dt} |\psi(t)\rangle\rangle = H^\dagger |\psi(t)\rangle\rangle, \quad |\psi(t)\rangle\rangle \in \mathcal{H}_{math}. \quad (33)$$

In the review [3] it has been emphasized that the use of the QH language helped to elucidate several deep and long-standing unresolved conceptual as well as technical puzzles, say, in the field of relativistic quantum mechanics or in quantum cosmology. In several other physical contexts, unfortunately, the abstract version of the QH theory has been found “very difficult to implement”, with reasons explained on p. 1216 of ref. [3]. In this context, our present letter is to be read as the description of one of the innovative simplification strategies.

In a final remark let us add that the main starting-point technicality of the GSP approach, *viz.* the choice of the positive-definite solution  $\Theta_N = \Theta_N(H)$  of the constraint (26) is, in general, ambiguous [22]. In this sense, any choice can be considered and used as a starting point of the GSP-based metric-multiplication strategy as summarized in table 1. Naturally, every such an initial selection of the physics-determining operator  $\Theta_N(H)$  must satisfy all of the obligatory mathematical properties

as listed and discussed, say, in review [2]. In [13,14] we emphasized that among them a key role is played by the mathematical requirements imposed upon the separate factors  $Z_k$  of the metric. In the present paper the emphasis has been shifted to the physical aspects of these factors. We revealed that the phenomenological information carried by these factors is given by theorem 2, *i.e.*, by their re-interpretation as factors in the candidates for observables,

$$\Lambda_k = Z_k Z_{k-1} \dots Z_2 Z_1, \quad (34)$$

with  $k = 1, 2, \dots, N$ .

*Data availability statement:* No new data were created or analysed in this study.

## REFERENCES

- [1] BENDER C. M., *Rep. Prog. Phys.*, **70** (2007) 947.
- [2] SCHOLTZ F. G., GEYER H. B. and HAHNE F. J. W., *Ann. Phys. (N.Y.)*, **213** (1992) 74.
- [3] MOSTAFAZADEH A., *Int. J. Geom. Methods Mod. Phys.*, **7** (2010) 1191.
- [4] MOSTAFAZADEH A., *J. Math. Phys.*, **43** (2002) 205.
- [5] ZNOJIL M. and GEYER H. B., *Fortschr. Phys. - Prog. Phys.*, **61** (111) 2013.
- [6] BENDER C. M., *PT Symmetry in Quantum and Classical Physics* (World Scientific, Singapore) 2018.
- [7] BISHOP R. F. and ZNOJIL M., *Eur. Phys. J. Plus*, **135** (2020) 374.
- [8] BAGCHI B. and FRING A., *Phys. Lett. A*, **373** (2009) 4307.
- [9] ZNOJIL M., SEMORÁDOVÁ I., RŮŽIČKA F., MOULLA H. and LEGHRIB I., *Phys. Rev. A*, **95** (2017) 042122.
- [10] DYSON F. J., *Phys. Rev.*, **102** (1956) 1230.
- [11] JANSSEN D., DÖNAU F., FRAUENDORF S. and JOLOS R. V., *Nucl. Phys. A*, **172** (1971) 145.
- [12] BISHOP R. F. and ZNOJIL M., *Acta Polytech.*, **54** (2014) 85.
- [13] ZNOJIL M., *Phys. Lett. A*, **421** (2022) 127792.
- [14] ZNOJIL M., *Eur. Phys. J. Plus*, **137** (2022) 335.
- [15] BENDER C. M. and BOETTCHER S., *Phys. Rev. Lett.*, **80** (1998) 5243.
- [16] BENDER C. M., BRODY D. C. and JONES H. F., *Phys. Rev. Lett.*, **89** (2002) 270401; **92** (2004) 119902.
- [17] MESSIAH A., *Quantum Mechanics I* (North Holland, Amsterdam) 1961.
- [18] ZNOJIL M., *Phys. Rev. D*, **78** (2008) 025026.
- [19] DIEUDONNE J., in *Proceedings of the International Symposium on Linear Spaces* (Pergamon, Oxford) 1961, pp. 115–122.
- [20] BAGARELLO F., GAZEAU J.-P., SZAFRANIEC F. and ZNOJIL M., (Editors) *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects* (Wiley, Hoboken) 2015.
- [21] MOSTAFAZADEH A. and BATAL A., *J. Phys. A: Math. Gen.*, **37** (2004) 11645.
- [22] ZNOJIL M., *SIGMA*, **4** (2008) 001 (arXiv:0710.4432v3).