# An analytical solution for the Haissinski equation with purely inductive wake fields

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# An analytical solution for the Haissinski equation with purely inductive wake fields

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**Abstract.** – The static solution of the Fokker-Planck equation, known as the Haissinski equation and governing the longitudinal evolution of an electron beam moving in a Storage-Ring which is subject to an inductive wake field, can be expressed in an analytical form, using the Lambert $W$-function. We show how the use of this analytical tool allows a better understanding of the validity of the numerical solutions and the distribution normalization properties.

*Introduction.* – The longitudinal phase-space distribution of electrons in a storage ring is given by the Fokker-Planck equation. Looking at the stationary solution of the Fokker-Planck equation an expression is found for the longitudinal distribution which is the Haissinski equation [1]. The form of this equation depends on the form of the wake field, which is given by the Fourier transform of the storage ring impedance. The case of the purely inductive impedance has been observed in storage rings like the SLC damping rings [2] and at KEK [3].

Approximated solutions of the Haissinski equation have been extensively investigated numerically [2–5]. We note that an analytical solution of this equation exists. This solution is given by a particular expression of the so-called Lambert $W$-function [6], which appears frequently in applied mathematics and has important application in many other fields [7, 8]. In this paper we present the Lambert $W$-function and analyze the nature of the analytical solution of the Haissinski equation in the case of a purely inductive impedance.

The Haissinski equation in the pure inductive case can be written in the form

$$\rho' = -\frac{\xi}{1 - S\rho}\rho, \tag{1}$$

where $\rho$ represents the beam distribution and the derivative is taken with respect to $\xi$, linked to the position $z$ of the electron with respect to the synchronous particle by

$$\xi = \frac{\omega_{\mathrm{s}}}{\alpha_{\mathrm{c}} c \sigma_\epsilon} z, \tag{2}$$

with $\omega_s$ being the synchrotron frequency, $\alpha_c$ the momentum compaction factor, $c$ the speed of light, and $\sigma_\epsilon$ the natural energy spread of the beam.

The parameter $S$ is specified in terms of the inductance of the wake field $L$, the revolution period $T_0$, the nominal energy of the particles $E_0$, the number of particles $N$, and also the synchrotron frequency, the natural energy spread, $\sigma_\epsilon$, and the elementary charge $e$, as

$$|S| = \frac{e^2 L N \omega_s}{\alpha_c^2 \sigma_\epsilon^2 T_0 E_0}. \tag{3}$$

Equation (1) can be rewritten in the following more convenient form:

$$\ln(\rho) - S\rho = -\frac{\xi^2}{2} + \ln(A), \tag{4}$$

where $A$ is the normalization constant defined by

$$\ln A = \ln\left(\rho\left(0\right)\right) - S\rho\left(0\right). \tag{5}$$

Equations of the type above, *i.e.* eq. (1), have a natural analytical solution in terms of the function known as the Lambert $W$-function [6]. This function appears in many branches of pure and applied mathematics, may be a useful tool for the solution of many dynamical problems [8], and is not as widely known as it should be.

The Lambert $W$-function, $W(z)$, is implicitly defined as the root of the following equation:

$$W(z)\exp[W(z)] = z, \tag{6}$$

and explicitly by the series expansion

$$W(z) = \sum_{n=0}^{\infty} \frac{(-n)^{n-1}}{n!} z^n, \tag{7}$$

converging for $|z| < \frac{1}{e}$. In the forthcoming sections other expansions of $W$, holding in a wider range, will be exploited.

*The Haissinski equation for an inductive wake and singularities.* – It is well known and evident that if $S$ is negative, eq. (1) has no singularity and there is always a unique continuous solution [5].

In the case of positive $S$ the solution exists, but the presence of a singularity point limits the validity of the solution to a restricted range of $S$ values, which will be specified in the following.

According to eqs. (6)-(7) the solution of eq. (1) can be written as an infinite sum of Gaussians, namely

$$\rho = -\frac{W(-AS\exp[-\frac{\xi^2}{2}])}{S} = \frac{1}{S}\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(AS)^n \exp\left[-n\frac{\xi^2}{2}\right]. \tag{8}$$

The argument of the Lambert $W$-function has to be real and greater than $-\frac{1}{e}$ to define a singled-valued function as a real physical distribution, which is not assured by the multivalued Lambert function $W$ defined for any complex argument.

The convergence of the above series depends on the real value of $AS$. According to the indications of the previous section we find that the validity of the solution does not extend to all the values of the constants $A$ and $S$, but holds for

$$AS \leq \frac{1}{e}. \tag{9}$$

The upper limit, associated with a branch point of the Lambert $W$-function, clarifies the role of the singularity.

We can now exploit eq. (8) to specify the normalization of the distribution $\rho$, and the role of $A$ and $S$.

It is evident that from eq. (8) the normalized distribution is given by the value $A$ solving for a given $S$

$$S = \int_{-\infty}^{\infty} W\left( - SA \exp\left[ - \frac{\xi^2}{2} \right] \right) \mathrm{d}\xi = \sqrt{2\pi} \sum_{n=1}^{\infty} \frac{n^{\frac{2n-3}{2}}}{n!} (AS)^n. \tag{10}$$

The r.h.s of eq. (10) is a series converging with the same range imposed by (9).

By taking for $AS$ the upper limit of convergence and by using the Stirling approximation

$$n! \simeq \sqrt{2\pi n} n^n e^{-n}, \tag{11}$$

we find the maximum value of $S$:

$$S^* \simeq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{12}$$

This is a rough approximation and the exact computation (given by the computing algebra software Maple) yields

$$S^* \simeq 1.550608\ldots, \tag{13}$$

which is very close to the value given in [5], and more accurate than the previous value given in the literature [9].

It is important to emphasise that the distribution (8) exhibits a r.m.s value below $\sigma_\xi < 1$. In physical terms, this means that an electron bunch experiencing a purely inductive wake with $S > 0$, due to, *e.g.*, a negative-momentum compaction factor, may have a length lower than the natural value, provided by $\sigma_z = \dfrac{c\alpha_{\mathrm{c}}}{\omega_{\mathrm{s}}}\sigma_\epsilon$.

The behavior of $\sigma_\xi$ as a function of $AS$ is given in fig. 1 and the limiting value, calculated with the Stirling approximation, but very close to the exact value, is

$$\sigma_\xi^* = \sqrt{\frac{6}{\pi^2} \sum_{n=1}^{\infty} n^{-3}} \simeq 0.854846\ldots. \tag{14}$$

The results obtained so far, apart from providing an analytical solution for eq. (1), have clarified the nature of the singularity associated with the limits of validity of the Taylor expansion of the Lambert $W$-function and the range of $S$ values ($S < S^*$) for which eq. (1) admits a normalizable solution.

Before concluding this section, let us note that the series expansion of the solution (8) can be extended to negative values of $S$ too, provided that $SA < \frac{1}{e}$. An idea of the behavior of the solution, for $AS \to \frac{1}{e}$ and for $AS \to \frac{-1}{e}$, is given in fig. 2. As is evident for positive $(AS)$ values the distinction is clearly similar to the Gaussian with a r.m.s. slightly larger than the natural value. On the contrary for negative $(AS)$ values the shape is significantly different from a Gaussian.

The possibility of obtaining a valid solution for a positive $AS$ in a larger interval will be discussed in the following concluding section.

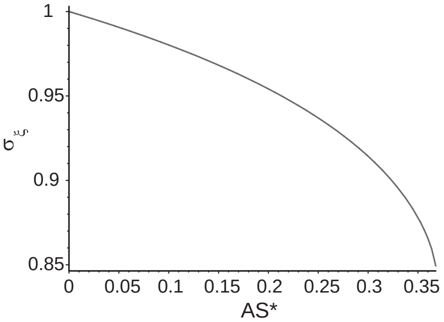


Fig. 1

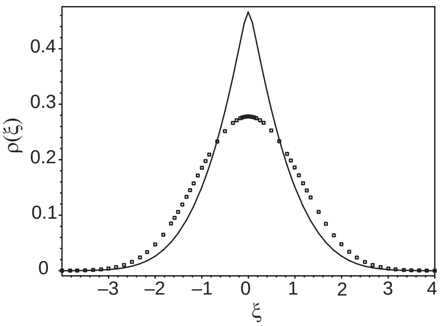


Fig. 2

Fig. 1 – The bunch length, $\sigma_\xi$, as a function of $AS^*$.

Fig. 2 – The bunch distribution in the case $AS \simeq \frac{1}{e}$ (squares) and in the case $AS \simeq -\frac{1}{e}$ (line).

*Concluding remarks.* – In the previous section we have explored the solution of the Haissinski equation for a purely inductive wake using the Taylor expansion of the Lambert $W$-function which has a limited convergence radius. In this section we will see how a different expansion, admitting a larger radius of convergence, can be exploited to get a non trivial and useful form of solution valid for negative $S$-values. To this aim we note that, for the present problem, a natural alternative to the Taylor expansion is provided by [8]

$$\begin{aligned} W(\exp[z]) = \quad & 1 + \frac{1}{2}(z-1) + \frac{1}{16}(z-1)^2 - \frac{1}{192}(z-1)^3 - \\ & -\frac{1}{3072}(z-1)^4 + \frac{13}{61440}(z-1)^5 + O((z-1)^6)\,, \end{aligned} \tag{15}$$

whose radius of convergence is $\sqrt{4+\pi^2}$.

According to the previous relation we can write the solution of (1) in the form ($\Lambda = -S$)

$$\begin{aligned} W\left(A\Lambda \exp\left[-\frac{\xi^2}{2}\right]\right) = \quad & 1 + \frac{1}{2}\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right) + \frac{1}{16}\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right)^2 - \\ & -\frac{1}{192}\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right)^3 - \frac{1}{3072}\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right)^4 + \\ & +\frac{13}{61440}\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right)^5 + O\left(\left(-\xi^2/2 + \ln\frac{A\Lambda}{e}\right)^6\right). \end{aligned} \tag{16}$$

The above solution shows that the charge distribution, in the case of a perfect inductor, is symmetric about $\xi = 0$ and tends to a parabolic shape for $A\Lambda \gg 1$.

A comparison between analytical and numerical solution is offered by fig. 3 and the agreement is more than satisfactory.

As a further comment we remark that the series converges for $A\Lambda \leq 41.4$, which is a good range for the specific problem we are considering.

It is also worth noting that it can be easily verified that the normalization constant can

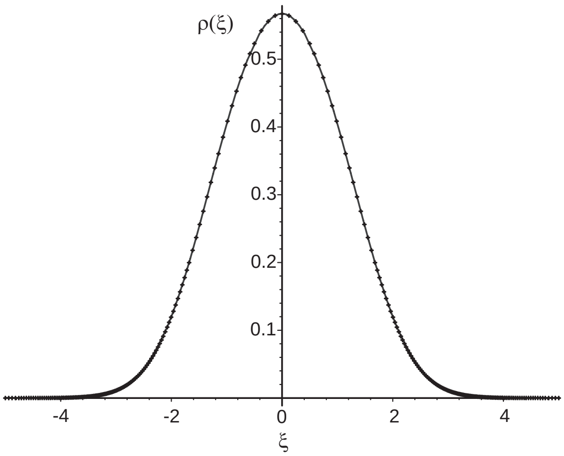


Fig. 3 – Comparison between analytical (line) and numerical (points) solution $\rho\,(\xi)$ of the Haissinski equation, for $S = -1$ and $A = 1$. The maximum error is less than $10^{-6}$.

be directly inferred and reads ($\delta = A\Lambda$)

$$\tilde{N} = \int_{-\infty}^{\infty} \rho(\xi \mathrm{d}\xi) \simeq 3.1\delta^{\frac{1}{3}}, \tag{17}$$

while the second-order normalized momentum can be written as

$$\sigma_\xi^2 = \frac{1}{\tilde{N}} \int_{-\infty}^{\infty} \xi^2 \rho(\xi)\mathrm{d}\xi \simeq 1.058(\delta + 5)^{\frac{1}{6}}. \tag{18}$$

Therefore for large $\delta$, and thus for large current, the r.m.s. of the distribution scales roughly as $\tilde{N}^{\frac{1}{4}}$.

In conclusion it is also interesting to mention the possibility of exploiting an asymptotic solution of the Haissinski equation valid for large $\delta$ values, namely [6]

$$\begin{aligned} W(z(\xi,\delta)) \approx \quad & \ln z(\xi,\delta) - \ln\left(\ln\left(z(\xi,\delta)\right)\right) + \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln\left(z(\xi,\delta)\right)^n} \sum_{m=1}^{\infty} \frac{(-1)^m n! \ln\left(\ln\left(z(\xi,\delta)\right)\right)^m}{(n-m+1)!m!}\,, \end{aligned} \tag{19}$$

with $z(\xi,\delta) = \delta \exp\left[-\frac{\xi^2}{2}\right]$.

The above expression can be used as bench-marking of the numerical solutions for values above the convergence radius of the expansion (16).

We must however underline that the series (19), being an asymptotic expansion, cannot be viewed as a real distribution since we cannot define a normalization factor, although the series is convergent.

In a forthcoming note we will see how a two-variable generalization of the Lambert $W$-function can be exploited to study problems associated with the beam distribution distortion due to microwave effects.

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