Geometrical exponents in the integer quantum Hall effect

To cite this article: I. Bratberg *et al* 1997 *EPL* **37** 19

EUROPHYSICS LETTERS 1 January 1997

Europhys. Lett., **37** (1), pp. 19-24 (1997)

Geometrical exponents in the integer quantum Hall effect

I. Bratberg, A. Hansen and E. H. Hauge

Institutt for Fysikk, Norges Teknisk-naturvitenskapelige Universitet
N-7034 Trondheim, Norway

(received 6 December 1995; accepted in final form 18 November 1996)

PACS. 73.40Hm– Quantum Hall effect (including fractional).
PACS. 05.30−d – Quantum statistical mechanics.
PACS. 73.50Jt – Galvanomagnetic and other magnetotransport effects (including thermomagnetic effects).

Abstract. – We point out that the extended Chalker-Coddington model in the "classical" limit, *i.e.* the limit of large disorder, shows crossover to the so-called "smart kinetic walks". The reason why this limit has previously been identified with ordinary percolation is, presumably, that the localization length exponents ν coincide for the two problems. Other exponents, like the fractal dimension D, differ. This gives an opportunity to test the consistency of the semiclassical picture of the localization-delocalization transitions in the integer quantum Hall effect. We calculate numerically, using the extended Chalker-Coddington model, two exponents τ and D that characterize critical properties of the geometry of the wave function at these transitions. We find that the exponents, within our precision, are equal to those of two-dimensional percolation, as predicted by the semiclassical picture.

Fifteen years after its discorvery [1], the quantum Hall effect [2] continues to generate a large body of theoretical work. A sizable subset of this work concerns the scaling properties of the localization-delocalization transitions that occur between subsequent plateaus in the Hall conductance [3], [4]. These scaling properties are usually characterized by a small number of critical exponents. Emphasis has been put on the exponent ν, which is the exponent governing the divergence of the localization length ξ. It is believed to be universal. Koch *et al.* [5] measured it experimentally to be $\nu = 2.3 \pm 0.1$, while Wei *et al.* [6] found the value $\nu = 2.4 \pm 0.2$. This latter value was not measured directly, however, but relies on assumed values of other exponents. More recently, Dolgopolov *et al.* [7] have found $\nu \approx 1$. This value is based on gradient percolation arguments equivalent to those of Sapoval *et al.* [8], which in fact are incompatible with the original percolation picture of the integer quantum Hall effect [9], [10]. Attempts at deriving ν from field theory have so far been unsuccessful, but controversial semiclassical scaling arguments exist which suggest that $\nu = \nu_{\mathrm{p}} + 1 = 7/3$, where $\nu_{\mathrm{p}} = 4/3$ is the classical percolation correlation length exponent [11], [12]. Several numerical simulations have been performed to determine ν. Models based directly on continuum descriptions of the problem, using long-wavelength Gaussian potentials or short-range impurity potentials, result in $\nu \approx 2.3$–2.4 [13]-[18]. Network models, originally introduced by Chalker and Coddington [19], give a similar value for ν [20], [21].

We address here the controversial question whether the agreement between the value of ν found from the semiclassical picture, and that determined both numerically and experimentally (with exceptions), is fortuitous. We do this by recalling that the semiclassical picture, by a very direct argument [12], provides predictions for *other* exponents, beyond ν. We test these predictions numerically by simulations on the extended Chalker-Coddington model.

The semiclassical picture is based on viewing the localization-delocalization transition in the Hall system as a percolation transition [9], [10]. It assumes the disordered potential V in which the electrons move, to be slowly varying on the scale of the magnetic length $l = \sqrt{\hbar/eB}$, where B is the magnetic field ([1]). This assumption constitutes the high-field model. Interactions between electrons are ignored. In this approximation, the electron wave function ψ is localized to equipotential curves, forming "ribbons" of width l. Changing the magnetic field B results in the localized wave function moving up or down in this landscape. Typically, the ribbons constituting the wave function are restricted to encircle mountains or valleys in the energy landscape. However, scattering between closed ribbons is possible when they overlap at saddle points. There is a typical distance a between saddle points. In a window ΔB centered at a magnetic field B_c, a sample-spanning cluster of ribbons overlapping at saddle points exists, resulting in delocalization. The width of the window ΔB is determined by the magnetic length l and decreases with increasing sample width W—or, more precisely, W/a. This description becomes semiclassical once the question of overlap across saddle points takes tunneling into account. Overlap is then no longer characterized as overlap within the magnetic length l, but by a tunneling probability [23], [24]

$$P = \frac{e^{-1/\chi}}{1+e^{-1/\chi}} = \frac{1}{1+e^{1/\chi}}\,; \quad \chi = \frac{ml^2|V_{xx}V_{yy}|^{1/2}}{he|B-B_\mathrm{c}|}\,. \tag{1}$$

Here V_{xx} and V_{yy} are the principal double spatial derivatives of the disordered potential at the saddle point, and m is the effective electron mass. The semi-classical picture provides arguments that, taking tunneling into account, one effectively changes the basic length scale of the percolation problem from a to χa, leading to $\nu = \nu_\mathrm{p} + 1 = 7/3$. In this letter, these arguments are not discussed. On the contrary, the *conclusion* of a basic rescaling is taken for granted, and consequences beyond the shift in ν are explored and checked.

The correlation length exponent ν is not sufficient to determine all critical properties of a percolation transition; one more exponent is needed. In ordinary percolation, the cluster size distribution

$$N(s,B) = s^{-\tau} F\left(\frac{s}{\xi^D(B)}\right)\,, \tag{2}$$

giving the density of clusters of size s, may be used to determine the complete set of exponents. F is a scaling function that approaches a constant for small arguments and falls off faster than any power law for large arguments. The two exponents appearing in eq. (2) are known for ordinary percolation in two dimensions [25]: $\tau_\mathrm{p} = 187/91$ and $D_\mathrm{p} = 91/48$. A natural definition of a cluster of size s in the "classical" (*i.e.* ignoring tunneling) high-field model is s closed loops in contact, in the sense of overlap, across saddle points. Thus, the cluster size distribution in this model describes the geometry of the localized wave function ψ.

From a practical point of view, it is convenient to study the cluster size distribution $N(s,B)$ by measuring its moments. We define the k-th moment as

$$M_k(\xi) = \left\langle \left(\int \mathrm{d}\mathbf{x} |\psi|^2 \right)^k \right\rangle\,, \tag{3}$$

([1]) It has, however, been shown recently that delocalization of electrons in a strong magnetic field moving among randomly placed hard scatterers constitutes a percolation problem [22].

where $\langle \ldots \rangle$ denotes sample averaging. The wave function is calculated by specifying its value on the borders of the system. Thus, $\psi(\mathbf{x}_\mathrm{s})$ is not an eigenstate of the time evolution operator, but rather describes a stationary state where current flows through the system. At this stage it is important to note the difference between the definition (3) of $M_k(\xi)$ and the moments $N_k(\xi) = \langle \int \mathrm{d}\mathbf{x} |\psi|^{2k} \rangle$, used by Klesse and Metzler [26]. Their focus was the multifractality of the wave function. In contrast, we are interested in the essentially geometric cluster structure near the percolation threshold, and the exponents describing that structure. Our moments $M_k(\xi)$ provide a means to measure precisely those exponents, and can say nothing about the multifractality of the wave function. We also note that their wave functions were eigenstates of the time evolution operator of the system, making direct comparison with our results difficult. Using (2), we find that M_k is related to ξ as

$$M_k(\xi) \sim \int^{\xi^{D_\mathrm{p}}} \mathrm{d}s\; s^{k-\tau_\mathrm{p}} \sim \xi^{D_\mathrm{p}(k+1-\tau_\mathrm{p})} \,. \tag{4}$$

When tunneling is included, the purely geometric definition of the cluster size s is no longer practical. However, if the size distribution is defined in terms of the moments, (3), the notion of size is still viable. In [12], it was argued that tunneling does not change the two exponents appearing in $N(s,B)$, $\tau = \tau_\mathrm{p}$ and $D = D_\mathrm{p}$, since they only measure *relative* sizes of clusters. This is a highly nontrivial prediction of the semiclassical picture: *The exponents τ and D should remain invariant when tunneling is taken into account, whereas other exponents, like ν, are considerably changed.*

We test this prediction in this letter, by numerically determining the exponents τ and $(\tau - 1)D$ of eqs. (3) and (4) for the extended Chalker-Coddington model. We sketch this model in the following . Imagine a square network oriented at 45° with respect to two boundaries L lattice spacings apart. The lattice is periodic in the orthogonal direction. The bonds of the network represent the paths the electrons may take in the disordered energy landscape. Each bond can only be traversed in one direction. The possible directions are chosen so that the network forms a grid of loops of alternating handedness. The nodes represent saddle points, and the lattice constant corresponds to a. The wave function itself is represented by a complex variable z. At a given saddle point, z_1 and z_2 are ingoing amplitudes, and z_3 and z_4 outgoing amplitudes as shown in fig. 1. The amplitudes are related through the relations $z_3 = z_1 \tanh\gamma + z_2(1/\cosh\gamma)$ and $z_4 = z_1(1/\cosh\gamma) - z_2 \tanh\gamma$. In addition, random phases are picked up along the bonds joining the nodes, representing the variation in path length from loop to loop. The control parameter γ determines the scattering properties at the saddle point. When $\gamma \to 0$, the transmission probabilities are $P_{1\to4} = P_{2\to3} = 1$ and $P_{1\to3} = P_{2\to4} = 0$. Likewise, when $\gamma \to \infty$, we have that $P_{1\to3} = P_{2\to4} = 1$ and $P_{1\to4} = P_{2\to3} = 0$. When $\gamma = \gamma_\mathrm{c} = \ln(1+\sqrt{2})$, one finds $P_{1\to4} = P_{1\to3} = P_{2\to4} = P_{2\to3} = 1/2$. When $\gamma < \gamma_\mathrm{c}$, $P_{1\to3} = P_{2\to4} = 1/(1+\sinh^{-2}\gamma)$ are due to tunneling, while $P_{1\to4} = P_{2\to3} = 1 - P_{1\to3}$ are direct channels. When $\gamma > \gamma_\mathrm{c}$, $P_{1\to4} = P_{2\to3} = 1/(1+\sinh^{+2}\gamma)$ are due to tunneling while the two remaining channels are direct. This is illustrated in fig. 1. Comparing these results with eq. (1) derived in the high-field model, we identify [4], [23]

$$\chi = \frac{1}{2|\ln\sinh\gamma|} \,. \tag{5}$$

While Chalker and Coddington kept γ equal for all nodes, with disorder in the phases ϕ_i only, thus making it difficult to interpret the model in a percolation context, we follow Lee *et al.* [20], and extend the model by writing $\gamma = \gamma_\mathrm{c} e^{\mu - v}$. Here μ is a fixed parameter equal for all nodes in the network, while v is chosen randomly on the interval $[-w/2, +w/2]$. In the case when $|\mu - v| \ll 1$, we may relate $|\mu - v| \propto |B - B_\mathrm{c}|$ of the high-field model.

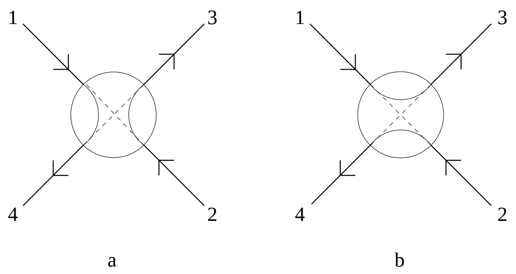

Fig. 1. – Illustration of scattering at a given node in the Chalker-Coddington model: *a*) shows the case when $\gamma \leq \gamma_c$ while *b*) shows the case $\gamma \geq \gamma_c$. The dashed curves denote the case $\gamma = \gamma_c$.

For finite μ and $v \to \infty$, the open channels are $z_1 \to z_4$ and $z_2 \to z_3$, and when $v \to -\infty$ they are $z_1 \to z_3$ and $z_2 \to z_4$. In this extreme case, there is no contact across the saddle points. However, delocalization is still possible. The correlation length exponent in the extreme case when $w \to \infty$ has been measured by Lee *et al.* [20] to be 1.29 ± 0.05 in agreement with classical percolation theory (predicting 4/3). However, it is important to realize that the system in this limit is *not* equivalent to the percolation picture of Kazarinov and Luryi and Trugman [9], [10]. Instead, this limit reproduces a system of square tiles, in which each tile is chosen at random from two possible elements, defining which channels are open. The resulting curves are known as *Dragon curves* [27]. The critical properties of this tiling were studied by Roux *et al.* [28]. They found that it is in the universality class of smart kinetic walks (SKW). Weinrib and Trugman [29] determined the universality class of SKW, finding, *e.g.*, $\nu_{\rm skw} = \nu_{\rm p} = 4/3$. The distribution of the length s of closed SKWs —loops— is $\tilde{N}(s) \sim s^{-\tau_{\rm skw}}$, where $\tau_{\rm skw} = 15/7 = 195/91$, in comparison to $\tau_{\rm p} = 187/91$ in the cluster size distribution of percolation, (2). The fractal dimension of the SKW is $D_{\rm skw} = 7/4 = 84/48$, while in classical percolation the fractal dimension of the clusters is $D_{\rm p} = 91/48$ [28].

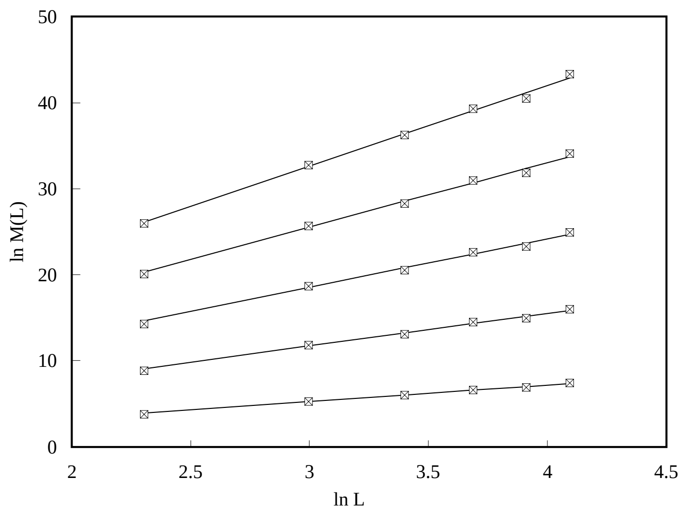

Fig. 2. – The moments $M_k(L)$ as a function of L. The straight lines are least-square fits from which $y(k)$ is determined.

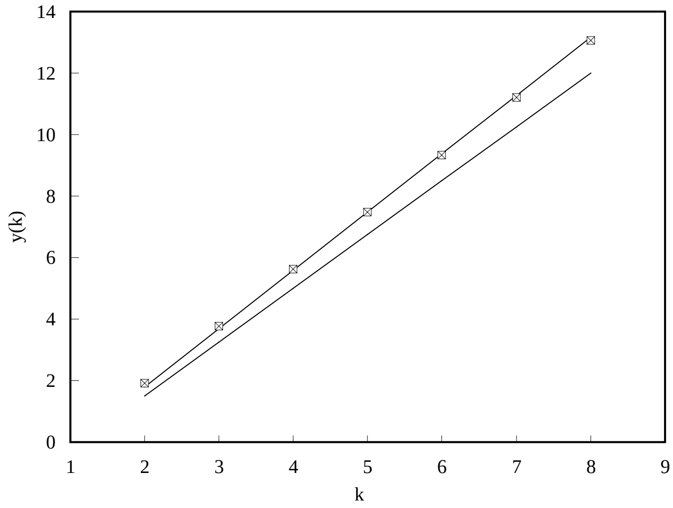

Fig. 3. – $y(k)$ as determined from the data of fig. 2. The upper continuous line is $y(k) = (91/48)k - 2$, while the lower straight line is $y(k) = (7/4)k - 2$.

Thus, the criticality of the Chalker-Coddington model in the limit $w \to \infty$ comes from the loop size distribution, and not from joining of overlapping loops at saddle points, as there are no saddle point crossings.

We now turn to the case of finite w. For small enough w and μ, tunneling becomes important, and the tunneling amplitude of a particular node depends on the value of the stochastic variable v in the range set by w. The wave function is no longer confined to non-intersecting loops. Rather, one has clusters of loops, connected across nodes through tunneling. Thus, we are back to a percolation situation. It is this situation we study numerically: disorder both in phase factors and in γ.

We fix the value of z to unity on the bonds entering the network along the left boundary, and to zero on the bonds entering on the right. Thus, we are describing a stationary state where current flows through the network. We then solve for the wave function inside the network using a biconjugate gradient algorithm for inverting the linear set of equations that ensue [30]. We generated 2000 10×10 lattices, 1200 20×20 lattices, 2000 30×30 lattices, 600 40×40 lattices, 600 50×50 lattices, and 500 60×60 lattices setting $w = 1$. With $\mu = 0$, the correlation length diverges, and its role is taken over by the finite linear dimension L of the lattice. From eq. (4), the moments should scale like $M_k(L) = \langle (\sum_i |z_i|^2)^k \rangle \sim L^{y(k)}$, where the sum is over nodes and where $y(k) = D(k+1-\tau) = D_{\rm p}(k+1-\tau_{\rm p}) = (91/48)k - 2 \approx 1.89k - 2$. In fig. 2 we plot $M_k(L)$ as a function of L. From the slopes of the least-square fits indicated in this figure, we determine $y(k) = Ak - B$, where $A = 1.86 \pm 0.10$ and $B = 1.83 \pm 0.10$. In fig. 3, we show $y(k)$ as a function of k. We also show in this figure $y(k) = (91/48)k - 2$, expected from the semiclassical arguments, and the SKW result $y(k) = (7/4)k - 2$. Clearly, our numerical data fit the semiclassical result well. This confirms the semiclassical prediction: With a strictly *finite* width w, *i.e.* with tunneling through saddle points of stochastic height, the values of the exponents τ and D are universal and equal to $\tau_{\rm p}$ and $D_{\rm p}$. As a check on consistency, we also simulated the extended Chalker-Coddington model with a *large* (*i.e.* essentially infinite) w. The exponents beyond ν were found to be consistent with the SKW exponents, *not* with the percolation exponents.

In conclusion, we have addressed the controversial question whether or not the semiclassical picture is capable of explaining the nature of the localization-delocalization transitions. By numerically determining the scaling properties of moments of the spatial distribution of the wave function in the extended Chalker-Coddington model, we find, within the error bars, agreement with the predictions from the semiclassical picture. Furthermore, we note that the SKW to which the extended Chalker-Coddington model crosses over at large disorder yields results outside the error bars. Our results lend strong support to the semiclassical picture.

***

We thank J. S. Høye, B. Huckestein, J. M. Leinaas, C. A. Lütken, F. A. Maaø, J. Myrheim and K. Olaussen for valuable discussions. This work was supported by the Norwegian Supercomputing Committee (TRU) and the Norwegian Research Council (NFR).

REFERENCES

[1] von Klitzing K., Dorda G. and Pepper M., *Phys. Rev. Lett.*, **45** (1980) 494.
[2] Prange R. E. and Girvin S.M. (Editors), *The Quantum Hall Effect* (Springer, Heidelberg) 1987.
[3] Janssen M., Viehweger O., Fastenrath U. and Hajdu J., *Introduction to the Integer Quantum Hall Effect* (VCH Verlag, Weinheim) 1994.
[4] Huckestein B., *Rev. Mod. Phys.*, **67** (1995) 357.
[5] Koch S., Haug R. J., von Klitzing K. and Ploog K., *Phys. Rev. Lett.*, **67** (1991) 883.
[6] Wei H. P., Tsui D. C., Paalanen M. A. and Pruisken A. M. M., *Phys. Rev. Lett.*, **61** (1988) 1294.
[7] Dolgopolov V. T., Shashkin A. A., Kravchenko G. V., Emeleus C. J. and Whall T. E., *JETP Lett.*, **62** (1995) 168.
[8] Sapoval B., Rosso M. and Gouyet J. F., *J. Phys.* (*Paris*) *Lett.*, **46** (1985) L149.
[9] Kazarinov R. F. and Luryi S., *Phys. Rev. B*, **25** (1982) 7626.
[10] Trugman S. A., *Phys. Rev. B*, **27** (1983) 7539.
[11] Mil'nikov G. V. and Sokolov I. M., *JETP Lett.*, **48** (1988) 536.
[12] Hansen A. and Lütken C. A., *Phys. Rev. B*, **51** (1995) 5566.
[13] Mieck B., *Europhys. Lett.*, **13** (1990) 453.
[14] Huckestein B. and Kramer B., *Phys. Rev. Lett.*, **64** (1990) 1437.
[15] Huo Y. and Bhatt R. N., *Phys. Rev. Lett.*, **68** (1992) 1375.
[16] Huckestein B., *Europhys. Lett.*, **20** (1992) 451.
[17] Liu D. and Das Sarma S., *Phys. Rev. B*, **49** (1994) 2677.
[18] Gammel B. M. and Brenig W., *Phys. Rev. Lett.*, **73** (1994) 3286.
[19] Chalker J. T. and Coddington P. D., *J. Phys. C*, **21** (1988) 2665.
[20] Lee D.-H., Wang Z. and Kivelson S., *Phys. Rev. Lett.*, **70** (1993) 4130; **72** (1993) 3918.
[21] Lee D. K. K. and Chalker J. T., *Phys. Rev. Lett.*, **72** (1994) 1510.
[22] Bobylev A., Maaø F. A., Hansen A. and Hauge E. H., *Phys. Rev. Lett.*, **75** (1995) 197.
[23] Fertig H. A., *Phys. Rev. B*, **38** (1988) 996.
[24] Hansen A., *Springer Lect. Notes Phys.*, **437** (1994) 331.
[25] Stauffer D. and Aharony A., *Introduction to Percolation Theory*, second edition (Taylor and Francis, London) 1992.
[26] Klesse R. and Metzler M., *Europhys. Lett.*, **32** (1995) 229.
[27] Wells D., *The Penguin Dictionary of Curious and Interesting Geometry* (Penguin, London) 1991.
[28] Roux S., Sornette D. and Guyon E., *J. Phys. A*, **21** (1988) L475.
[29] Weinrib A. and Trugman S. A., *Phys. Rev. B*, **31** (1985) 2993.
[30] Press W. H., Teukolsky S. A., Vetterling W. T. and Flannery B.P., *Numerical Recipes* (Cambridge University Press, Cambridge) 1992.