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Perspective

Amorphous topological matter: Theory and experiment

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Abstract – Topological phases of matter are ubiquitous in crystals, but less is known about their existence in amorphous systems, that lack long-range order. We review the recent progress made on defining amorphous topological phases, their new phenomenology. We discuss the open questions in the field which promise to significantly enlarge the set of materials and synthetic systems benefiting from the robustness of topological matter.

Introduction. – The quantum Hall state, the first topological phase ever observed, was discovered in crystalline heterostructures [1], even though its existence does not require an underlying crystalline lattice. Indeed, a two-dimensional free electron gas under a perpendicular magnetic field displays Landau levels. Its associated metallic topological edge states and quantized conductance arise in a confining potential, with no assumption of an underlying crystalline lattice. The quantum Hall displays a continuous translational invariance, and the corresponding electron's momentum $\mathbf{p}$ enters the parabolic dispersion relation $\mathbf{p}^2/2m$, with m being the electron's mass. By promoting m to the effective mass of the electron within a medium, the parabolic dispersion and its corresponding Landau levels can be thought of as arising from the long-wavelength limit of a lattice tight-binding crystalline model [2]. With this notion of translational invariance in place, the condensed matter community discovered how to dispose of magnetic fields and define topological states in crystalline systems [3], establishing topological phases in crystals of any dimension, irrespective of their insulating, conducting or superconducting nature [4–6].

Topological phases do exist in the absence of long-range periodicity, as we are not forced to regularize a continuum theory using a periodic lattice. This observation is at the heart of this perspective article. Our goal is to summarize the recent progress made to understand how topological phases emerge on the largest class of noncrystalline systems, amorphous systems [7–10]. Characterizing topology in amorphous matter, without the convenience of Bloch's theorem, has led to the emergence of new phenomenology, unique to amorphous matter. Topology remains largely unexplored in this class of solids, which may offer different functionalities compared to crystals.

We start by discussing the main properties of amorphous and topological matter, followed by a review of the progress made in combining these two fields. We finish by summarizing the experimental status and offering some perspectives on the main open questions. For a more technical review we refer the reader to ref. [11].

Basic properties of amorphous matter. Amorphous materials are defined by their lack of long-range order [12]. However, they display short- and even medium-range order, as well defined nearest and next-to-nearest neighbour distances, respectively. The short-range order manifests itself as preferred bond lengths and angles, peaked around the values of its crystalline counterpart. Due to the short-range order, amorphous materials have a well-defined coordination environment with a distinct number of nearest neighbours. In solid-state systems this is a result of the electronic configuration of the atoms involved in bonding. Hence, amorphous solids remain locally ordered [12,13]. The absence of Bragg peaks in the diffraction pattern, and thus the absence of long-range order, determines which solids are amorphous.

Amorphous materials are commonplace in science and technology [12]. Their applications range from common

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objects such as window glass to technological devices like computer memories or solar cells [14,15]. Similarly to crystals, amorphous materials can be insulators, semiconductors, metals, and superconductors [12]. Amorphous oxides used in glassware, such as silicon oxide or lead glass, are century-old insulators. Amorphous semiconductors, such as silicon or germanium, have also been extensively studied, due to their possible use in electronic devices [13]. Amorphous metals are exceptionally hard and can display unique magnetic properties [16]. Amorphous superconductors can also be synthesised [17], as conventional superconductivity is robust to disorder, an observation known as Anderson's theorem [18]. Remarkably, the critical superconducting temperature has been observed to be higher in several amorphous materials compared to their crystalline counterpart.

The existence and robustness of topological phases [19] poses the natural question of whether they can be realized in amorphous systems. Before reviewing how topological amorphous phases were first achieved [7–10,20] and extended, we revisit the main properties of topological phases.

Basic properties of topological matter. The discovery of the quantum Hall effect and its quantized Hall conductance [21–23] introduced the field of topological matter; phases of matter characterized by their metallic boundary states and quantized responses to external fields, which are robust against impurities and local perturbations [19]. Two states are defined to be in the same topological phase if they can be adiabatically perturbed into one another smoothly without closing the conduction gap and not breaking the underlying symmetries, while keeping the number of orbitals fixed during the process.

Translational invariance in crystal lattices allows use of Bloch's theorem to define crystal momentum, simplifying the characterization of the topological phases and yielding closed-form momentum-space expressions of the topological invariants. For example, the Chern number [24] characterizing the quantum Hall phase in two dimensions (2D) is evaluated as the integral of the Berry curvature [25] over the first Brillouin zone. In general, crystalline symmetries simplify how to identify topological phases, through the concept of symmetry indicators [26–29] —eigenvalues of point group operators whose products determine topological invariants.

Although translation invariance simplifies describing and classifying topological phases, it is not necessary for their existence. For example, strong topology is protected by local symmetries, irrespective of the lattice details. Nontrivial topology only requires the existence of a mobility gap, and not a spectral gap. However, characterizing topological phases of matter far from the crystalline limit, notably for noncrystalline lattices, requires new tools, as the known momentum space expressions for topological invariants are no longer applicable. We describe these tools and the models introduced to study amorphous topological matter next.

Theory of amorphous topological matter. – There are a variety of amorphous models displaying topological phases, ranging from strong topological states to spatial-symmetry–protected topological phases. Amorphous strong topological states include 2D Chern insulators in class A [7–10,20,30–34], 2D and 3D time-reversal invariant topological insulators in class AII [7,33,35–38], and 2D time-reversal breaking topological superconductors in class D [39,40]. Amorphous structures also support phases *a priori* protected by crystalline symmetries, such as 2D reflection-symmetry–protected topological insulators [41], 2D and 3D higher-order topological insulators [42–44], 2D and 3D obstructed insulators [45], and 3D topological metals [46]. While structural disorder is detrimental to some of these states, it can also induce nontrivial phases when starting from a trivial crystalline state [38,43,46], and it can give rise to new phenomenology intrinsically associated with amorphous topological matter and phase transitions [33,34,41,45,46].

A common starting point is a crystalline tight binding Hamiltonian known to host a topologically nontrivial phase. The hopping terms are generalized to account for arbitrary angles and distances between sites. For example the angular dependence can be modelled using the Slater-Koster parametrization [47], and the readial dependence can be accounted for by an exponential [7,33, 34,40–43,46,48] or polynomial [38] decay with the radial distance. There are several ways to introduce structural disorder, including lattices with uncorrelated random sites [7,33,34,39–43,46,48], more realistic models which preserve the local coordination number [8,30,32,45], and lattices with controllable deviations from the crystalline limit [9,10,38,43].

Characterizing topology without translational symmetry. Among the different methods to characterize topological phases far from translationally invariant limits topological markers are a wide-spread tool. Topological marker is a unifying term that includes the local markers [49–56], the spectral localizers [57–60], the nonlocal (spin) Bott indices [46,61–69], and similar generalizations of the winding of the quadrupole and octupole moment [42,43]. Markers characterizing the two-dimensional quantum Hall phase [70,71] are especially well explored, including the local Chern marker [49,50] and the nonlocal Bott index [63]. The local Chern marker [49,50] is the Fourier transform of the Chern character. For a crystalline lattice it quantizes to the Chern number at each lattice point. For non-crystalline lattices quantization requires averaging over a large enough region, where the size of the region is model dependent [50,53]. The chiral and Chern-Simons markers [53] are local markers analogous to the Chern marker in odd dimensions. The chiral marker characterizes the $\mathbb{Z}$ invariant topological phases with chiral symmetry, whilst the Chern-Simons marker characterizes

$\mathbb{Z}_2$ invariant phases with either time reversal or particle-hole symmetry, depending on the dimension.

Topological states often display a characteristic transport or electromagnetic response, such as quantized longitudinal conductance, the Hall conductivity, and the Witten effect [48,72–74], which can also be used to characterize the topological phase. The local marker in ref. [54] is for example based on the local Hall conductivity measured in the bulk of the system. Alternatively, the scattering matrix can determine topological indices without relying on the Hamiltonian eigenstates [75].

Topological phases can be detected by the presence of anomalous boundary states in the local density of states calculated with open boundary conditions [4,5]. Neural networks can also detect nontrivial topology, by efficiently learning features associated with topology from for example the wave functions [35], and the flow of the entanglement spectrum [76]. Other approaches include the effective Hamiltonian [32,77], symmetry indicators [32], and the structural spillage [78], which take advantage of the gap closing and band inversion in a topological phase transition.

Amorphous models with strong topology. The Chern insulator was the first amorphous topological phase to be characterized [7,9,10,30–32]. Reference [7] introduced a random lattice implementation of a model that displays a Chern insulator phase on a square lattice. The random lattice exhibits a gapped topological phase characterized by a nontrivial Bott index, edge states, and a quantized longitudinal conductance, which are all hallmarks of a Chern insulator, where the nontrivial phase is separated from trivial atomic insulators by bulk gap closings. There exists a similar random lattice implementation of a quantum Hall state, but in the presence of a magnetic field [79]. The three- and fourfold-coordinated Weaire-Thorpe amorphous lattices [13] with complex intrasite hoppings [32], provide a more realistic model for covalently bonded amorphous solids. The local symmetries of these models make it possible to compute symmetry indicators analogous to the ones defined for crystals [80], which predict a Chern insulator phase. Amorphous Chern insulators are also present in artificial systems, such as mechanical metamaterials [8], gyromagnetic photonic lattices [9,10,81,82], and magnetic impurities on the surface of topological insulators [30]. The Chern insulator phase also survives in an atomic liquid, defined via tight-binding molecular dynamics, which not only lacks long-range order, but has thermally moving atoms [31].

Amorphous quantum spin Hall insulators [7,36–38,76,83] are characterized by a nonzero spin Bott index and edge states carrying a quantized $2e^2/h$ conductance. Reference [7] realized a quantum spin Hall phase by placing the Bernevig-Hughes-Zhang model [84] on a random lattice. References [36,37] performed a realistic modelling of amorphous monolayer Bismuth using density functional theory, showing that the topology of the crystal survives in the amorphous structure. Based on both tight-binding and density functional theory calculations, ref. [78] showed that the amorphous Bismuth bilayer remains topological, as indicated by the structural spillage and the conductance. Reference [38] demonstrated a structural-disorder–induced quantum spin Hall phase, constructing a phase diagram as a function of spin-orbit coupling and disorder strength, by modelling the disorder by Gaussian deviations from an initial triangular lattice.

Amorphous structures also display 3D time-reversal–invariant topological insulators [7,35,48]. Reference [7] described a 3D random lattice model with exponentially decaying hoppings that, for appropriate onsite energy M and range of the hopping r_0, displays surface states. Reference [48] further characterized the r_0-M phase diagram of the same model, and found that the phase with surface states features the Witten effect —due to the axion electromagnetic term in the action, a magnetic monopole binds a half-odd integer electric charge, forming a dyon [72–74].

Finally, refs. [39,40] have reported gapped time-reversal–breaking 2D amorphous topological superconductors in class D. Reference [39] studied a Shiba glass, an ensemble of randomly distributed magnetic moments on a gapped superconducting surface with Rashba spin-orbit coupling. In contrast to the long-range pairing in this system, ref. [40] realised a topological superconductor in 2D Dirac models in random lattices with local pairing.

Spatial-symmetry–protected topological amorphous models. Amorphous systems support and induce topological phases beyond strong topological states, including systems protected by spatial symmetries. [41–43,45,46]. The appearance of these phases is related to the concept of statistical topological insulators [85–88], which are spectral insulators protected by an average symmetry. They display gapless boundary states pinned to the critical point of a topological phase transition, and protected from localization by the average symmetry.

Based on this idea, ref. [41] has classified all 2D amorphous statistical topological insulators protected by the average continuous rotation and reflection symmetries present in amorphous matter. Unlike in crystals, where reflection-symmetry–protected topological insulators display edge states only on the boundaries respecting the symmetry, their amorphous counterparts show delocalized boundary states at all edge terminations. Furthermore, they are characterized by a bulk $\mathbb{Z}_2$ topological invariant that can be defined from the effective Hamiltonian.

Higher-order topological insulators are another example of topological insulators protected by combinations of crystalline and discrete onsite symmetries, whose amorphous counterparts have also been reported [42–44].

Obstructed atomic insulators are a class of insulators that are topologically trivial, in the sense of being described by exponentially localized and symmetric wave functions, but are not adiabatically connected to the trivial atomic limit [89–95]. The simplest example is the half-filled inversion-symmetric Su-Schrieffer-Heeger chain [96]. Reference [45] suggested that phase-change materials, whose amorphous form can exhibit an average dimerization characterized by a double-peak structure in the three-particle correlation function [97], can controllably realize an obstructed amorphous phase, with flatbands of fractional charges at all terminations.

Finally, there are amorphous generalizations of Weyl semimetals, dubbed a topological amorphous metal [46]. In crystals their topological charge can be measured by the Chern number of a surface enclosing the node in momentum space [6]. Reference [46] defined the amorphous counterpart based on a known time-reversal–breaking two-band Weyl semimetal model defined on a random lattice. The topological amorphous metal is signaled by the nonzero Bott index and Hall conductivity in the planes perpendicular to the Weyl node separation in the crystal, and by the boundary states at these planes. Furthermore, in contrast to its crystalline version, the topological amorphous metal displays diffusive metallic behaviour.

Amorphous topological phase transitions. Motivated by the different nature of the disorder and of the driving parameter of the transition, refs. [33,34] numerically analyzed the quantum Hall plateau transitions in amorphous lattices. In particular, they considered both continuum 2D random geometries as well as discrete (square and triangular) 2D lattices with randomly occupied sites, as studied in percolation theory. In their models, a Chern insulator in class D appears above a critical density, dependent on the parameters of the Hamiltonian. They examined the critical scaling of both the Chern number and the conductance, as well as the conductance distribution curves. While their analysis is compatible with the standard two-parameter scaling form, the localization length critical exponent ν is highly nonuniversal. The exponents interpolate between a geometric classical percolation transition [98] and a standard Anderson localization transition [99]. While these differences with standard theory of disordered systems remain to be fully understood, it is possible that changing the density of sites introduces a variable length scale that modifies the range of the geometric correlations in the system, which are believed to affect the critical exponents of the transition [100–104].

Strongly interacting amorphous topological models. All the above phases concern amorphous but noninteracting systems. The first step towards topological amorphous many-body systems was taken by Prodan [105], who defined toric code models, which display topological order with anyonic excitations, in random triangulations. The ground state degeneracy and anyonic excitation survive amorphization, even if some commutation relations of the Hamiltonian terms are modified.

Electron-electron interactions could also lead to many-body amorphous topological phases. However, identifying these phases is challenging due to the lack of local topological markers for interacting systems. Reference [106] circumvented this issue by solving an amorphous Chern insulator model [7] with strong Hubbard interactions using a parton construction. Fractionalizing the electron into a neutral fermion f and a charged boson b leads to a mean-field phase diagram with a phase displaying protected electrically neutral chiral edge modes of f, dubbed the fractionalized amorphous Chern insulator. Recently, it was shown that the Kitaev spin-liquid is exactly solvable in a three-fold coordinated amorphous lattice [107], which survives even if the lattice is not fully amorphous [108].

Experimental status of amorphous topological matter. – Amorphous topological matter has been experimentally studied in both synthetic and solid-state systems. The first experimental observation was reported in a mechanical system of coupled gyroscopes [8]. Later on, the observation of spin-momentum locked surface states was reported in an amorphous electronic system [109], as well as topological edge states in amorphous photonic lattices [110–112].

In solid-state systems, amorphous phases of topological materials have been studied both before and after the discovery of the quantum spin Hall effect [113]. However experimental studies of amorphous materials did not address the survival of topological properties. For example, phase-change materials have been studied extensively, with $GeSb_2Te_4$ being one of the most widely studied representative [114]. Interestingly, $GeSb_2Te_4$ is also a topological insulator in its crystalline phase [115]. Amorphous Bi_2Se_3 has also been studied long before it was predicted to be a 3D topological insulator in its crystalline form [116]. Amorphous and structurally disordered counterparts of crystalline topological materials have provided materials systems that show large spin-orbit torque efficiencies, but the existence and role of topological surface states have not been explored [117,118].

Using fixed-coordination amorphous structures of coupled gyroscopes, generated from different point sets such as hyperuniform or jammed, Mitchell *et al.* [8,119] showed the existence of a mechanical amorphous Chern insulator with chiral, propogating edge modes. The authors used d.c. motors that interacted via a magnetic interaction, finding that the local connectivity, which is predictive of the global density of states, is crucial for the existence of topological states in amorphous systems. Similar findings were reported in photonic systems [110–112]. By placing an amorphous arrangement of gyromagnetic rods into a waveguide and biasing them with a magnetic field, the authors observe photonic topological edge states. Interestingly, topological states exist while the system has short-range order, and disappear at the glass-to-liquid transition [111]. Moreover, lattice disorder [112] enhances light confinement increasing the generation rate of correlated photon pairs by an order of magnitude compared to periodic topological platforms.

Regarding electronic materials, physical vapor deposition (PVD) is a particularly useful growth technique for amorphous materials and has been found to make amorphous materials which are not available by liquid quenching. PVD has several advantages since it allows to control a variety of different properties, such as the substrate temperature, growth rate (which affects the time absorbed atoms have to diffuse to ideal positions), irradiation, and chemical dopants to frustrate crystallization. Modifying the substrate temperature enables the growth of amorphous films with different local ordering and produces what is called an “ideal glass” [120,121].

Growth conditions are critical for achieving high-quality amorphous films, especially amorphous topological materials. Growing the amorphous phase of a known topological crystal does not always preserve topological properties. Several groups have grown amorphous counterparts of known crystalline topological insulators finding no evidence for topological surface states, but rather a highly insulating, localized state [122,123]. First-principles calculations indicate that the local environment plays an important role in the electronic structure in three-dimensional solids [124]. If the disorder associated with the new atomic positions (new atomic environment) closes the mobility gap, the system can be trivial. These subtleties might explain why some amorphous versions of known crystalline topological insulators do not display evidence for a topological bulk.

Focusing on electronic systems, the first demonstration of topological properties in an amorphous solid-state system was inspired by a known crystalline topological insulator, Bi_2Se_3. Using PVD, Corbae *et al.* [109] grew amorphous Bi_2Se_3 thin films with short- and medium-range order (next-to-nearest neighbours) as well as no van der Waals gap. In transport measurements, the films showed an increased bulk resistance that was largely temperature independent, and the weak-antilocalization effect resulting from quantum interference in two dimensions in the presence of spin orbit coupling. Using ARPES/SARPES the authors showed that two-dimensional surface states cross the bulk electronic gap and are spin polarized. The spin polarization switches multiple times as a function of binding energy matching the spin resolved spectral function from an amorphous topological model. These results contrast data taken on nanocrystalline samples which show a lack of disperson in ARPES and an insulating resistivity, consistent with earlier works [125]. Amorphous Bi_2Se_3 in this study possesses a local environment similar to the crystal, as seen in Raman measurements, suggesting that by preserving a similar local environment to that of the crystal the topological bulk mobility gap is not closed preserving the topological nature. In contrast, the atomic environment at grain boundaries in nanocrystalline systems is quite disordered, providing a possible explanation for the absence of topological features.

Perspective and open questions. – The growing field of topological phases in amorphous matter is an opportunity to establish a deeper understanding of topological phases and the systems that host them. In particular, the quest to define real-space topological markers and invariants to characterize topological phases is an ongoing quest. Defining topological indicators that signal non-crystalline topological metals remains an open question. Specifically, generalizations of Weyl semimetals to amorphous systems that respect time-reversal symmetry cannot be described by the Bott index or the Chern marker, and thus require the development of new tools.

An important open question is the lack of experimental evidence for solids that are both amorphous and topological, relating to the theoretical challenge of how to efficiently find them. The field would benefit from a *textbook* amorphous topological material, where topology is unambiguously confirmed by combining different experiments. However, we lack a criterion with which to establish a hierarchy of amorphous materials where to find topological phases. Currently, we draw from criteria applicable to crystals, such as large spin-orbit coupling. However, this methodology precludes reaching the major milestone of finding materials that are only topological when grown amorphous, and that are otherwise trivial crystals. A promising possibility is to integrate methods such as the structural spillage and symmetry indicators, with realistic molecular dynamics predictions based on first-principle calculations. Developing these may establish a pipeline to manufacture candidate material databases which can guide experiments.

A related open problem is the prediction of an amorphous topological superconductor beyond toy models. Such an achievement could widen the search for platforms useful for topological quantum computing. While topological superconductivity has been found by assuming a finite pairing [39,40] its appearance in a self-consistent calculation is yet to be demonstrated. Engineering the interactions to obtain a self-consistent topological ground state is a nontrivial problem since Anderson’s theorem [18] is strictly applicable only to conventional s-wave pairing.

The search for novel topological phases and phenomenology should also incorporate phenomena familiar from crystals. Amorphous topological states have for example not been fully explored in amorphous interacting [11,105–107], driven, or non-Hermitian systems [126].

In summary, amorphous solids are central to fundamental science and technology. We are confident that research in this direction will bring a deeper understanding of condensed matter, as well as novel and interesting phenomenology.

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