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LETTER

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Thermodynamic studies of a rotating polytropic black hole: Outer and interior regions

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Abstract – In this letter, considering the metric of a rotating polytropic black hole in the Boyer-Lindquist coordinates, at first, we derive the thermodynamic parameters such as entropy S, Helmholtz free energy F, internal energy U and Gibbs free energy G and study its dependence on the outer horizon by depicting suitable graphs. Then after reconstruction of the metric of the same in the Eddington-Finkelstein coordinates, we establish the interior volume of the black hole. We further analyze the variations of the interior volume with the small change of the advanced time with respect to the radius. Here we show the existence of a certain value of the radius for which this variation becomes maximum. Moreover, we show the dependence of this maximum value of the radius on the mass of the black hole. We derive the differential form of the interior volume for this limit of the radius and hence the maximal interior volume of the said black hole. Finally, we analyze the same thermodynamic parameters inside the black hole and present a comparative study between the parameters in the outer and interior regions of the black hole.

Introduction. – Simply, black holes (BHs) are defined as compact objects, formed out of the catastrophic collapse of post-main sequence supermassive stars. However, new research shows that the primordial BHs could form at times from the Planck time to 1 second after the Big Bang, or later [1]. Simply stated, the BH is a space-time region, that can be considered as the solution to Einstein's general relativity. Besides this, other modified gravities like Lovelock gravity [2], Gauss-Bonnet gravity [3], etc. also have BH solutions. The BHs produce a highly strong gravitational field. Due to this strong field, not even light can come out of the BHs. The proposal that a BH can radiate via quantum tunnelling [4,5], initiated to consider a BH as a thermodynamic system. The study of BH thermodynamics is not only one of the most important and fruitful research in the area of theoretical physics but also offers fundamental and deep insights into our understanding of the real connection between gravitation and quantum mechanics [6]. Now as a BH may evaporate over huge time spans via this process, it is quite impossible to describe a BH by its basic parameters completely and so it is always in a perturbed state. The mass of a Schwarzschild-AdS is an ever-increasing function of the radius of the event horizon. The surface gravity is a non-negative quantity associated with a BH and is treated as the BH's temperature, familiar as the Hawking temperature. The surface area of the BH is proportional to the entropy of the BH. The thermodynamic properties of the same BHs do not provide identical results in AdS and dS spaces. In AdS space, the large Schwarzschild-AdS BHs show a positive heat capacity, whereas, for small Schwarzschild-AdS BHs, it will be negative. Therefore, the large Schwarzschild-AdS BHs will be thermally stable and get cooler due to their loss of mass. On the other hand, the small Schwarzschild-AdS BHs get hotter and eventually evaporate away. In dS space, BHs are always thermodynamically stable.

The thermodynamics of various types of BHs have been studied in refs. [7,8]. Moreover, in the articles [9,10], the authors have analyzed the thermodynamic properties of different BHs applying the modified entropy. To study the effects of quantum correction of the BH thermodynamics, the Cardy formula has been used in the article [11]. Furthermore, the corrected thermodynamics has been studied under the effect of the matter field around the BHs in many articles [12,13].

The boundary of the BHs is not well specified in general relativity and the interior volume is not well defined. Moreover, it depends on the choice of a particular

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space-like hypersurface. A proper tool or method for choosing such hypersurfaces is not yet available. However, it has a special physical significance. The author in the article [14] has proposed a definition of interior volume which is independent of the choice of stationary time-slicing for stationary spacetimes. Other authors also have discussed the same in many works [15–17]. Reference [18] has proposed that the volume of the BH acts as a space-like surface with spherical nature in the interior region. For an asymptotically flat Schwarzschild BH having ADM mass M, the maximal interior volume is calculated as $V \sim 3\sqrt{3}\pi M^2 v$, where $v = t + \int \frac{1}{f(r)}\mathrm{d}r$ is advanced time and t the proper time. In the classical limit, the Schwarzschild BH remains static from the outside point of view and hence it has the fixed area $16\pi M^2$. Reference [19] has generalized the above result to the higher-dimensional charged BHs. Moreover, this result has been extended to the RN [20] and Kerr BHs [21].

The concept of "vector volume" has been explained in the article [22]. For a stationary non-degenerate BH, a definition of the volume rate has been raised in the article [23]. In the articles [24,25], the analysis of entropy of massless scalar particles inside the BHs has been established, which may provide us with implications for discussions of the information lost paradox [26] since a larger volume can accommodate huger information. Since the interior volume is always an increasing function of the advanced time v, it may be a candidate to resolve the information paradox problem. Thus the BH can have a large volume to hide the information that may be available at the end of the evaporation [20]. In this context, such a large volume may contain the entropy that may relate to the Bekenstein-Hawking entropy. Therefore, it is necessary to analyze the entropy of the BH of the hidden modes inside the BH [27]. The entropy for a Schwarzschild BH [24] and the interior entropy of a Kerr BH [28] have been investigated further.

The rotating polytropic BH solution is the newly suggested BH solution [29]. The authors have used some fundamental tools such as the Newman-Janis algorithm without complexification, the construction of unstable null orbits, etc. to construct this solution. After that, the GUP–modified Hawking radiation and transmission/reflection coefficients have been studied [30].

In this letter, we are inspired to study the rotating polytropic BHs. The thermodynamic parameters (T, S, F, U, H and G) in the outer region of the BH and the effects of logarithmic correction have been studied in many articles [9,10]. But, in the interior region, the same is still not studied deeply. This fact mainly motivates us to concentrate on the investigation of such parameters in the interior region of the said BH.

This letter is organized as follows: in the next section, we will present the metric of the rotating polytropic BHs in the Boyer-Lindquist coordinates (t, r, θ, ϕ) and will study the thermodynamic parameters (T, S, F, U, H and G) in the outer region of the BH. We will transform the BH metric in the Eddington-Finkelstein coordinates (v, r, θ, φ) and will compute the interior volume in the third section. In the fourth section, the lost mass rate of the BH and hence the maximal interior volume will be calculated. The thermodynamic parameters associated with the interior volume will also be investigated. Finally, we will conclude the letter briefly. Throughout this letter, we use the Planck units, *i.e.*, $G = c = \hbar = \kappa_B = 1$ and signature $(- + + \ldots +)$.

Rotating polytropic BH. – The metric of the polytropic BHs in the Boyer-Lindquist coordinates (t, r, θ, ϕ) can be obtained by applying the Newmann-Janis algorithm without complexification [29]. For this, one can start with a static spherically symmetric metric given as

$$\mathrm{d}s^2 = -g(r)\mathrm{d}t^2 + f(r)^{-1} + h(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \quad (1)$$

Now, introducing the advanced null coordinates (u, r, θ, ϕ) defined as $\mathrm{d}u = \mathrm{d}t - \frac{\mathrm{d}r}{\sqrt{fg}}$ and complex transformations as $r \to r + ia\cos\theta$, $r \to r - ia\cos\theta$, considering $g(r) \to X(r,\ \theta,\ a)$, $f(r) \to Y(r,\ \theta,\ a)$, $h(r) \to \psi(r,\ \theta,\ a)$, further applying the global coordinate transformations $\mathrm{d}u = \mathrm{d}t + \lambda(r)\mathrm{d}r$, $\mathrm{d}\phi = \mathrm{d}\phi + \chi(r)\mathrm{d}r$, taking

$$\begin{aligned} X(r,\theta,a) &= \frac{f(r)h(r) + a^2\cos^2\theta}{\left(\sqrt{\frac{f(r)}{g(r)}}h(r) + a^2\cos^2\theta\right)^2}\psi, \\ Y(r,\theta,a) &= \frac{f(r)h(r) + a^2\cos^2\theta}{\psi} \end{aligned} \quad (2)$$

and finally choosing $f(r) = g(r) = \frac{r^2}{l^2} - \frac{2M}{r}$, $h(r) = r^2$ and $\psi = r^2 + a^2\cos^2\theta$, the metric (1) reduces to the metric of the polytropic BHs in such coordinates as [30]

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{\mathcal{F}}{\rho_p^2}\right)\mathrm{d}t^2 + \frac{\rho_p^2}{\Delta_p}\mathrm{d}r^2 + \rho_p^2\mathrm{d}\theta^2 \\ &+ \frac{\Sigma_p}{\rho_p^2}\sin^2\theta\mathrm{d}\phi^2 - \frac{2a\mathcal{F}}{\rho_p^2}\mathrm{d}t\mathrm{d}\phi, \end{aligned} \quad (3)$$

where $\mathcal{F} = r^2(1 - \frac{r^2}{l^2} + \frac{2M}{r})$, $\rho_p^2 = r^2 + a^2\cos^2\theta$ and

$$\Delta_p = a^2 + r^2\left(\frac{r^2}{l^2} - \frac{2M}{r}\right),\ \Sigma_p = \left(r^2 + a^2\right)^2 - a^2\Delta_p\sin^2\theta. \quad (4)$$

Here, $l^2 = -\frac{3}{\Lambda}$, where l represents the AdS radius of the BH, Λ depicts the cosmological constant and a is the spin parameter which is defined as the angular momentum per unit mass of the BH (for more details follow ref. [29]).

The horizons (inner and outer) of the BH are obtained for the root $\Delta_p = 0$ which yields the outer horizon (also known as the event horizon of the BH) as

$$r_h = \frac{1}{3^{\frac{1}{3}}2^{\frac{5}{6}}}\left\{\left(\frac{6l^2M}{\mathcal{A}} - 2\mathcal{A}^2\right)^{\frac{1}{2}} + \mathcal{A}\sqrt{2}\right\}, \quad (5)$$

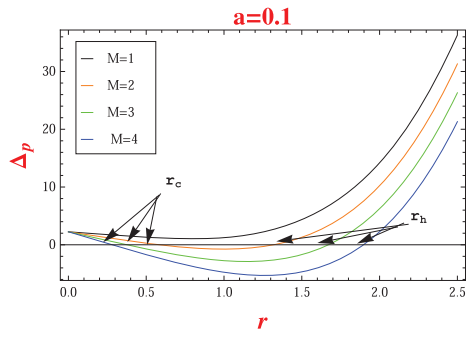

Fig. 1: Variations of the metric function Δ_p with r, keeping $l = 1$. r_c and r_h denote the BH inner Cauchy horizon and the BH outer event horizon.

where $\mathcal{A} = \sqrt{\frac{B}{2} + \frac{6^{\frac{1}{3}} a^2 l^2}{B}}$ and $B = \{9l^4M^2 + \sqrt{3}l^3 (27l^2M^4 - 16a^6)^{\frac{1}{2}}\}^{\frac{1}{3}}$.

Thus, the spin parameter is bounded as

$$a < \sqrt{3} l^{\frac{1}{3}} \left(\frac{M}{2}\right)^{\frac{2}{3}}. \tag{6}$$

It is evident from the above equation that the result differs from the result which is obtained in the case of the Kerr solution, where the constraint is computed as [29]

$$a < M. \tag{7}$$

We also investigate the inner and the outer horizon by plotting Δ_p *vs.* r curve taking different mass, shown in fig. 1. From the diagram we find the two roots of the metric function, which represent the BH inner Cauchy horizon r_c and the BH outer event horizon r_h for massive BHs [31] but when $M \leq 1$, that is for the low mass of the BHs, there is no horizon, and we have a naked singularity. This may be an interesting result for this kind of BHs. We also note that for a particular M, r_c and r_h coincide.

The Hawking temperature and the area of the BH are obtained by using the well-known formulae, namely [29,30]

$$T_h = \frac{1}{4\pi} \lim_{r \to r_h} \frac{\partial_r g_{tt}}{\sqrt{g_{tt} r_{rr}}}, \quad A_h = \int_0^{\pi} \int_0^{2\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} \mathrm{d}\theta \mathrm{d}\phi, \tag{8}$$

which exhibit

$$T_h = -\frac{a^2 l^2 - 3r_h^4}{4a^2 l^2 \pi r_h + 4l^2 \pi r_h^3}, \quad A_h = 4\pi \left(a^2 + r_h^2\right). \tag{9}$$

Thus, the Bekenstein-Hawking entropy of the said BH is also computed as

$$S_h = \frac{A_h}{4} = \pi \left(a^2 + r_h^2\right). \tag{10}$$

Moreover, the thermodynamic parameters ($F_h = -\int S_h \mathrm{d}T_h, U_h = F_h + T_h S_h, H_h = U_h + PV$ and $G_h = H_h - T_h S_h$), in the outer region of the BH are calculated as

$$F_h = -\frac{2a\left(l^2 - 3a^2\right)\tan^{-1}\left(\frac{r_h}{a}\right) - \frac{a^2 l^2}{r_h} + 6a^2 r_h + r_h^3}{4l^2},$$

$$U_h = \frac{1}{2l^2}\left[a\left(3a^2 - l^2\right)\tan^{-1}\left(\frac{r_h}{a}\right) - 3a^2 r_h + r_h^3\right],$$

$$H_h = \frac{1}{4l^2}\left[2\left\{a\left(3a^2 - l^2\right)\tan^{-1}\left(\frac{r_h}{a}\right) - 3a^2 r_h + r_h^3\right\} - \frac{\left(2a^2 + r_h^2\right)\left(a^4 l^2 + 3a^2 l^2 r_h^2 + 9a^2 r_h^4 + 3r_h^6\right)}{2a^4 r_h - 3a^2 r_h^3 - 3r_h^5}\right],$$

where

$$V = \frac{2\pi}{3r_h}\left(a^2 + r_h^2\right)\left(2a^2 + r_h^2\right)$$

and

$$P = -\frac{\mathrm{d}F_h}{\mathrm{d}V} = -\frac{3\left(a^4 l^2 + 3a^2 l^2 r_h^2 + 9a^2 r_h^4 + 3r_h^6\right)}{8l^2\pi\left(a^2 + r_h^2\right)\left(2a^4 - 3r_h^2\left(a^2 + r_h^2\right)\right)}$$

represent the volume and pressure of the BH and

$$G_h = \frac{a}{2l^2}\left[\left(3a^2 - l^2\right)\tan^{-1}\left(\frac{r_h}{a}\right) + \frac{r_h\left\{6a^5 + 5a^3 l^2 + a r_h^2\left(a^2 - 3r_h^2 + 3l^2\right)\right\}}{3r_h^2\left(a^2 + r_h^2\right) - 2a^4}\right]. \tag{11}$$

The heat capacity C_h [32,33] in the outer region of the BH is calculated as

$$C_h = \frac{\partial M_h}{\partial T_h} = -\frac{2\pi\left(a^2 + r_h^2\right)^2\left(a^2 l^2 - 3r_h^4\right)}{a^4 l^2 + 3a^2 l^2 r_h^2 + 9a^2 r_h^4 + 3r_h^6}. \tag{12}$$

The dependence of the thermodynamic parameters (S_h, F_h, U_h, H_h and G_h) on the outer horizon r_h are depicted in the diagrams in figs. 2(a)–(e), while the value of l is taken as unity. Here we observe that the variations of the parameters S_h and U_h with r_h are of similar nature *i.e.*, increasing function of the outer horizon r_h. This is obvious. H_h *vs.* r_h curve shows that for very small r_h, H_h is negative, reaches a local minimum then increases sharply, reaches a local maximum at $r_h = r_{hc}$ as r_h increases. If we increase r_h further we observe that H_h decreases rapidly and then increases with comparatively slow rate. The variations of F_h and G_h show the BH supports a mandatory second-order phase transition. When we vary the heat capacity C_h of the BH with r_h, we note that it is always positive and increases as r_h increases (fig. 2(f)). This means that the BH is thermodynamically stable. When we study the variations of all the thermodynamic parameters (T, C, S, U, G and F) with r_h according to the changes of the parameters (a and l) for possible and specific positive and negative values, we find graphs of the same nature as shown in figs. 2(a)–(f). When we study the same for Kerr BH, we observe that S_h varies with r_h in the same way as the polytropic BH. Moreover, the changes of U_h and H_h with r_h are similar and are increasing for large r_h but at low horizon the nature is

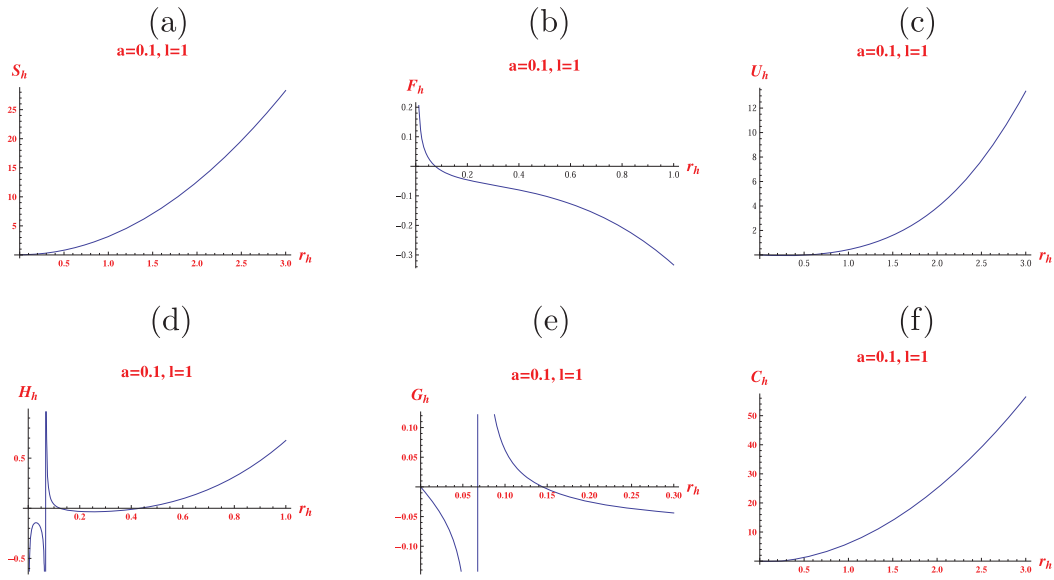

Fig. 2: (a)–(f) Variations of S_h, F_h, U_h, H_h, G_h and C_h with r_h, keeping $l = 1$.

quite different, the variations of F_h and G_h are also similar for Kerr BH, *i.e.*, both the parameters become positive from their negative value as r_h increases which is different from the polytropic BH. For Kerr BH C_h is always negative, *i.e.*, unlike the polytropic BH the Kerr BH is thermodynamically unstable. For Schwarzschild BH the entropy is an ever increasing function of r_h as usual, the other thermodynamic parameters (F_h, U_h, H_h, G_h) are linearly proportional to r_h. Therefore those are always positive and increase linearly due to increment of r_h. But the C_h is negative and it varies with r_h like Kerr BH.

At the limit $\Lambda \to 0$, *i.e.*, $l \to \infty$, the temperature of the BH (9) becomes negative, which is unphysical. Thus all the thermodynamic parameters (S_h, F_h, U_h, H_h and G_h) even the heat capacity C_h, obtained in eqs. (11) and (12) will be also unphysical at this limit.

Interior volume. – Using the coordinate change

$$\mathrm{d}t = \mathrm{d}v - \frac{r^2 + a^2}{\Delta_p}\mathrm{d}r \quad \text{and} \quad \mathrm{d}\phi = \mathrm{d}\varphi - \frac{a}{\Delta_p}\mathrm{d}r, \tag{13}$$

the metric (3), in the Eddington-Finkelstein coordinates (v, r, θ, φ) is described as

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathcal{F}}{\rho_p^2}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + \rho_p^2\mathrm{d}\theta^2 + \frac{\Sigma_p}{\rho_p^2}\sin^2\theta\mathrm{d}\varphi^2 - 2a\sin^2\theta\mathrm{d}r\mathrm{d}\varphi - \frac{2a\mathcal{F}}{\rho_p^2}\mathrm{d}v\mathrm{d}\varphi, \tag{14}$$

where v and φ in the line element (14) represent the advanced time and azimuthal angle, respectively.

In this section, we have derived the maximal interior volume of the rotating polytropic BH. Let us choose an arbitrary vector on the hypersurface. This vector can be divided into two parts —one is normal and the other is tangent to the hypersurface given as $\tau^a = \xi l^a + \xi^a$, where ξ is the lapse function and ξ^a is the shift-vector. The covector on the hypersurface is defined as $l_a = -\xi\Delta_a r \equiv -\xi(\mathrm{d}r)_a$, where $\Delta_a r \equiv (\mathrm{d}r)_a$ is the normal covector. For the space-like hypersurface, one can use the property of the normal vector as $l_a l^a = \xi^2 g^{ab}(\mathrm{d}r)_a(\mathrm{d}r)_b = -1$. Using the above equations, we get $g^{rr} = -\frac{1}{\xi^2}$

Now, from the definition, the induced metric on the hypersurface at constant r is computed as

$$s = \xi^{-2}(-g) = g^{rr}g = \sqrt{-\Delta_p}\rho^2\sin^2\theta. \tag{15}$$

We choose ϑ as the induced volume on the hypersurface. From the metric (14), the volume of an arbitrary hypersurface at constant r may be obtained as

$$\mathcal{V} \sim \int \vartheta = 2\pi\sqrt{2Mr - \frac{r^4}{l^2} - a^2} \times \left[\sqrt{a^2 + r^2} + \frac{r^2}{2a}\ln\left(\frac{\sqrt{a^2 + r^2} + a}{\sqrt{a^2 + r^2} - a}\right)\right] v. \tag{16}$$

This result is considered as a maximal value of the volume when r is taken as $r = r_{max}$, the largest hypersurface inside the BH. Thus, this maximal value is assumed as the interior volume of the polytropic BH.

The rate of change of the interior volume of the BH $\mathcal{V}$ with the advanced time v is calculated as

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}v} \sim 2\pi\sqrt{2Mr - \frac{r^4}{l^2} - a^2} \times \left[\sqrt{a^2 + r^2} + \frac{r^2}{2a}\ln\left(\frac{\sqrt{a^2 + r^2} + a}{\sqrt{a^2 + r^2} - a}\right)\right]. \tag{17}$$

It is worth noting that the above result (17) will be non-negative for $2Mr > \frac{r^4}{l^2} + a^2$. This means that the interior

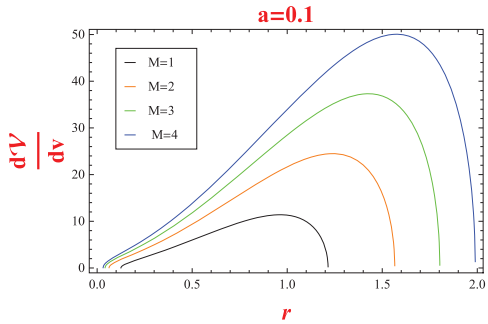

Fig. 3: Variations of $\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}v}$ with r, keeping $l = 1$.

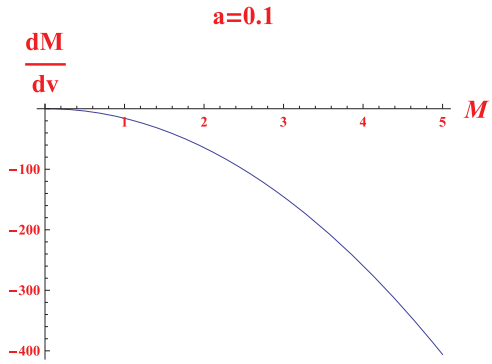

Fig. 4: Variations of $\frac{\mathrm{d}M}{\mathrm{d}v}$ with M, keeping $l = 1$ and $\gamma = 0.01$.

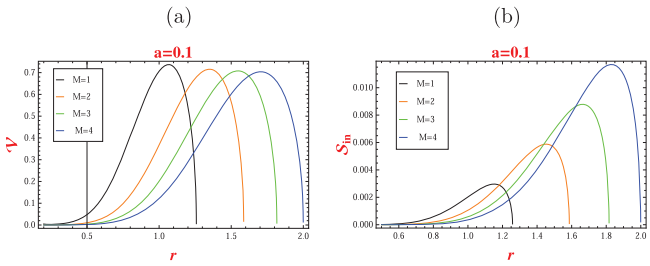

Fig. 5: (a), (b): variations of $\mathcal{V}$ and $\mathcal{S}_{in}$ with r, keeping $l = 1$ and $\gamma = 0.01$.

volume of the said BH will be increased only when the mass of the BH is large enough such as Kerr BH [21]. In the differential form of eq. (16), M may be considered as a constant and v is taken as a variable.

From fig. 3, we observe that there is a maximal value of the rate of increment for a certain value of r, keeping M constant as we assumed earlier. We also show that if M is increased, the rate of increment is also increased and its position shifts towards the larger radius zone. Initially, the increment in mass increases the radius of the outer horizon and hence the interior volume. Besides these, due to the increment in mass, the gravitational attraction power of the BH increases. Again, the gravitational attraction leads to further increment in volume which increases the rate of increment of mass. When r exceeds the critical value, $\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}v}$ falls abruptly and becomes zero for a certain value of r at constant mass. We may interpret this behavior as, when the BH grows bigger, the rate of increment of the interior volume increases, reaches a maximal for a particular volume, and with further growth, this rate falls abruptly and becomes zero for a particular mass. This means for a fixed mass, there is a certain limit to the interior volume of the BH. This volume is nothing but the maximal interior volume of the concerned BH.

Thermodynamic studies at the interior of the BH. – According to the classical point of view, the interior volume of the BHs increases with advanced time v. The thermodynamic parameters inside the BH are different from those of the volume of the same on which an outside observer is organized. In this region of the BH, the statistical quantities of the quantum field may be influenced by the special character of the interior volume and this may be useful to solve an important puzzle, *viz.*, the information lost paradox of a BH. Thus, it is important to study how the special character of the internal volume affects the statistical quantities of the quantum fields.

The number of quantum states with energy less than E is expressed as [19]

$$g(E) = \frac{E^3\mathcal{V}}{12\pi^2}, \tag{18}$$

and ignoring the exotic features of the interior volume temporarily, the Helmholtz free energy and the pressure are computed as [19]

$$F_{in} = -\frac{\pi^2\mathcal{V}}{180}T_h^4 \qquad \text{and} \qquad P_{in} = -\left(\frac{\partial F}{\partial \mathcal{V}}\right)_{T_h}. \tag{19}$$

(For a detailed discussion see ref. [28].)

The measure of the entropy of a BH in its interior region is based on mainly two postulates [24,34], i) the Hawking radiation from a BH may be assumed as the emission from a black body and hence, the temperature of the BH at the outer horizon may be taken and ii) the evaporation process from the BH is assumed as slow as occurred within the quasi-static process. Thus the thermal equilibrium may be established between the outer horizon and the scalar field interior of the BH. The first assumption says the temperature of the event horizon is nothing but the Hawking temperature of the BH, whereas the second assumption says when the thermal equilibrium in an infinitesimal process, the temperature of both the scalar field and the horizon will be the same. So, the temperature of the scalar field inside the BH may be taken as the Hawking temperature.

Due to the first assumption, the lost mass rate of the BH can be derived by the Stefan-Boltzmann law [35] as

$$\frac{\mathrm{d}M}{\mathrm{d}v} = -\frac{1}{\gamma}T_h^4 A_h, \quad \gamma > 0, \tag{20}$$

where γ is a positive constant. It depends on the number of quantized matter fields coupling with gravity. The lost mass rate for this BH is obtained by applying eqs. (9) and (11) in eq. (20). Here, the mass of the BH does not remain constant, rather it changes with the advanced time.

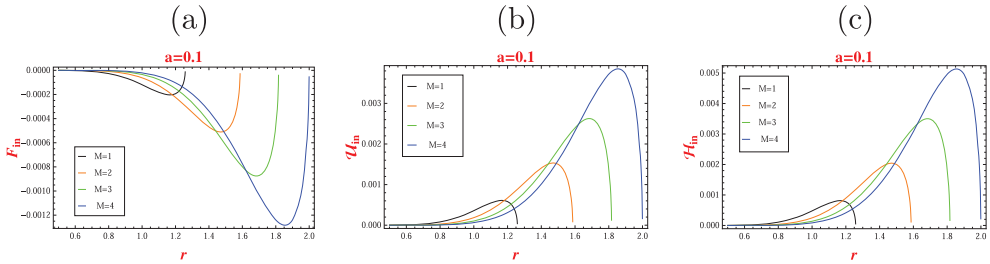

Fig. 6: (a)–(c) Variations of F_{in},$\mathcal{U}_{in}$ and $\mathcal{H}_{in}$ with r, keeping $l = 1$ and $\gamma = 0.01$.

In the diagram in fig. 4, we notice that this rate decreases as M increases. This means with an increasing mass, the evaporation rate of the BH also increases as expected.

To calculate the change in the entropy of the scalar field in the BH evaporation process, generally, one can apply two methods: i) integral method and ii) the equilibrium statics method. Since both methods are very complicated, here we apply the differential form directly.

According to this statement, the differential form of the maximal interior volume ($r = r_h = r_{max}$) is obtained from eqs. (17) and (20) and integrating this one we can compute the maximal interior volume of the BH as

$$\mathcal{V} = -\left[\frac{8\gamma l^6\pi^4\left(a^2+r_{max}^2\right)^8\sqrt{2Mr_{max}-a^2-\frac{r_{max}^4}{l^2}}}{3\left(a^2-r_{max}^2\right)\left\{2a^2\ln\left(r_{max}\right)+r_{max}^2\right\}}\right]$$
$$\times\left[\frac{\sqrt{a^2+r_{max}^2}+\frac{r_{max}^2}{2a}\ln\left\{\frac{2a\left(\sqrt{a^2+r_{max}^2}+a\right)+r_{max}^2}{r_{max}^2}\right\}}{\left\{l^2Mr_{max}^2+r_{max}^5-a^2\left\{l^2\left(r_{max}+M\right)-2r_{max}^3\right\}\right\}^3}\right]. \tag{21}$$

The entropy inside the BH is obtained as

$$\mathcal{S}_{in} = -\frac{\partial F}{\partial T_h} = \frac{\pi^2\mathcal{V}}{45}T_h^3. \tag{22}$$

Substituting eqs. (9) and (21) into eq. (22), we have the interior entropy of the BH associated with the maximal interior volume.

From fig. 5(a), it is clear that there is a certain value of the radius (here we define it as $r = r_{max}$) of the BH for which the interior volume will be maximal whatever the mass of the BH. The variations of the entropy inside the BH with the radius r are shown in fig. 5(b). Here we find that the peak value of $\mathcal{S}_{in}$ is increased and shifts towards the high radius region as M increases. This surprising behavior of $\mathcal{S}_{in}$ is because, in such BHs, the maximal interior volume exists for $r = r_{max}$ and beyond this limit, the interior volume shrinks.

The other thermodynamic parameters ($\mathcal{U}_{in}, \mathcal{H}_{in}$ and $\mathcal{G}_{in}$) in this region of the BH are computed by applying eqs. (19) and (22) as [19]

$$\mathcal{U}_{in} = \frac{\pi^2\mathcal{V}}{60}T_h^4, \quad \mathcal{H}_{in} = \frac{\pi^2\mathcal{V}}{45}T_h^4, \quad \mathcal{G}_{in} = 0. \tag{23}$$

It is noticed from eqs. (19) and (23) that at the maximal interior volume of the BH, all the thermodynamic parameters are proportional to the fourth power of the Hawking temperature of the BH.

Now we substitute the values of T_h and $\mathcal{V}$ in eqs. (18) and (23) in the maximal limit and study the variations of the thermodynamic parameters (F_{in},$\mathcal{U}_{in}$ and $\mathcal{H}_{in}$) with the radius r of the BH.

If we investigate the thermodynamic parameters (F_{in},$\mathcal{U}_{in}$ and $\mathcal{H}_{in}$) inside the BH, we observe that due to the increment of the radius, initially, F_{in} decreases, and reaches a minimum. Upon further increment of r, F_{in} increases sharply and becomes zero (fig. 6(a)). The reverse nature is shown for $\mathcal{U}_{in}$ *vs.* r curves and $\mathcal{H}_{in}$ *vs.* r curves (figs. 6(b), (c)). This peculiar behavior of the curves is due to the existence of the maximal interior volume of the BH at a particular value of r.

Conclusions. – In this letter, we consider the metric of a rotating polytropic BH solution in the Boyer-Lindquist coordinates and investigate the thermodynamic parameters such as entropy, Helmholtz free energy, internal energy and Gibbs free energy at the outer horizon of the BH. In this observation, we find that S_h, U_h and H_h change with r_h in similar fashion at large event horizon, *i.e.*, these parameters increase as r_h increases (figs. 2(a), (b) and (d)). The variations of F_h and G_h with r_h show that the BH supports a mandatory second-order phase transition. The heat capacity of the BH at the outer horizon is also found positive and increasing function of r_h (fig. 2(f)), *i.e.*, the BH is thermodynamically stable. When analyzing the inner and the outer horizon of the BHs in more detail by depicting the variations of the metric function Δ_p with respect to r taking different mass (fig. 1) an interesting result is found, *i.e.*, for smaller BHs there is no horizon, and we have a naked singularity. We also note that the temperature of the BH at the outer horizon becomes negative at the limit $\Lambda \to 0$, *i.e.*, $l \to \infty$ and therefore, the discussion of all the thermodynamic parameters including the heat capacity of this kind of BHs at the horizon has no meaning at the same.

To study the same inside the BH, we reconstruct the metric in the Eddington-Finkelstein coordinates by applying the suitable coordinate transformations (13). At first, we try to calculate the interior volume trapped inside the event horizon of the BH by introducing advanced time.

For a particular radius and a constant mass, the maximal interior volume is proportional to the advanced time v. If we include the massless scalar field inside the BH, the number of quantum states of it is proportional to the interior volume. Thus, due to the increment of the advanced time, when the interior volume of the BH increases the number of quantum states also increases. It would be a sign to think about the BH information lost paradox [26]. We may conclude from fig. 3 that when the BH is large enough with a fixed mass, the interior volume does not change with advanced time. This volume is actually the maximal interior volume of the concerned BH. Figure 4 signifies that the massive BH evaporates at a high rate. We find that $\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}v}$ is non-negative for $2Mr > \frac{r^4}{l^2} + a^2$, which signifies that the interior volume of the said BH will be increased only when the mass of the BH is large enough such as Kerr BH [21].

The variations of the entropy inside the BH with the radius r show that, at $r = r_{max}$, it reaches a maximum and this maximum value increases with the increment of the mass of the BH. Beyond this limit, *i.e.*, $r > r_{max}$, $\mathcal{S}_{in}$ began to decrease and become almost zero. It is because, in such BHs, the maximal interior volume exists for $r = r_{max}$ and beyond this limit, the interior volume starts to shrink. The thermodynamic parameters (F_{in},$\mathcal{U}_{in}$ and $\mathcal{H}_{in}$) inside the BH also obey Stefan's law for a fixed interior volume. We observe that due to the increase in the span of the space-time confined in a BH's event horizon, more amount of internal energy is trapped inside it.

Though the metric of the rotating polytropic BH is described in Kerr coordinates, this metric solution contains an important parameter called the AdS radius of the BH, which is related to the cosmological constant as well as the thermodynamic pressure. Therefore, the study of thermodynamic parameters with the presence of this parameter may be significant as compared to studying the same for the Kerr BH and Schwarzschild BH. In the future, one may be motivated to extend this work in the light of logarithmic corrections (first-order as well as higher correction term) for the rotating polytropic BH.

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