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Exact Bianchi type-I solutions of the Einstein equations with Dirac field

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Abstract. – We present the solutions of the Einstein-Dirac equations in a Bianchi type-I space-time with cosmological constant Λ. According to the value of a dimensionless parameter Φ, three classes of solutions of the Einstein-Dirac equations exist. If $\Phi < 1$, these solutions have two singularities t_1 and t_2: as $t \to t_1, t_2$ the metric tensor is of Kasner type, whereas, as $t \to \pm\infty$, it becomes isotropic and, because of Λ, has a stationary-type behaviour.

According to the Fridmann-Robertson-Walker (FRW) cosmological models, the Universe is homogeneous and isotropic and has evolved from a singular state, always maintaining its symmetries [1]. In such a way the FRW models do not investigate the causes of homogeneity and isotropy. As a consequence, the question arises whether the Universe has always been like the present one or has assumed the actual symmetries, evolving from a less symmetric initial state. A way to solve the problem of the causes of isotropy is to represent the Universe by means of an anisotropic and homogeneous cosmological model, which, at late times, acquires the symmetries observed at the actual cosmological epoch, as the result of a dynamical process. In effect, the Bianchi spaces contain in some particular cases the FRW models, and to these they reduce themselves at late times: the FRW models with $k = 0, 1, -1$ are contained, respectively, in Bianchi spaces classified as Bianchi type-I, Bianchi type-IX and Bianchi type-V [2], [3].

In particular, in this paper, we consider a homogeneous and anisotropic cosmological model of Bianchi type-I, in which the source of the gravitational field is an unquantized Dirac field.

Moreover, following Raychaudhuri [1], we assume that the cosmological constant Λ is not zero *a priori*.

The Lagrangian density of the gravitational field plus the Dirac field minimally coupled to gravity (we use units such that $\hbar = c = 1$) is [4]

$$\mathbf{L} = e\Big(\frac{\mathcal{R} + 2\Lambda}{16\pi G} + \mathcal{L}_{\mathrm{Dirac}}\Big) = e\Big[\frac{\mathcal{R} + 2\Lambda}{16\pi G} + \frac{1}{2} i\left(\bar{\psi}\gamma^\alpha \nabla_\alpha \psi - \left(\nabla_\alpha \bar{\psi}\right)\gamma^\alpha \psi + 2im\bar{\psi}\psi\right)\Big], \tag{1}$$

where $e = \sqrt{-g} = \det\|B^\alpha{}_i\|$, $\mathcal{R}$ is the curvature scalar of a V_4 space-time, $\|B^\alpha{}_i\|$ is the orthonormal vierbein field, such that

$$g_{ij} = B^{\alpha}{}_{i} B^{\beta}{}_{j} \eta_{\alpha\beta}, \tag{2}$$

$\|\eta_{\alpha\beta}\| = \mathrm{diag}(1,-1,-1,-1)$ is the Minkowski metric, g_{ij} is the metric tensor, ψ is the Dirac field with mass m, $\bar{\psi} \equiv \psi^{+}\gamma^{0}$ is the Dirac adjoint of ψ, $\{\gamma^{\alpha}\}$ are the special-relativistic Dirac matrices, satisfying the anticommutation rules

$$\{\gamma^{\alpha}, \gamma^{\beta}\} = 2\eta^{\alpha\beta}\mathbf{I}, \tag{3}$$

$$\nabla_{\alpha}\psi = A_{\alpha}{}^{i}\partial_{i}\psi - \frac{1}{8}\Gamma_{\alpha\sigma\beta}[\gamma^{\sigma},\gamma^{\beta}]\psi,\ \ \nabla_{\alpha}\bar{\psi} = A_{\alpha}{}^{i}\partial_{i}\bar{\psi} + \frac{1}{8}\Gamma_{\alpha\sigma\beta}\bar{\psi}[\gamma^{\sigma},\gamma^{\beta}], \tag{4}$$

are, respectively, the covariant derivatives of ψ and $\bar{\psi}$ with $\|A_{\alpha}{}^{i}\| \equiv \|B^{\alpha}{}_{i}\|^{-1}$,

$$\Gamma_{\alpha\beta\sigma} = \Gamma_{\alpha[\beta\sigma]} = (-\eta_{\mu\sigma}A_{\alpha}{}^{i}A_{\beta}{}^{j} + \eta_{\mu\alpha}A_{\beta}{}^{i}A_{\sigma}{}^{j} - \eta_{\mu\beta}A_{\sigma}{}^{i}A_{\alpha}{}^{j})\partial_{[i}B^{\mu}{}_{j]} \tag{5}$$

is the connection of a V_4 space-time, valued on an orthonormal tetrad [5]. Varying the action integral $I \equiv \int \mathbf{L}\mathrm{d}^4x$, built up by the Lagrangian density (1), with respect to $\bar{\psi}$ and $B^{\alpha}{}_{i}$, and equating such variations to zero , we get, respectively, the Dirac equation and the Einstein equations with cosmological constant:

$$i\gamma^{\alpha}\,\nabla_{\alpha}\,\psi - m\psi = 0, \tag{6}$$

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\mathcal{R} - \Lambda\eta_{\alpha\beta} = 4\pi G i[-\bar{\psi}\gamma_{(\beta}\nabla_{\alpha)}\psi + \nabla_{(\alpha}\bar{\psi}\gamma_{\beta)}\psi]\,, \qquad [6] \tag{7}$$

where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor with respect to $\Gamma_{\alpha\beta\sigma}$ and Λ is assumed to be positive.

Now, we assume that the Universe is a Bianchi type-I space-time: its space-time separation is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (a_1)^2(\mathrm{d}x^1)^2 - (a_2)^2(\mathrm{d}x^2)^2 - (a_3)^2(\mathrm{d}x^3)^2\,, \tag{8}$$

where a_1, a_2, a_3 depend on time t only.

Since such space-time is spatially homogeneous, it is reasonable to suppose that ψ does not depend on spatial coordinates:

$$\psi = \psi(t). \tag{9}$$

Consistently with (2), we choose the only non-zero elements of $\|B^{\alpha}{}_{i}\|$ as

$$B^{0}{}_{0} = 1, \quad B^{i}{}_{i} = a_i, \quad i = 1,2,3. \tag{10}$$

The representation of Dirac matrices $\{\gamma^{\alpha}\}$ we choose is

$$\gamma^{0} = \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{I} \end{bmatrix}, \quad \gamma^{b} = \begin{bmatrix} \mathcal{O} & \sigma^{b} \\ -\sigma^{b} & \mathcal{O} \end{bmatrix}, \tag{11}$$

where $\mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and σ^b are the Pauli matrices ($b = 1,2,3$).

In a Bianchi type-I space-time, according to (9) and (10), the Dirac equation becomes

$$\gamma^{0}\,[i\,\dot{\psi} + \frac{i}{2}\,(H_1 + H_2 + H_3)\psi - m\gamma^{0}\psi] = 0, \tag{12}$$

where $\dot{\psi} \equiv \frac{\mathrm{d}\psi}{\mathrm{d}t}, H_i \equiv \frac{\dot{a}_i}{a_i}$ (Hubble functions), $\dot{a}_i \equiv \frac{\mathrm{d}a_i}{\mathrm{d}t}, i = 1,2,3.$

Its solution is [7]

$$\psi = \frac{1}{(-g)^{\frac{1}{4}}} \begin{bmatrix} b_1 e^{-im\,t} \\ b_2 e^{-im\,t} \\ b_3 e^{im\,t} \\ b_4 e^{im\,t} \end{bmatrix}, \tag{13}$$

where $\sqrt{-g}$ has to be valued from the Einstein equations (see eq. (22)) and b_i are integration complex constants.

Now, after deriving $\dot{\psi}_i$ and $(\dot{\psi}_i)^*$ from the Dirac equation (12) and replacing them in the Einstein equations, we obtain that the field equations $G_{0b} - 8\pi G\sigma_{0b} = 0, b = 1,2,3$ are identically satisfied and

$$G_{00} - \Lambda\eta_{00} - 8\pi G\sigma_{00} = 0 : H_1H_2 + H_1H_3 + H_2H_3 - \Lambda - \frac{2\beta}{\sqrt{-g}} = 0, \tag{14}$$

$$G_{bb} - \Lambda\eta_{bb} - 8\pi G\sigma_{bb} = 0 : \dot{H}_c + \dot{H}_d + (H_c)^2 + (H_d)^2 + H_cH_d - \Lambda = 0, \tag{15}$$

$$b, c, d = 1,2,3, \quad c, d \neq b,$$

$$G_{12} - \mathcal{K}\sigma_{12} = 0 : \frac{H_1 - H_2}{4a_3}(|b_1|^2 - |b_2|^2 + |b_3|^2 - |b_4|^2) = 0, \tag{16}$$

$$G_{13} - \mathcal{K}\sigma_{13} = 0 : i\frac{H_3 - H_1}{4a_2}(b_2(b_1)^* + b_4(b_3)^* - b_1(b_2)^* - b_3(b_4)^*) = 0, \tag{17}$$

$$G_{23} - \mathcal{K}\sigma_{23} = 0 : \frac{H_3 - H_2}{4a_1}(b_2(b_1)^* + b_4(b_3)^* + b_1(b_2)^* + b_3(b_4)^*) = 0, \tag{18}$$

where

$$\beta = 4\pi Gm(|b_3|^2 + |b_4|^2 - |b_1|^2 - |b_2|^2). \tag{19}$$

If we require that $H_1 \neq H_2, H_1 \neq H_3, H_2 \neq H_3$, then from eqs. (16)-(18) we get the following constraints on the b_i constants:

$$b_1(b_2)^* + b_3(b_4)^* = 0, \quad |b_1|^2 - |b_2|^2 + |b_3|^2 - |b_4|^2 = 0. \tag{20}$$

Now, by operating suitably on eq. (15), this can be put into the form

$$(y\,H_b)^{\cdot} - \Lambda y - \beta = 0, \quad b = 1,2,3, \tag{21}$$

with $y \equiv \sqrt{-g}$, such that

$$\ddot{y} - 3\Lambda y - 3\beta = 0, \tag{22}$$

obtained by means of eq. (21) and taking into account that

$$H_1 + H_2 + H_3 \equiv \frac{\dot{y}}{y}. \tag{23}$$

From eq. (21) it follows that

$$H_b = \frac{k_b + \beta t + \Lambda \int y\,\mathrm{d}t}{y}, \tag{24}$$

where $k_b, b = 1, 2, 3$, are three real integration constants and

$$y \equiv \sqrt{-g} = a e^{\sqrt{3\Lambda}\, t} + b e^{-\sqrt{3\Lambda}\, t} - \frac{\beta}{\Lambda} \tag{25}$$

is the general solution of (22) where, due to eqs. (14)-(15),

$$a\, b = \frac{\beta^2}{4\Lambda^2}(1 - \Phi), \quad \Phi = \frac{\Lambda({k_1}^2 + {k_2}^2 + {k_3}^2)}{2\beta^2}. \tag{26}$$

Substituting (24) and (25) in the identity (23) we get that

$$k_1 + k_2 + k_3 = 0. \tag{27}$$

According to the value of the dimensionless parameter Φ, eq. (24) has three classes of solutions: they correspond to $\Phi > 1, \Phi = 1, \Phi < 1$, respectively.

Now, if $\Phi > 1$, then Λ is inferiorly limited by a positive constant, so that for such solutions it makes no sense to consider the limit $\Lambda \to 0$.

Moreover, if $\Phi = 1$, then $\Lambda = 0$ implies the constraint $\beta = 0$.

For these reasons the only solutions for which we can take the limit $\Lambda \to 0$, without having any unwanted constraint, are those corresponding to $\Phi < 1$: one has that

$$H_b = \frac{\sqrt{\frac{\Lambda}{3}\,(1 - \Phi)}\, \sinh(x) + \frac{k_b \Lambda}{\beta}}{\sqrt{1 - \Phi}\, \cosh(x) - 1}, \tag{28}$$

$$\sqrt{-g} = \frac{\beta}{\Lambda}(\sqrt{1 - \Phi}\, \cosh(x) - 1), \tag{29}$$

where we have set $x \equiv \sqrt{3\Lambda}(t + d)$, d being a real integration constant.

From eq. (28) we obtain

$$a_b(t) = a_{b0} |\sqrt{1 - \Phi}\, \cosh(x) - 1|^{\frac{1}{3}} \left| \frac{e^x - z_1}{e^x - z_2} \right|^{p_b}, \tag{30}$$

where

$$z_1 = \frac{1 + \sqrt{\Phi}}{\sqrt{1 - \Phi}}, \quad z_2 = \frac{1 - \sqrt{\Phi}}{\sqrt{1 - \Phi}}, \tag{31}$$

$$p_b = \frac{k_b}{\mathrm{sign}\,(\beta) \sqrt{\frac{3}{2}((k_1)^2 + (k_2)^2 + (k_3)^2)}}, \tag{32}$$

$$p_1 + p_2 + p_3 = 0, \tag{33}$$

a_{10}, a_{20}, a_{30} are integration real constants satisfying

$$a_{10} a_{20} a_{30} = \frac{\beta}{\Lambda} \mathrm{sign}\,(\sqrt{1 - \Phi}\, \cosh(x) - 1). \tag{34}$$

It is easy to verify that, as $\Lambda \to 0$, the solutions (31) reduce to the ones corresponding to $\Lambda = 0$ (in the absence of scalar field) as in paper [7], by assigning suitable values to k_b and d.

Moreover, since $\sqrt{-g} > 0\ \forall t$, we conclude that

a) $t \in]-\infty\, , t_2 [\cup] t_1\, , +\infty [$ if $\beta > 0$,

b) $t \in]\, t_2\,,\, t_1\, [$ if $\beta < 0$.

The metric tensor has two singularities (one, if $k_1 = k_2 = k_3 = 0$):

$$t_1 = -d + \frac{1}{\sqrt{3\Lambda}} \ln\left(\frac{1+\sqrt{\Phi}}{\sqrt{1-\Phi}}\right),\; t_2 = -d + \frac{1}{\sqrt{3\Lambda}} \ln\left(\frac{1-\sqrt{\Phi}}{\sqrt{1-\Phi}}\right). \tag{35}$$

They are physical singularities because the curvature scalar

$$\mathcal{R} = -2(\dot{H}_1 + \dot{H}_2 + \dot{H}_3 + (H_1)^2 + (H_2)^2 + (H_3)^2 + H_1H_2 + H_1H_3 + H_2H_3) \tag{36}$$

diverges at $t = t_1, t_2$.

The singularity $t = t_1$ is a zero of a_1, a_2 and a_3, if one has that, respectively, $p_1 > -\frac{1}{3}, p_2 > -\frac{1}{3}, p_1 + p_2 < \frac{1}{3}$ (let us remember that $p_3 = -p_1 - p_2$).

On the other hand, $t = t_1$ is an infinity of a_1, a_2 and a_3, if one has that, respectively, $p_1 < -\frac{1}{3}, p_2 < -\frac{1}{3}, p_1 + p_2 > \frac{1}{3}$: we notice that these conditions cannot be verified together, so at $t = t_1$ the three scale factor a_i cannot diverge together.

Furthermore, the singularity $t = t_2$ is a zero of a_1, a_2 and a_3, if one has that, respectively, $p_1 < \frac{1}{3}, p_2 < \frac{1}{3}, p_1 + p_2 > -\frac{1}{3}$.

On the contrary, $t = t_2$ is an infinity of a_1, a_2 and a_3, if one has that, respectively, $p_1 > \frac{1}{3}, p_2 > \frac{1}{3}, p_1 + p_2 < -\frac{1}{3}$: we notice again that, since these conditions cannot be verified together, at $t = t_2$, a_i cannot diverge together.

Moreover, at $t = t_1$, a_1, a_2 and a_3 take over a finite non-zero value respectively if $p_1 = -\frac{1}{3}, p_2 = -\frac{1}{3}, p_1 + p_2 = \frac{1}{3}$: we see that no more than two scale factors can have together a finite non-zero value at $t = t_1$, *i.e.* at least one of them will be singular at $t = t_1$.

Similarly, at $t = t_2$, a_1, a_2 and a_3 take over a finite non-zero value respectively if $p_1 = \frac{1}{3}, p_2 = \frac{1}{3}, p_1 + p_2 = -\frac{1}{3}$: once again we have that no more than two scale factors can have together a finite non-zero value at $t = t_2$, *i.e.* at least one of them is singular at $t = t_2$.

We see that if $|p_1 + p_2| < \frac{1}{3}$, with $|p_1| < \frac{1}{3}, |p_2| < \frac{1}{3}$, in the interval $]\, t_2\,, t_1\, [$ the Universe expands from $t = t_2$ up to the instant $t = -d$, at which the scale factors a_i reach a maximum, and then collapses up to $t = t_1$.

In such a case, the Universe expands from $t = t_1$ up to infinity in the interval $]\, t_1,\, +\infty\, [$ and contracts from $t = -\infty$ up to $t = t_2$ in the interval $]\, -\infty\,,\, t_2\, [$.

From eq. (24) we see that, at $t = t_1$ and $t = t_2$, H_i^2, H_iH_j and $\dot{H}_i$ diverge as $\frac{1}{(\sqrt{-g})^2}$, indeed, in the field equations (14), (15) the cosmological constant Λ and the mass-energy term $\frac{2\beta}{\sqrt{-g}}$ become negligible in a neighbourhood of the singularities and we have an empty universe whose metric is of the Kasner type: as $t \to t_1$, developing $e^x - z_1$, it turns out that

$$a_i \approx |t - t_1|^{q_i}, \tag{37}$$

where $q_i \equiv \frac{1}{3} + p_i$ are such that

$$q_1 + q_2 + q_3 = 1,\; (q_1)^2 + (q_2)^2 + (q_3)^2 = 1; \tag{38}$$

similarly, as $t \to t_2$, developing $e^x - z_2$, it turns out that

$$a_i \approx |t - t_2|^{r_i}, \tag{39}$$

where $r_i \equiv \frac{1}{3} - p_i$ are such that

$$r_1 + r_2 + r_3 = 1,\; (r_1)^2 + (r_2)^2 + (r_3)^2 = 1. \tag{40}$$

This strong initial anisotropy is increasingly damped as $|t| \gg |t_1|, |t_2|$ by the mass-energy of the Dirac field and by Λ: finally, at infinity, the metric has an isotropic stationary-type behaviour and we have an empty universe with constant Hubble functions: $H_i = \sqrt{\frac{\Lambda}{3}}$.

Finally, if $k_1 = k_2 = k_3 = 0$, one has $\Phi = 0$ and we get the solution of Isham-Nelson with cosmological constant in a spatially flat FRW universe [4], which, in turn, becomes the solution of Davis-Ray [8] as $\Lambda \to 0$.

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