A new hypernetted-chain treatment for Laughlin quantum Hall states

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A new hypernetted-chain treatment for Laughlin quantum Hall states

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Abstract. – The hypernetted-chain theory is applied to study the fractional quantum Hall effect with the Laughlin wave functions A new method is proposed to include the effect of the elementary diagrams, which improves upon the commonly used modified hypernetted-chain approximation The correlation energy, the pair distribution function, as well as the magnetoroton excitation spectrum have been computed within this method The results obtained are in very good agreement with the available Monte Carlo estimates The method is generalizable to treat other wave functions, like those corresponding to the hierarchical states or those of composite fermion type

The fractional quantum Hall effect (FQHE) [1] is one of the most remarkable many-body phenomena discovered in recent years [2]-[4] It occurs in a two-dimensional electronic system in the extreme quantum limit of strong perpendicular magnetic field ($B > 5\,\mathrm{T}$), low temperature ($T < 2\,\mathrm{K}$) and high mobility of electrons ($\mu > 10^5\ \mathrm{cm}^2/\mathrm{Vs}$) The magnetic field is so strong that mixing of Landau levels by disorder or by electron-electron interaction is a very weak perturbation Due to the absence of an energy scale, like the bandwidth of a periodic solid or the Fermi energy of an electron liquid in the absence of a magnetic field, the Coulomb interaction induces such strong correlations amongst the electrons that cannot be accounted for in a perturbative way. It has been shown that these correlations are very well approximated by *Jastrow* factors, similarly to the case of liquid ^{4}He and ^{3}He

Integral equation techniques, such as hypernetted chain (HNC) for bosons or Fermi hypernetted chain (FHNC) for fermions, allow for realistic evaluations of the distribution functions and related quantities for Jastrow correlated wave functions In fact, they are particularly useful when calculations must be performed strictly in the thermodynamic limit. They have been extensively and successfully used in the study of quantum liquids. HNC theory has also

been adopted in the study of the FQHE, to evaluate the pair distribution function for the variational wave function proposed by Laughlin [5]

$$\psi_m = \prod_{j<k}^{N} (z_j - z_k)^m \prod_{j=1}^{N} \exp\left[-|z_j|^2/4l_0^2\right], \tag{1}$$

where m is an odd number. Here z_j is the complex coordinate $z_j = x_j + iy_j$ and $l_0 = \sqrt{\hbar/eB}$ is the magnetic length This wave function describes a translationally invariant incompressible liquid at a density $\rho = 1/(2\pi m l_0^2)$, corresponding to a lowest Landau-level filling factor $\nu = 1/m$

It is well known that the (F)HNC techniques are intrinsically approximated because there is a set of cluster terms (corresponding to the so-called *elementary diagrams*) which cannot be fully included in any closed form. Whereas the approximation of totally neglecting these terms ((F)HNC/0) leads to reliable results for the ground-state energies, realistic evaluations of other quantities, like for instance the pair distribution function or the magnetoroton spectrum, require better approximations Therefore, it is important to find numerical procedures to include efficiently the main contributions from the elementary diagrams

In this paper we propose a new *scaling* procedure for the inclusion of elementary diagrams in (F)HNC calculations to be used in the study of the FQHE, and we apply it to the case of the Laughlin wave function of eq (1)

The modulus square of the wave function can be viewed as a product of a Jastrow factor times a single-particle term, namely

$$|\Psi|^2 = \exp\left[-\sum_{i<j=1}^{N} u(r_{ij})\right] \prod_{i=1}^{N} \phi^2(r_i)\,, \tag{2}$$

where

$$u(r) = -2m \ln(r)\,, \tag{3}$$

and

$$\phi^2(r) \equiv \rho_0(r) = \frac{\exp\left[-r^2/2l_0^2\right]}{2\pi l_0^2} \tag{4}$$

Using the HNC theory, one can express the pair function $g(r)$ as a series of cluster terms, associated with linked diagrams The difference with respect to the case of the standard Jastrow wave function, in which the single-particle term is not present, is that the diagrams are not irreducible and each vertex brings the *uncorrelated* one-body density $\rho_0(r)$ as a vertex correction It has been proved [6] that such a series can be recast into a series of irreducible diagrams with the full one-body density $\rho(r)$ being the vertex correction Since the full density is a constant, then the HNC equations for the pair function are exactly the same as for a Jastrow wave function without the single-particle term and at density $\rho = 1/(2\pi m l_0^2)$.

The pair distribution function is expressed in terms of the functions $N(r_{ij})$ and $E(r_{ij})$, which give the sum of the *chain* and *elementary* diagrams, respectively, namely

$$g(r_{ij}) = \exp[-u(r_{ij}) + N(r_{ij}) + E(r_{ij})]\,, \tag{5}$$

where the function $N(r)$ satisfies the following convolution equation.

$$N(r_{ij}) = \rho \int \mathrm{d}^2 r_k \; X(r_{ik}) \; (X(r_{kj}) + N(r_{kj}))\,, \tag{6}$$

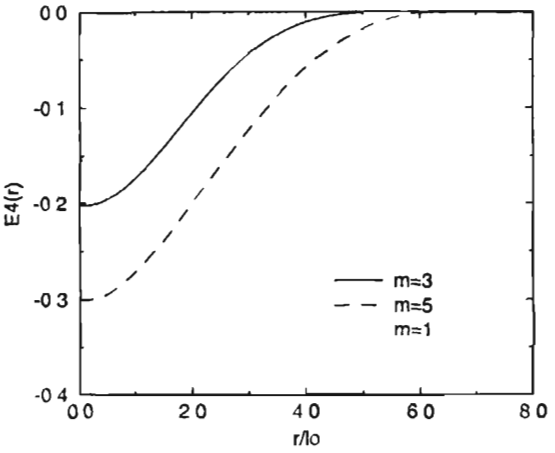

Fig 1

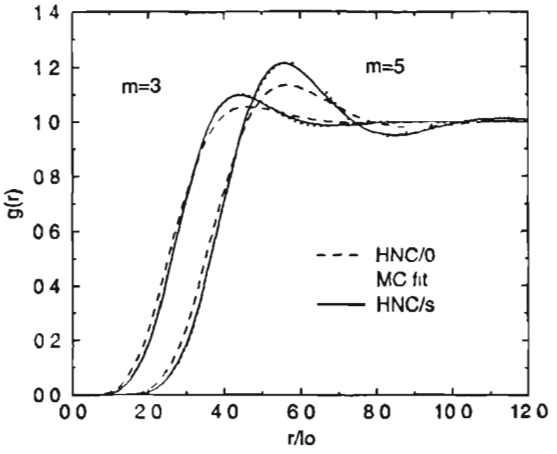

Fig 2

Fig 1 – Lowest-order elementary function $E_4(r)$ for states $\nu = \frac{1}{3}$ (solid) and $\nu = \frac{1}{5}$ (dashed) compared to the exact $E(r)$ for the state $\nu = 1$ (dotted)

Fig 2 – Comparison of the pair distribution function $g(r)$ obtained with HNC/0 (dashed), HNC/s (solid) and the best MC data fitting, after Girvin *et al* ref [10] (dotted), for the Laughlin states $\nu = \frac{1}{3}$ and $\nu = \frac{1}{5}$

and

$$X(r_{ij}) = g(r_{ij}) - 1 - N(r_{ij}) \tag{7}$$

is the function which sums up all the *composite* diagrams. The function $E(r)$ is not given in a closed form, like $N(r)$, and is in general approximated The HNC/0 approximation neglects $E(r)$ completely The HNC/4 approximation includes the elementary diagrams of the simplest structure, namely the four-point elementary diagram $E_4(r)$ Higher-order approximations include the five-point elementary structures (HNC/5), and so on It is known that the series HNC/0, HNC/4, . , converges very slowly to the exact result. It is also known that the various elementary structures $E_4(r)$, $E_5(r)$, ..., roughly scale with each other Based on this property, the scaling approximation

$$E(r) = \alpha E_4(r) \tag{8}$$

has been successfully used [7] in variational calculations on liquid ^{4}He

In this paper we apply this approximation to calculate the pair function, the energy per particle and the magnetoroton spectrum for the Laughlin wave function given in eq. (1). For a given filling factor $1/m$ we solve the HNC equations in HNC/0 approximation Then we

TABLE I – *Energies per particle in units of* $e^2/\epsilon l_0$

m	HNC/0	HNC/4	HNC/s	MHNC	MC
3	−0 4055	−0 4063	−0 4100	−0 4156	−0 410
5	−0 3240	−0 3247	−0 3274	−0 3340	−0 3277

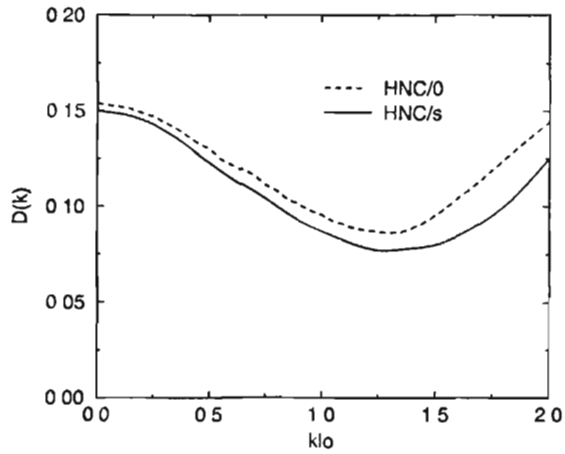

Fig 3 – The SMA collective excitation gap $D(k)$ in units of $e^2/\epsilon l_0$ obtained using the HNC/0 and HNC/s pair distribution function $g(r)$ for the Laughlin state $\nu = \frac{1}{3}$

compute the four-point elementary diagram

$$E_4(r_{ij}) = \frac{\rho^2}{2} \int [h(r_{ik})h(r_{il})h(r_{jk})h(r_{jl})h(r_{kl})] \mathrm{d}^2 r_k \mathrm{d}^2 r_l, \tag{9}$$

where $h(r) = g(r) - 1$. From the correspondence [5] of $|\Psi|^2$ with the partition function of a *charge neutral* two-dimensional plasma, it follows that [8] HNC/0 theory automatically satifies the *charge neutrality* and the *perfect screening* sum rules, but violates the *compressibility* sum rule. Therefore, we fix the scaling parameter α by imposing the *compressibility* sum rule [8]

$$\rho \int \mathrm{d}^2 r \; r^4 (g(r) - 1) = -16 l_0^4 (1 - m/2) \tag{10}$$

With the estimated $E(r)$ we solve again the HNC equation for a new $g(r)$ and we iterate the process until convergence is reached

In all the cases the pair function satisfies almost exactly the *charge neutrality* and the *perfect screening* sum rules as in the first treatment

This approximation improves significantly upon that used by Laughlin [5] in his original calculations, named Modified HNC (MHNC) [8] MHNC approximation assumes that, for any given filling factor, $E(r)$ scales with respect to the corresponding function $E^{(m=1)}(r)$ for filling one, which is exactly known. In fact the assumption that the shape of $E(r)$ does not change significantly with the filling factor is not fully justified, as shown in fig 1 The results obtained for the energy per particle for $m = 3, 5$ are given in table I Our scaling approximation (HNC/s) is compared with HNC/0, HNC/4 and with the MHNC results of ref [5] and the Monte Carlo (MC) results of ref [9] One can see that the HNC/s results are in much better agreement with the MC ones than the other approximation schemes.

The values found for the scaling coefficient α, 6 25 for filling 1/3 and 5 2 for filling 1/5 are quite large, consistently with the fact that HNC/4 is a rather poor approximation MHNC approximation gets worse for higher values of m

In fig. 2 we plot the pair distribution functions obtained for $\nu = 1/3$ and $\nu = 1/5$ by using HNC/0 and HNC/s techniques together with that given in ref. [10] which fits the Monte Carlo data. The agreement between the HNC/s pair function and the MC one is impressive

especially in the small-r regime The r^2, r^4 and r^6 coefficients of $g(r)$ at small r compare very well with the MC results of ref. [10]

We also investigated the collective excitations in the single-mode approximation (SMA), using the wave function [10], [11]

$$\Phi_{\mathbf{k}}^{(m)} = \bar{\rho}_{\mathbf{k}} \Psi_m, \tag{11}$$

where $\bar{\rho}_{\mathbf{k}}$ is the projection of the density operator $\rho_{\mathbf{k}} = \sum_j \exp[i\mathbf{k} \cdot \mathbf{r}_j]$ onto the subspace of the lowest Landau level In fig 3 we plot the excitation energies $D(k)$ obtained with both HNC/s and HNC/0 at filling $\nu = 1/3$ One can see that the effect of the elementary diagrams is quite sizeable for such a quantity, and HNC/0 approximation gives a higher gap Similar results are found also at filling $\nu = 1/5$

We have presented a new scheme to solve the HNC equations for the Laughlin wave function in the FQHE. This is based on the assumption that the sum of all the elementary diagrams scales with respect to the simplest four-point elementary diagram E_4 This scheme provides better results than the largely used MHNC approximation for the energy per particle, the pair distribution function and magnetoroton spectrum Our method can be easily generalized to multicomponent HNC or to FHNC and, therefore, can be applied to calculations with other variational wave functions, like for instance the hierarchical wave functions [12], [13] or the composite fermion wave function [14] Work in this direction is in progress.

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