

## Dynamic scaling and universality in evolution of fluctuating random networks

To cite this article: M. Kotrla et al 2002 EPL 60 14

View the article online for updates and enhancements.

## You may also like

- Active to absorbing state phase transition in the presence of a fluctuating environment: feedback and universality Niladri Sarkar and Abhik Basu
- <u>A HIGH-RESOLUTION SURVEY OF H i</u> <u>ABSORPTION TOWARD THE CENTRAL</u> 200 pc OF THE GALACTIC CENTER Cornelia C. Lang, W. M. Goss, Claudia Cyganowski et al.
- <u>The Effect of Shear Rate on the Shear</u> <u>Strength of Saturated Coal-Measure Soil</u> Kun Chen, Hong Zhang, Can Wu et al.





EUROPHYSICS LETTERS Europhys. Lett., **60** (1), pp. 14–20 (2002)

## Dynamic scaling and universality in evolution of fluctuating random networks

M. Kotrla<sup>1</sup>(\*), F. Slanina<sup>1</sup> and J. Steiner<sup>2</sup>

 <sup>1</sup> Institute of Physics, Academy of Sciences of the Czech Republic Na Slovance 2, 182 21 Prague 8, Czech Republic
 <sup>2</sup> Charles University at Prague, Department of Theoretical Physics V Holešovičkách 2, 180 00 Prague 8, Czech Republic

(received 18 April 2002; accepted 1 July 2002)

PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion. PACS. 89.75.Fb – Structures and organization in complex systems. PACS. 05.10.-a – Computational methods in statistical physics and nonlinear dynamics.

**Abstract.** – We found that models of evolving random networks exhibit dynamic scaling similar to scaling of growing surfaces. It is demonstrated by numerical simulations of two variants of the model in which nodes are added as well as removed (*Phys. Rev. Lett.*, **83** (1999) 5587). The averaged size and connectivity of the network increase as power laws in early times but later saturate. Saturated values and times of saturation change with paramaters controlling the local evolution of the network topology. Both saturated values and times of saturation obey also power law dependences on controlling parameters. Scaling exponents are calculated and universal features are discussed.

Introduction. – In the last two years, there has been an increasing activity in the study of the structure and the time evolution of complex random networks [1-3]. Real data have been analyzed and at the same time many simplified models have been formulated and investigated. The applications in different contexts are ranging from evolution of WWW and Internet, over evolution of metabolic networks to the structure of social networks, or linguistic networks. Detailed mechanisms of evolution differ from one problem to another, nevertheless, some common features have already been identified. For example, it was revealed that many networks are *scale free*, *i.e.*, the distribution of the connectivity of nodes (vertices) is a power law. It was shown that the scale-free structure can be reproduced in the case of growing systems by models with a preferential attachment for adding new nodes [4].

The challenging open problem is to clarify what are the universal features of network dynamics, if any. The concept of scale-free networks does not provide a classification into universality classes. A wide range of exponents was found by analysis of the real data, and the exponents in models with preferential attachment can take an arbitrary value larger than two [5,6]. Moreover, this concept deals with the internal structure of the networks rather than with possible types of dynamics.

<sup>(\*)</sup> E-mail: kotrla@fzu.cz

It is of interest to explore the time dependence of global characteristics of networks. Two such global quantities are the size of the network N(t) and the mean connectivity  $\overline{k}(t)$ . The size of the network is the total number of nodes and the mean connectivity is defined as the average of the degrees of nodes (the local connectivities) over all nodes in the network. Different types of network evolution can be encountered. In some cases, one observes a rapid increase of the network size, the typical example being the Internet. Other systems, *e.g.* linguistic networks, seem to be in a stationary state with an approximately constant size. Similarly, the mean connectivity can also increase or saturate. The situation clearly depends on the time scale considered. A network which is rapidly growing in a certain time interval may after some transient reach a stationary state. A system which looks to be stationary or only slowly varying in a limited time interval may exhibit strong fluctuations on a long time scale.

The network size and the mean connectivity are expected to fluctuate in real systems. The fluctuation and/or shortage of reliable real data in a sufficiently long time interval can result in difficulty or even impossibility of tracing the general trends. However, the fluctuations can be reduced by averaging over many independent realizations when the evolution is explored by the simulations of simplified models. Then the information about the time behaviour can be extracted, and bias or saturation can be revealed. One can ask what are the time dependences of the global quantities, whether there are generic scenarios of the network evolution, etc.

In this letter, we study the time evolution of global characteristics in the model which has been recently introduced by us in the context of study of the biological evolution [7,8]. We found that the averaged network size and the averaged mean connectivity initially increase as power laws and later saturate. We show that these quantities exhibit dynamical scaling analogous to the scaling revealed previously in the study of surface growth [9]. The existence of scaling invariance suggests a possible universal behaviour. Therefore, we check the sensitivity to modifications of the model.

*Model.* – To fix ideas, we briefly describe our model. The model is based on extremal dynamics [10] with particular rules for adding and removing a node. The system is composed of N nodes, connected by links. The state of each node is described by a single dynamical variable  $b \in (0, 1)$ , called barrier. In each step, the following dynamical rules [7] are applied:

i) The node with minimum b is found and mutated. The barrier of the mutated node is replaced by a new random value b'.

ii) The barriers of all nodes linked to the selected unit are also replaced by new random numbers. If b' is larger than the barriers of all its linked neighbours, the node gives birth to a new node (speciation). If b' is lower than the barriers of all linked neighbours, the node dies out (extinction). In the remaining cases, the number of nodes remains unchanged.

The motivation of these rules is the following. Well-adapted nodes proliferate more rapidly and the chance for speciation is higher. However, if the local diversity, measured by connectivity of the node, is bigger, there are fewer available possibilities for the change and the probability of speciation is lower. On the other hand, poorly adapted nodes are more vulnerable to extinction, but at the same time larger diversity (larger connectivity) may favour the survival.

iii) Speciation means that a new node is added into the system, with a random barrier.

iv) Links are established between the new node and the neighbours of the mutated node: each link of the "mother" node is inherited with probability p by the "daughter" node. The rule iv) reflects the fact that the new node is to a certain extent a copy of the original, so the relations to the environment will be initially similar to the ones the old node has. Moreover, if a node which speciates has only one neighbour, a link between "mother" and "daughter" is also established. Similar models with duplication of the local geometry were recently used for modeling of protein interaction network [11–13].



Fig. 1 – Time evolution of the averaged number of nodes  $\langle N \rangle$  (upper curves) and the averaged mean connectivity  $\langle \overline{k} \rangle$  (lower curves) for different values of the parameter p. Values of p from bottom to top are p = 0.85, 0.9, 0.95, 0.97, 1 for the number of nodes and p = 0.85, 0.9, 0.95, 0.97 for connectivity. Data are averaged over 1000 independent runs. The straight lines are guides for the eye, with slopes 0.35 (dashed line) and 0.24 (dot-dashed line).

Fig. 2 – Data collapse of time evolution of the number of nodes  $\langle N \rangle$  for different values of the parameter p from fig. 1. Inset: saturated values of the number of nodes  $N^{\text{sat}}$  (crosses) and times of saturation  $t_N^{\text{sat}}$  (pluses) vs. (1-p). The straight lines are guides for the eye, with slopes 0.8 (full line) and 2.3 (dashed line).

v) When the node is extinct, it means that the node is removed without any substitution and all links it has are broken. As a boundary condition, we use the following exception: if the network consists of a single isolated node only, it never dies out. We suppose that a node completely isolated from the rest of the system has lower chance to survive. This leads to the following rule.

vi) If a node dies out, all its neighbours which are not connected to any other node die out with probability  $p_{sing}$ . We call this kind of extinctions singular extinctions.

We previously showed that the model is self-organized critical [7,8]. Here, we focus on scaling with time. In all simulations below, the evolution starts with a single node.

Dynamic scaling of evolving network. – In this part, we restrict ourselves to the situation  $p_{\text{sing}} = 1$  (all isolated nodes will die out), but we consider different values of the parameter p. The size N as well as the connectivity  $\overline{k}$  averaged over all nodes strongly fluctuate during the evolution (cf. fig. 2 in [7]), and no clear tendency can be deduced. However, two different regimes are identified after averaging over many independent runs (fig. 1). The crossover time  $t^{\text{sat}}$  between two regimes depends on the parameter p. In the initial stage  $t \ll t^{\text{sat}}$ , both averaged quantities increase as power laws:

$$\langle N \rangle \propto t^{\beta_N}, \qquad \langle \overline{k} \rangle \propto t^{\beta_k}, \tag{1}$$

with the exponents  $\beta_N \simeq 0.35$  and  $\beta_k \simeq 0.24$ . Here  $\langle ... \rangle$  is a statistical average, and the bar denotes an average over nodes. In the second regime, the averaged number of nodes  $\langle N \rangle$ 



Fig. 3 – Data collapse of time evolution of the averaged mean connectivity  $\langle \overline{k} \rangle$  for different values of parameter p from fig. 1. Inset: saturated values of the averaged mean connectivity  $\overline{k}^{\text{sat}}$  (crosses) and time for saturation  $t_k^{\text{sat}}$  (pluses) vs. (1-p). The straight lines are guides for the eye, with slopes 0.6 (full line) and 2.5 (dashed line).

Fig. 4 – Time evolution of the averaged number of nodes  $\langle N \rangle$  in the lattice variant of the model with 8 neighbours and for different values of the parameter  $p_{\rm sing}$ . Values of  $p_{\rm sing}$  from top to bottom are  $p_{\rm sing} = 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1$ . Data are averaged over 1000 to 10000 independent runs. The straight dashed line is a guide for the eye, with slope 0.41. Inset: time evolution of the averaged number of nodes  $\langle N \rangle$  in the lattice variant of the model with 4 neighbours. Values of  $p_{\rm sing}$  from top to bottom are  $p_{\rm sing} = 0.05, 0.1, 0.2, 0.4, 0.6, 0.8$ . The straight dashed line is a guide for the eye, with slope 0.41.

and therefore also the averaged mean connectivity  $\langle \overline{k} \rangle$  saturate after times  $t_N^{\text{sat}}$  and  $t_k^{\text{sat}}$ . The crossover times  $t_N^{\text{sat}}$  and  $t_k^{\text{sat}}$  are different ( $t_k^{\text{sat}}$  being smaller than  $t_N^{\text{sat}}$ ). The saturated values  $N^{\text{sat}}$ ,  $\overline{k}^{\text{sat}}$  and the times of saturation  $t_N^{\text{sat}}$  and  $t_k^{\text{sat}}$  increase with p.

The saturated values  $N^{\text{sat}}$ ,  $k^{\text{sat}}$  and the times of saturation  $t_N^{\text{sat}}$  and  $t_k^{\text{sat}}$  increase with p. The parameter p controls correlations between the structure of the neighbourhood of the "mother" node and the "daughter" node. We define  $\xi = 1/(1-p)$  as an analog of the correlation length. We have found that the saturated values  $N^{\text{sat}}$ ,  $\overline{k}^{\text{sat}}$  depend on  $\xi$  as powers (insets in figs. 2 and 3),

$$N^{\text{sat}} \propto \xi^{\alpha_N}, \qquad \overline{k}^{\text{sat}} \propto \xi^{\alpha_k}$$

$$\tag{2}$$

with the exponents  $\alpha_N = 0.8$ ,  $\alpha_k = 0.6$ . The exponents  $\alpha_N$  and  $\alpha_k$  approximately agree with the exponents obtained in [8] by the calculation of the mean values using the stationary distribution  $\mathcal{P}(k)$ . Both crossover times also fulfil the power law,

$$t_N^{\text{sat}} \propto \xi^{z_N}, \qquad t_k^{\text{sat}} \propto \xi^{z_k}.$$
 (3)

The exponents are  $z_N = 2.3$  and  $z_k = 2.5$  (insets in figs. 2 and 3). The behaviour for p = 1 is different. There is no saturation for p = 1 on the scale of our simulations and the model is critical [7,8].

This scaling behaviour is similar to the well-known dynamic scaling for kinetic roughening during surface growth [9]. Data for  $\langle N \rangle(t,\xi)$  as well as for  $\langle \overline{k} \rangle(t,\xi)$  can be rescaled to a single

curve (figs. 2 and 3). Hence, both  $\langle N \rangle$  and  $\langle \overline{k} \rangle$  fulfil the following scaling relations:

$$\langle N \rangle \propto \xi^{\alpha_N} f_N(t/\xi^{z_N}), \qquad \langle \overline{k} \rangle \propto \xi^{\alpha_k} f_k(t/\xi^{z_k}), \qquad (4)$$

where  $f_N(x)$  and  $f_k(x)$  are two different scaling functions with properties  $f_N(x) = \text{const}_1$ ,  $x \gg 1$ ,  $f_N(x) \propto x^{\beta_N}$ ,  $x \ll 1$  and  $f_k(x) = \text{const}_2$ ,  $x \gg 1$ ,  $f_k(x) \propto x^{\beta_k}$ ,  $x \ll 1$ . Exponents  $\beta_N$ ,  $\alpha_N$ ,  $z_N$  ( $\beta_k$ ,  $\alpha_k$ ,  $z_k$ ) are not independent, but obey the relation  $\beta_N = \alpha_N/z_N$  ( $\beta_k = \alpha_k/z_k$ ) (for reviews on kinetic roughening see [14, 15]). The relations between exponents are well satisfied for the above-obtained values.

Lattice variant of the model. – If  $p_{\rm sing} < 1$ , then more and more isolated nodes will be generated. The above rules do not allow to connect an isolated node to any other node. It will remain isolated forever. This may be an unrealistic situation. One rather expects that isolated nodes can be, with a certain probability, reconnected. In this case an additional rule for reconnection would be needed, *e.g.*, based on the list of links of extinct nodes. This would imply the need of complicated bookkeeping. In order to assess the effect of singular extinctions, we use a different variant of the model which allows to avoid the above unrealistic situation.

We suppose that nodes are now living on a regular lattice. It means that only some sites of the given lattice are occupied by nodes at the given time. The occupation may change during evolution. The lattice sites can represent the different positions in the Euclidean space or in some abstract space. The extremal dynamics is again employed, nodes are added or removed using the rule ii). However, a node can speciate only if there is an unoccupied site in the neighbourhood of the selected extremal node. A new "daughter" node is placed on a randomly selected unoccupied site within the neighbourhood.

The structure of links of the "daughter" node is now not inherited, but it is automatically given by existing nodes in her neighbourhood. The links are established from a given node to all neighbouring nodes (occupied sites within a given neighbourhood). Hence, the above rule iv) does not apply and is replaced by automatic connection within the given neighbourhood. The network is formed by clusters of connected nodes on the lattice. All links of an extinct node are broken. We apply the rule vi) for singular extinction with  $p_{\text{sing}} \in \langle 0, 1 \rangle$ . Therefore, isolated nodes can be created and can be reconnected. The rules also allow reconnection of previously disconnected clusters. The lattice variant of the model has similar self-organization properties as the above described off-lattice variant. In particular, it exhibits power law in the distribution of forward avalanches [16].

We consider for simplicity in the following the simple square lattice of size  $L \times L$  with periodic boundary conditions. The size L has been chosen so large that finite-size effects have no influence (L = 40-160 depending on  $p_{sing}$ ). We performed simulations for two definitions of the neighbourhood. The neighbourhood was composed of 4 nearest neighbours (4n-model) and 4 nearest neighbours plus 4 next-nearest neighbours (8n-model).

We found that the time behaviour of the network size and the averaged mean connectivity is qualitatively the same as in the off-lattice model. Both quantities are fluctuating; however, after averaging (over 1000 or more independent runs), two regimes can be clearly seen: the initial power law increase and the saturation at the late times. The time evolution of the network size for two modifications with 4 and 8 neighbours is displayed in fig. 4. An initial power law increase  $\langle N \rangle \propto t^{\beta_N}$  with exponent  $\beta_N = 0.41$  is observed in both cases.  $N_{\rm sat}$ increases with decreasing  $p_{\rm sing}$ . Saturated values are substantially lower for the 4n-model than for the 8n-model.

The dependence of saturated values and times of saturation on  $p_{\text{sing}}$  is shown in fig. 5. There are again power laws with more or less the same exponents for both considered neighbourhoods. Similarly as above, we define an analog of the correlation length  $\tilde{\xi} = 1/p_{\text{sing}}$ . Then



Fig. 5 – Saturated values of the number of nodes  $N^{\text{sat}}$  for the model with 4 neighbours (crosses) and with 8 neighbours (pluses) and time of saturation  $t_N^{\text{sat}}$  for the model with 4 neighbours (squares) and with 8 neighbours (asterisks) vs.  $p_{\text{sing}}$ . The straight lines are guides for the eye, with slopes 1.1 (full line) and 2.7 (dashed line).

Fig. 6 – Data collapse of time evolution of the number of nodes  $\langle N \rangle$  in the lattice variant of the model with 8 neighbours for different values of the parameter  $p_{\text{sing}}$  from fig. 4. Inset: data collapse of time evolution of the number of nodes  $\langle N \rangle$  in the lattice variant of the model with 4 neighbours.

we have

$$N^{\text{sat}} \propto \tilde{\xi}^{\alpha_N}, \qquad t_N^{\text{sat}} \propto \tilde{\xi}^{z_N},$$
(5)

with the exponents  $\alpha_N = 1.1$  and  $z_N = 2.7$ . The relation  $\beta_N = \alpha_N/z_N$  is well satisfied. We show the corresponding data collapse supporting the scaling relation  $\langle N \rangle \propto \tilde{\xi}^{\alpha_N} f_N(t/\xi^{z_N})$  in fig. 6. When  $p_{\text{sing}} = 0$ , the network grows until the evolution will be affected by the given finite lattice size.

Due to a low maximal possible value, the connectivity saturates very soon and good scaling of connectivity with  $p_{\text{sing}}$  has not been obtained. Therefore, data for the time evolution of connectivity are not presented. In order to study carefully scaling of connectivity in the lattice model, one should consider a larger size of neighbourhood —including also more distant sites or considering the model on the hypercube.

An external condition limiting the size of the network or the connectivity will lead trivially to the saturation of these quantities. One expects that there will be scaling with the limiting value. We verified that indeed there is dynamic scaling of the form

$$\langle N \rangle \propto L^{\alpha_N} g(t/L^{z_N}) \tag{6}$$

with time and the external parameter L. Here, g(x) is again a scaling function with the properties  $g(x) = \text{const}_3$  for  $x \gg 1$   $g(x) \propto x^{\beta_N}$  for  $x \ll 1$ . In the case of the 4n-model, we measured the exponents  $\alpha_N = 2.07$ ,  $\beta_N = 0.44$ ,  $z_N = 4.8$  fulfilling well the relation  $\beta_N = \alpha_N/z_N$ .

*Conclusion.* – We have studied the time dependence of the global characteristics of evolving random networks. We used the network model with the extremal dynamics and with

the variable rules for the evolution of the local geometry. We have found that the averaged network size and the averaged mean connectivity exhibit dynamic scaling with time and with the parameter controlling the form of dynamical rules which in turn affects spreading of the correlations during evolution. We considered two different variants of the model: the offlattice variant with varying reproduction of the local topology, and the lattice variant with variable amount of removal of isolated nodes. We have found that dynamic scaling is fulfilled in both variants of the model. However, the measured exponents are different. We attribute this variance to the difference in the mechanisms of evolution. On the other hand, we verified that there are universal features. The exponents in the lattice variant are robust to the change of size of the neighbourhood.

We expect that dynamic scaling can be observed also in other models of evolving networks with tunable internal dynamics. In general, dynamic scaling of evolving networks can be characterized by four independent exponents, two for the time dependence of the averaged size, and two for the evolution of the averaged mean connectivity. The exponents for the evolution of the network size may be trivial or not defined in models in which the number of nodes is only and permanently increasing. Nevertheless, the connectivity may saturate and exhibit nontrivial scaling. Varying the external parameters like the maximal size leads to a qualitatively different scaling. We believe that the study of dynamic scaling of evolving networks might provide a clue to the identification of universal features of network evolution. It is an open question if dynamic scaling allows the classification of types of network evolution into different universal classes as in the case of the surface growth, or if the evolution of global characteristics turns to be unique for each individual network as has been found for exponents describing the distribution of connectivities in scale-free networks.

\* \* \*

This work was supported by grant No. 202/01/1091 of the GA CR.

## REFERENCES

- [1] STROGATZ S. H., Nature, 410 (2001) 131.
- [2] ALBERT R. and BARABÁSI A.-B, Rev. Mod. Phys., 74 (2002) 47.
- [3] DOROGOVTSEV S. N. and MENDES J. F. F., cond-mat/0106144.
- [4] BARABÁSI A.-L. and ALBERT R., Science, 286 (1999) 509.
- [5] DOROGOVTSEV S. N., MENDES J. F. F. and SAMUKHIN A. N., Phys. Rev. Lett., 85 (2000) 4633.
- [6] KRAPIVSKY P. L., REDNER S and LEYVRAZ F., Phys. Rev. Lett., 85 (2000) 4629.
- [7] SLANINA F. and KOTRLA M., Phys. Rev. Lett., 83 (1999) 5587.
- [8] SLANINA F. and KOTRLA M., Phys. Rev. E, 62 (2000) 6170.
- [9] FAMILY F. and VICSEK T., J. Phys A, 18 (1985) L75.
- [10] BAK P. and SNEPPEN K., Phys. Rev. Lett., **71** (1993) 4083.
- [11] VAZQUEZ A., FLAMMINI A., MARITAN A. and VESPIGNANI A., cond-mat/0108043.
- [12] SOLÉ R. V, PASTOR-SATORRASF R., SMITH E. and KEPLER T. B., Adv. Complex Systems, 5 (2002) 43.
- [13] KIM J., KRAPIVSKY P. L., KAHNG B. and REDNER S., cond-mat/0203167.
- [14] BARABÁSI A.-L. and STANLEY H. E., Fractal Concepts in Surface Growth (Cambridge University Press, Cambridge) 1995.
- [15] HALPIN-HEALY T. and ZHANG, Y.-C., Phys. Rep., 254 (1995) 215.
- [16] STEINER J., Thesis, Charles University, Prague (2000).