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Propagating fronts on sandpile surfaces

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Abstract. – The flow of granular matter such as sand is often characterized by the motion of a thin superficial layer near the free surface, while the bulk of the solid remains immobile. A pair of equations called the BCRE equations (Bouchaud J-P., Cates M. E., Ravi Prakash J. and Edwards S. F. *J. Phys.* **4** (1994) 1383) have been proposed to model these flows and account for the dynamic exchange of mass between moving and stationary grains using the simplest kinematic considerations. We uncover a new conservation law for the BCRE equations and its variants that unifies a variety of recent special solutions and show that these equations support simple waves, and are capable of finite time singularities that correspond to propagating erosion fronts.

The flow of sand in a hourglass, the ripples on a beach and the clogging of a grain hopper are commonplace examples of our experience with particulate matter. The study of these materials cuts across the traditional boundaries of solids, fluids and gases; the finite angle of repose of a mound of sand is like that of a solid that preserves its shape, a snow avalanche is reminiscent of a flowing fluid, and the motion of a sediment suspension is like that of a dilute gas. Theoretical approaches to these problems at a macroscopic level use a combination of ideas from continuum mechanics and phenomenology, and at a microscopic level use inelastic molecular dynamics and statistical/kinetic approaches [1]. Of the phenomenological approaches, one model that accounts for the flow of granular materials in a thin superficial layer in such phenomena as avalanches [1] was first explicitly laid out in [2], although variants of these equations had been proposed earlier [3].

In its simplest form, the model characterizes the dynamics of these free-surface flows in terms of two dependent variables corresponding to the height $\widehat{H}(x,t)$ of a solid-like immobile phase and the effective height $R(x,t)$ of a liquid-like mobile phase that is akin to a free-surface shear band sliding on top of the solid-like phase. It is most convenient to think of the height $H(x,t)$ of immobile grains relative to a sandpile resting at the angle of repose so that $H(x,t)=\widehat{H}(x,t)-x\tan\theta$, where θ is the angle of repose. Although the original equations also

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included random forcing terms, we will deal with the deterministic equations in this paper. Then the so-called BCRE equations [2] are given by

$$\begin{aligned} H_t &= -\gamma R H_x\,, \\ R_t - v R_x &= \gamma R H_x\,. \end{aligned} \tag{1}$$

Here $(.)_a = \partial/\partial a$, and the equations as written are valid for flows on surfaces with a positive slope. For gravity-driven flows, rough estimates for the velocity of advection v and the frequency γ in terms of the grain diameter d are $v \sim \sqrt{gd}, \gamma \sim \sqrt{g/d}$, where g is the acceleration due to gravity. The above kinematic equations are multi-species analogs of the classical single-species kinematic relation embodied in Burgers' equation [4]. In fact they follow a long lineage of kinematic models that have been useful in analysing the evolution of surfaces in geomorphology [5], fluid mechanics [4] etc. Their main strength is the simplicity of the underlying assumptions. Here we focus on the simplest deterministic BCRE equations that embody the following phenomenological facts about the flow of granular materials:

1) When $R = 0$, $H(x,t) = \text{const.}$, *i.e.* the surface does not evolve unless there are grains in motion. When the surface does evolve, the rate of change of the height of the grains is proportional to the difference in the slope from the critical angle of repose θ_r. Thus these equations are valid only when the free-surface slope is in the vicinity of the angle of repose, when we can neglect higher-order gradients in R and H. Equivalently, the equations are valid when the height of the mobile grains $R(x,t)$ is small, and all mobile grains are assumed to be in contact with the immobile grains, justifying the form of the interaction term $\gamma R H_x$ in (1).

2) There is no velocity of convection for H. However, mobile grains are convected at a speed v in the negative x-direction. In general v must be determined using momentum balance and requires a constitutive equation that relates stress to strain rate. Here we will assume that v is constant, reflecting a balance between gravitational acceleration and inelastic collisions, and comment on a modification of this law at the end of the paper.

3) The equations when linearized about a steady state $H(x,t) = H_0(x), R(x,t) = R_0 =$ const. are hyperbolic with two distinct wave speeds v and γR_0. The first is simply the velocity of advection of the mobile species, while the second corresponds to waves traveling upward in the positive x-direction, *i.e.* in a direction opposing flow. This is in accordance with common experience in such instances as an upward propagating wave during an avalanche on an inclined plane [1].

Equations (1) are homogeneous and quasi-linear and therefore can be linearized via a hodograph transformation [6] that inverts the relationship between the dependent and independent variables using the transformation $x = x(H,R),\ t = t(H,R)$ [7]. Here we give a physically motivated derivation of the resulting simplification when the system (1) is viewed with respect to a coordinate frame moving with the rolling grains $\hat{x} = x + vt$. Then

$$H_t = -(v + \gamma R) H_{\hat{x}}, \qquad R_t = \gamma R H_{\hat{x}}\,. \tag{2}$$

Substituting for $H_{\hat{x}}$ in the first equation in terms of the second and integrating, we get

$$H(x+vt,t) = -\frac{v}{\gamma}\ln\left(\frac{\gamma R(x+vt,t)}{v}\right) - R(x+vt,t) + F(x)\,. \tag{3}$$

Thus, the conservation law (3) is valid in a frame moving with speed v in the negative x-direction, and relates the Riemann invariants of this system [6] while defining a relation between $H(x+vt,t)$ and $R(x+vt,t)$ for all time. It is sensible as long as $j = x_H t_R - x_R t_H$ and $J = H_x R_t - R_x H_t$ do not vanish, *i.e.* it is valid for all single-valued solutions of (1).

From (2) we also observe that along characteristic curves given by $\mathrm{d}\hat{x}/\mathrm{d}t = \gamma R + v$ $H(x+vt,t)$ is constant. From (3), it follows that $R(x+vt,t)$ is also constant along the curves. Thus, the

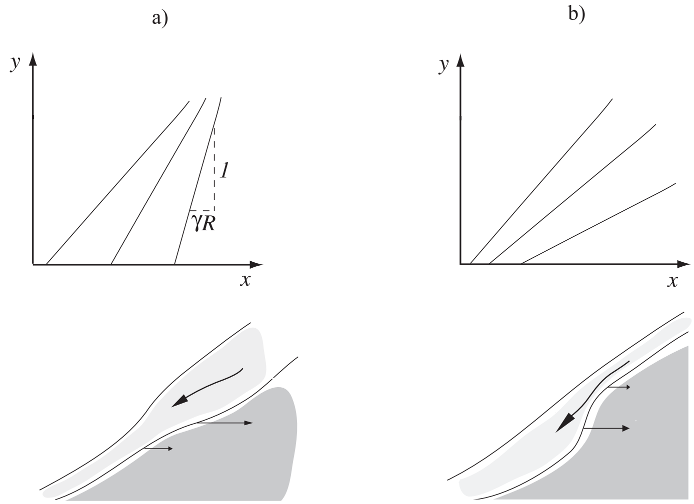

Fig. 1. – Schematic of the two scenarios possible for upward propagating waves far from any boundary. The slope of the characteristic lines $(\gamma R)^{-1}$ determines the scenario. (a) An erosion front forms when $R_x < 0$ since the characteristics become steeper as we move in the positive x-direction. (b) A rarefaction wave forms when $R_x > 0$ since the characteristics become less steep in the direction of propagation.

system (1) is hyperbolic, supports simple waves [6], and has characteristics that are straight lines. An interesting interpretation of (3) can be given by rewriting it as $R(x + vt, t) = \exp[\gamma(F(x) - H(x + vt, t) - R(x + vt, t))/v]$. In the frame of the rolling grains, this relation reflects an Arrhenius-like growth or decay of the mobile phase depending on the difference between the initial height F and the height of the free surface $H + R$ relative to the line at the angle of repose. This is consistent with the picture in [2] to describe the nucleation of motion as a first-order transition. This relation is only valid for small thicknesses of the mobile layer since it does not account for the diffusion of momentum associated with collisions, which would change v; we will not consider this question further in this paper.

A consequence of (3) is that it can be used to construct an infinite number of solutions by simply choosing a functional form for R that is consistent with some set of boundary conditions, and then evaluating H. A large class of simple boundary value problems are amenable to this approach. Recent work by de Gennes and his coworkers has lead to the solution of a number of interesting boundary-value problems for the BCRE equations and its variants [8]. All these special solutions fall naturally into the rubric provided by (3).

Instead of solving additional boundary-value problems in the context of the BCRE equations, we investigate the mathematical structure of these equations. Our goal in this is two-fold: 1) to look for singular solutions that allow us to objectively criticise the strengths and shortcomings of a model, and suggest regimes of applicability of the equations, 2) to look for similarity solutions that allow us to characterize certain regimes where the effects of boundaries and initial conditions are unimportant; thus they provide a different window to the applicability of the equations. Since (1) is a quasi-linear hyperbolic system [6], it is capable of forming singularities or shocks in finite time from smooth initial data beyond which the system is ill-posed. We will study the formation and evolution of these shock-like solutions that are manifested as propagating discontinuities in the height of the free-surface. To determine the condition for the formation of these shocks, we take advantage of the existence of simple wave solutions to (1). The characteristics are given by $x = x_0 + \gamma R(x_0, 0)t$; along these lines in space-time, both H and R are constant. Of course, this prescription is valid only as long as the characteristics do not intersect each other, since beyond an intersection, we cannot uniquely determine the evolution of the free-surface. These points of intersection which correspond to shocks lie on the envelope of the characteristics which is given by

$$1 + \gamma R_x(x_0, 0)t = 0, \qquad x = x_0 + \gamma R(x_0, 0)t\,. \tag{4}$$

The earliest time of formation of a shock is given by the minimum of the envelope of these characteristics in the (t, x)-plane. From (4) it follows that the times of shock formation are given by $t = -(\gamma R_x(x_0, 0))^{-1}$; thus the least time of shock formation is given by $\mathrm{d}t/\mathrm{d}x = 0$, and first occurs at locations in the initial profile where there is a point of inflection, *i.e.* at $R_{xx}(x_0, 0) = 0$. The physical mechanism of shock formation is as follows: at a location where the mobile layer is thick the wave speed γR is larger than at locations where the mobile layer is thin, as shown in fig. 1(a). Since the waves move in the positive x-direction, *i.e.* uphill, this requires $R_x < 0$ for the characteristics to intersect in the (x, t)-plane. This corresponds to a front of erosion that moves uphill as more and more of the immobile phase is entrained by the mobile phase. If $R_x > 0$, this results in a rarefaction wave that propagates upwards as shown in fig. 1(b). These upward moving fronts can be seen in a simple setting such as the flow of salt in a tilted salt-cellar.

In order to study the evolution of the shock, we modify the BCRE equations by introducing some diffusive terms to smooth out the discontinuity. Introducing terms similar to those proposed in [2, 9] we modify (1) to read

$$\begin{aligned} H_t &= -\gamma R H_x + \mu (R H_x)_x\,, \\ R_t - v R_x &= \gamma R H_x + \nu R_{xx}\,. \end{aligned} \tag{5}$$

Here $(RH_x)_x$ reflects the effects of collision-driven relaxation of the solid phase which is possible only when there are mobile grains, while R_{xx} reflects the effects of relaxation effects in the liquid phase in regions of high curvature. The associated diffusion constants are taken to be μ, ν, respectively. These terms are the simplest ones that respect the symmetry in the problem while conserving the mass. To determine the structure and velocity of the erosion front, we restrict ourselves to the case of a weakly nonlinear theory. Then we let $H(x, t) = \epsilon H_1(x - U_0 t) + O(\epsilon^2), R(x, t) = R_0 + \epsilon R_1(x - U_0 t) + O(\epsilon^2)$, where ϵ is small and characterizes the strength of the erosion front, and U_0 is the velocity of the front. On substituting this form of the solution into (5), at $O(\epsilon)$, we get

$$U_0 = \gamma R_0\ ,\ -U_0 R_1' = \gamma R_0 H_1' + v R_1', \tag{6}$$

where $(.)' = \partial/\partial\xi,\ \xi = x - U_0 t$. Thus to leading order the velocity of the front U_0 is equal to the velocity of upward propagating waves γR_0, and $R_1 = -\gamma R_0 H_1/(\gamma R_0 + v) + c_1$. At order $O(\epsilon^2)$ we get

$$\begin{aligned} \alpha R_0 H_1'' - \gamma R_1 H_1' &= 0\,, \\ -(v + \gamma R_0) R_2' - \gamma R_0 H_2' &= \gamma R_1 H_1' + \beta R_1'', \end{aligned} \tag{7}$$

by assuming a balance between the diffusive and nonlinear terms, so that $\alpha = \mu/\epsilon, \beta = \nu/\epsilon$. Substituting for R_1 from (6) into the first eq. (7) yields

$$\alpha H_1'' + \frac{\gamma^2 R_0}{v + \gamma R_0} H_1 H_1' = 0. \tag{8}$$

On integrating this equation for H, and substituting the result in the second equation in (7),

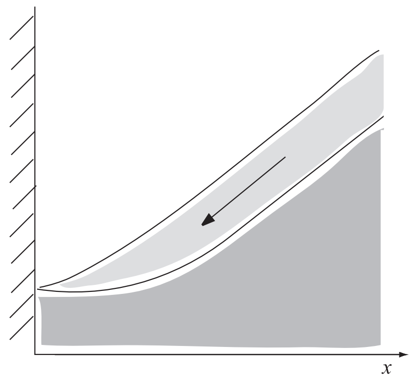

Fig. 2. – Schematic of the stoppage flow near a wall.

we get the profile of the traveling erosion front to order $O(\epsilon)$ [10],

$$\begin{aligned}\widehat{H}(x,t) &= x\tan\theta + \epsilon\tanh(\beta(x-\gamma R_0 t))\,,\\ R &= R_0\left[1 - \frac{\epsilon\gamma\tanh(\beta(x-\gamma R_0 t))}{\gamma R_0 + v}\right]\,.\end{aligned}\tag{9}$$

Here a constant of integration has been absorbed in a rescaling of the height, $\beta = \gamma^2 R_0/2\alpha(v+\gamma R_0)$ and ϵ is determined by the boundary conditions far from the front at $\xi \to \pm\infty$. This completes our description of the shock as an uphill-propagating front of erosion corresponding to a thinner immobile (thicker mobile) region downhill that invades a thicker immobile (thinner mobile) region.

The BCRE equations as they stand do not differentiate between thick and thin layer flows. In particular, there is no limit to the thickness of the mobile layer in the context of this theory. However, observations show that often $R(x,t)$ eventually saturates to a constant value, since all the grains do not interact with the solid phase. Then a model for thick avalanches consists of a layer with small vertical velocity gradients riding on the solid phase with a thin shear band separating the two. Various models have been proposed to take this into account [11,12], and can be written in the form

$$\begin{aligned}H_t &= -\gamma(R)RH_x\,,\\ R_t - vR_x &= \gamma(R)RH_x\,,\end{aligned}\tag{10}$$

where $\gamma(R)R$ saturates as R increases. We see that by changing to a frame that moves with the mobile phase, we can again simplify the equations to

$$\begin{aligned}\widehat{H}_t &= (v-\gamma(\hat{R})R)\widehat{H}_{\hat{x}}\,,\\ \hat{R}_t &= \gamma(\hat{R})\widehat{H}_{\hat{x}}\,,\end{aligned}\tag{11}$$

so that $H(\hat{x},t) + R(\hat{x},t) = v\int(\log R)_t \mathrm{d}t/\gamma(R)$.

Particular choices of $\gamma(R)$ that have been used include $\lambda R/(\lambda+R)$ [11] and $\gamma_0 - \gamma_1 R$ [12] and also lead to simple conservation laws.

We next use this generalized BCRE equation to look at a specific singular flow related to the stoppage that occurs when a thin mobile layer encounters a wall, leading to a front of accumulation that propagates upslope, as shown in fig. 2. A version of this problem has been recently treated in [13]. The relevant length scales in the problem are the grain size and the dimensions of the sand pile. On scales that are large compared to the grain size and small compared to the dimensions of the pile, we look for a similarity solution to describe this process. Substituting $H(x,t) = h(x/t^\alpha), R(x,t) = r(x/t^\alpha)$ into (1) and demanding that all terms in the equations be of the same order, we find that $\alpha = 1$. Then the resulting ordinary differential equations in terms of the similarity variable $\eta = x/t$ for $h(\eta), r(\eta)$ are

$$\eta h_\eta = \gamma r h_\eta \ , \ -\eta r_\eta = v r_\eta + \gamma r h_\eta \, . \tag{12}$$

Solving (12) with the proviso that $h_\eta \neq 0$ anywhere, and substituting $\eta = x/t$, we get an expression for the actual height of the immobile and mobile phases:

$$\begin{aligned} H(x,t) &= C + x \tan\theta - \frac{v}{\gamma} \ln \frac{x}{vt} - \frac{x}{\gamma t} \, , \\ R(x,t) &= \frac{x}{\gamma t} \, . \end{aligned} \tag{13}$$

We observe that (13) satisfies (3) for the particular choice $F(x) = 0$, $R(x,t) = x/\gamma t$. This solution breaks down in the neighborhood of the origin $\eta = 0$, corresponding to $x = 0$ or $t = \infty$. To estimate the size of the region near the wall where this happens, we use the regularized BCRE eqs. (5). Balancing the diffusive term $\mu(RH_x)_x$ with H_t, where H is given by (13), yields a characteristic length $H^* \sim \mu/\gamma$ in the neighborhood of the origin where (13) is not valid. Inside that region, $H \sim v \ln(\mu/\gamma v t)/\gamma, R \sim \mu/\gamma v t$. From an experimental point of view, this simple flow allows one to estimate the parameters v, γ and μ by looking at spatial and temporal gradients of the flow in the neighborhood of the wall [14].

We conclude with a summary and some general remarks. In this note, we have shown the existence of a new conservation law that unifies all the special solutions of the BCRE equations, which is a simplified model for the superficial flow of granular materials. We have also shown that for certain initial conditions, the equations predict the formation of shocks which correspond to upward-moving erosion fronts or equivalently, upstream hydraulic jumps induced by an obstacle in the flow. While the BCRE equations have been successful in qualitatively predicting a number of the features of simple free-surface granular flows [8], they have been experimentally probed carefully only in the simplest of situations [15]. Further experiments are required to quantitatively probe the existing predictions, and will allow us to modify the equations further to include 1) momentum balance to determine the average velocity of the mobile layer v [16], 2) a depth-dependent density in rapid flows [17, 18].

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