A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

Unified working regime of nonlinear systems rectifying thermal fluctuations

To cite this article: N. Sánchez Salas and A. Calvo Hernández 2003 *EPL* **61** 287

EUROPHYSICS LETTERS 1 February 2003

Europhys. Lett., **61** (3), pp. 287–293 (2003)

Unified working regime of nonlinear systems rectifying thermal fluctuations

N. SÁNCHEZ SALAS(*) and A. CALVO HERNÁNDEZ

Departamento de Física Aplicada, Universidad de Salamanca
37008 Salamanca, Spain

(received 12 July 2002; accepted in final form 25 November 2002)

PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.
PACS. 05.70.Ln – Nonequilibrium and irreversible thermodynamics.

Abstract. – We present the results of efficiency and net power output for Feynman's ratchet and pawl engine and its electric counterpart, the diode engine, calculated by means of a recently developed optimization criterion. In both cases a unified working regime between those predicted by maximum efficiency and maximum power calculations was found. These results extend previous findings for macroscopic heat engines and some molecular motors to mesoscopic energy converters.

Recently, we have reported [1] a unified optimization criterion (hereafter referred to as Ω-criterion) for energy converters in nonideal processes. If the process is characterized by an appropriate independent variable y and a set of parameters, or controls, $\{\gamma\}$, we define the Ω-criterion as a way to evaluate the best compromise between useful energy $E_{\mathrm{u}}(y;\{\gamma\})$ and lost useful energy $E_{\mathrm{u,L}}(y;\{\gamma\})$. Specifically, we take the Ω-function as the difference between these energies, *i.e.*, $\Omega(y;\{\gamma\}) \equiv E_{\mathrm{u}}(y;\{\gamma\}) - E_{\mathrm{u,L}}(y;\{\gamma\})$. Results for irreversible models of macroscopic thermal devices (heat engines, refrigerators and heat pumps) with both linear [1, 2] and nonlinear [2] heat transfer laws, as well as for isothermal linear models of biological motors [1], show that in all cases the Ω-based operation regime is intermediate between those arising from maximum useful energy and from maximum efficiency. Moreover, the application of this criterion is independent of environmental parameters (usually difficult to estimate) and does not require the explicit evaluation of entropy generation (a subtle issue in most cases). In particular, its implementation to heat engines only requires the knowledge of power output P and efficiency η. In this case it reads (see eq. (2) in [1])

$$\Omega(y;\{\gamma\}) = \frac{2\eta(y;\{\gamma\}) - \eta_{\max}(\{\gamma\})}{\eta(y;\{\gamma\})}\, P(y;\{\gamma\}), \tag{1}$$

where $\eta_{\max}(\{\gamma\})$ is the maximum value of the efficiency in the allowed range of values of y for given γ's. We stress that for endoreversible heat engines [3–5] (*i.e.* reversible models

(*) Also at Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Edif. No. 9, U. P. Zacatenco - 07738 México D. F., México.

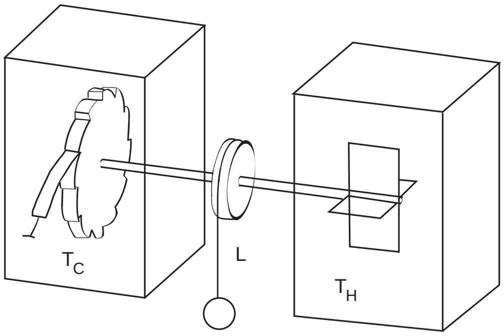

Fig. 1 – Sketch of Feynman's engine: the paddle reservoir is at temperature $T_{\rm H}$ and the ratchet reservoir at temperature $T_{\rm C} < T_{\rm H}$, in order to obtain work output of thermal fluctuations. L is the torque acting on the central wheel due to the external weight. From ref. [6].

where entropy generation only depends on the interaction of the working system with its surroundings), the Ω-criterion is equivalent to the so-called ecological optimization criterion [4], which represents the best compromise between maximum power output and minimum entropy production [4, 5].

The main goal of this paper is to apply the Ω-criterion to an entirely different class of heat engines: those producing power output by rectifying thermal fluctuations. In particular, we will focus on (mechanical) Feynman's ratchet and pawl engine [6] and its electric counterparts, the diode engines [7, 8]. Because of their relation to molecular motors, these mesoscopic, nonlinear systems have received much attention during the last few years [9–11]. However, only a few papers have been devoted to study the energetics of these systems. As has been pointed out recently [11], the concepts and strategies developed in Finite Time Thermodynamics (FTT) [12] to analyze optimum operating regimes of thermal devices can also be useful for ratchet systems. In this context, we recall that any regime in which the efficiency is greater than the efficiency at maximum power and the power is greater than the power at maximum efficiency is considered as an optimum operating regime in FTT [3].

We start with Feynman's ratchet and pawl engine [6, 10]. The following expressions for the dimensionless power output w and efficiency η were reported in [13]:

$$w(x;\tau,\alpha,\lambda) = {\rm e}^{-\alpha/\tau}[{\rm e}^{x_0-x}-1]x, \tag{2}$$

$$\eta(x;\tau,\alpha,\lambda) = \frac{[{\rm e}^{x_0-x}-1]x}{[{\rm e}^{x_0-x}-1](\alpha+x)+\lambda(1-\tau){\rm e}^{\alpha/\tau}}, \tag{3}$$

where

$$x=\frac{L\theta}{k_{\rm B}T_{\rm H}},\quad \alpha=\frac{\epsilon}{k_{\rm B}T_{\rm H}},\quad \tau=\frac{T_{\rm C}}{T_{\rm H}},\quad \lambda=\frac{t\sigma}{k_{\rm B}},\quad x_0=\frac{(1-\tau)\alpha}{\tau}. \tag{4}$$

In these equations (see fig. 1) ϵ is the energy needed to lift the pawl in a *backward* jump, θ is the angle between two consecutive teeth, L is the torque acting on the central wheel due to the external weight, $T_{\rm H}$ is the temperature of the hot reservoir (the paddle reservoir), $T_{\rm C}$ is the temperature of the cold reservoir (the ratchet reservoir), and σ is the thermal conductance characterizing the net energy flow from the hot reservoir to the cold one because of the mechanical link between the paddles and the ratchet [10]. Since $w \geq 0$, the dimensionless variable x in (4) takes values in the range $0 \leq x \leq x_0$, so that $L_0 = k_{\rm B}T_{\rm H}x_0/\theta$ is the torque at which the forward- and backward-jump rates are equal. Furthermore, the temperature ratio

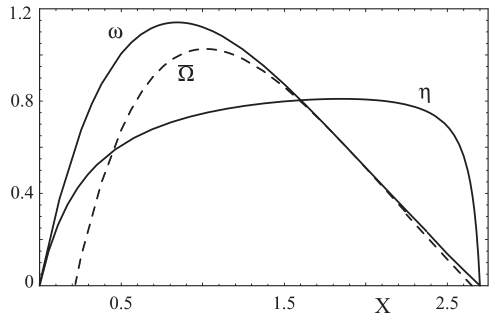

Fig. 2 – Dimensionless $\overline{\Omega}(\times 5)$, dimensionless power output $\omega(\times 5)$ and efficiency η *vs.* (dimensionless) x for Feynman's ratchet with $\alpha = 0.3$, $\tau = 0.1$, and $\lambda = 0.01$.

τ takes values in the range $0 \leq \tau \leq 1$ and $\alpha \geq 0$, $\lambda \geq 0$. For $x = x_0$ and $\lambda = 0$, eq. (3) becomes $\eta = \frac{x}{\alpha+x} \leq \frac{x_0}{\alpha+x_0} = 1-\tau$ and eq. (2) gives $w = 0$, *i.e.*, we obtain the Feynman engine under reversible conditions: Carnot efficiency and zero power output. In this system the role of the independent variable y in (1) is played by x and the controls $\{\gamma\}$ are (τ, α, λ). With this identification it is easy to evaluate Ω from eq. (1), while power and efficiency can be evaluated from eqs. (2) and (3), respectively. In fig. 2 we plot ω, η and (the dimensionless) $\overline{\Omega}$ in terms of x for a given set of controls ($\alpha = 0.3$, $\tau = 0.1$, $\lambda = 0.01$). We remark that the well-known parabolic-like behavior of the efficiency and power *vs.* an appropriate independent variable observed in irreversible models of macroscopic heat engines [2, 3, 12] also applies in this mesoscopic system. However, the main feature is that the efficiency under maximum $\overline{\Omega}$-conditions, $\eta_{\max\overline{\Omega}}$, lies between the maximum efficiency, $\eta_{\max}$, and the efficiency at maximum power, $\eta_{\max\omega}$. This behavior has been obtained for all the realistic values of the controls that we have checked. The behaviors with τ of the independent variable x giving maximum efficiency, $x_{\max\eta}$, maximum power, $x_{\max\omega}$, and maximum $\overline{\Omega}$, $x_{\max\overline{\Omega}}$, are plotted in fig. 3a. The behaviors with τ of the maximum power, $\omega_{\max}$, the power at maximum efficiency, $\omega_{\max\eta}$, and the power at maximum $\overline{\Omega}$, $\omega_{\max\overline{\Omega}}$, are plotted in fig. 3b. The maximum efficiency, $\eta_{\max}$, the efficiency at maximum power, $\eta_{\max\omega}$, and the efficiency at maximum $\overline{\Omega}$, $\eta_{\max\overline{\Omega}}$, are plotted against τ in fig. 3c. These figures clearly illustrate how the $\overline{\Omega}$-regime is intermediate between those of maximum efficiency and maximum power, *i.e.*, it can be considered as an optimum working regime. In particular, we note that the power at maximum $\overline{\Omega}$ conditions is very close to the maximum power (see fig. 3b) and $\eta_{\max\overline{\Omega}}$ is almost equal to the semisum of the maximum efficiency and the efficiency at maximum power (fig. 3c). The same features have been found for irreversible Carnot-like models of heat engines with linear and nonlinear heat transfer laws [2].

Next we analyze the diode engine [7, 8]. A detailed study of its maximum power and maximum efficiency regimes was reported in [14]. The basic expression is the stationary solution of the Fokker-Planck equation for the probability distribution $p(u)$ of the voltage u of the capacitor [8, 14]

$$p(u) = A \exp\left[-\int \mathrm{d}u \left[\frac{\left(\frac{1}{R_1(u)} + \frac{1}{R_2(u)}\right)u + i}{\frac{kT_1}{R_1(u)C} + \frac{kT_2}{R_2(u)C}}\right]\right], \tag{5}$$

where $R_j(u)$ and T_j ($j = 1, 2$) are the resistances and the temperatures of the nonlinear diodes and i denotes the voltage-independent current (A is a normalization constant). From $p(u)$ it

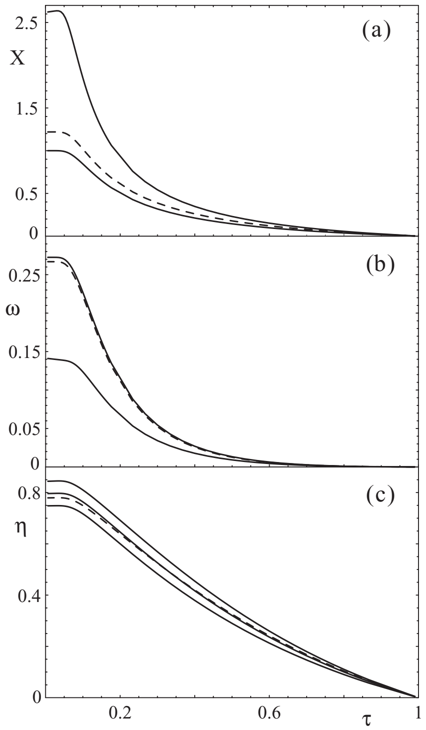

Fig. 3 – Feynman's ratchet. τ-behavior of: (a) the dimensionless x at maximum efficiency $x_{\max\eta}$ (upper solid line), maximum $\overline{\Omega}$, $x_{\max\overline{\Omega}}$ (dashed line), and maximum power $x_{\max\omega}$ (lower solid line); (b) maximum power, $\omega_{\max}$ (upper solid line), power at maximum $\overline{\Omega}$, $\omega_{\max\overline{\Omega}}$ (dashed line) and power at maximum efficiency, $\omega_{\max\eta}$ (lower solid line); (c) maximum efficiency, $\eta_{\max}$ (upper solid line), efficiency at maximum $\overline{\Omega}$, $\eta_{\max\overline{\Omega}}$ (dashed line), and efficiency at maximum power, $\eta_{\max\omega}$ (lower solid line). In all cases $\alpha = 0.3$ and $\lambda = 0.01$. The intermediate solid line in (c) is the semisum $[\eta_{\max\omega}(\tau) + \eta_{\max}(\tau)]/2$.

is straightforward to obtain the energetics of the system. In particular, the power output, P, is given by

$$P = i\langle u\rangle \equiv i\int_{-\infty}^{+\infty} up(u)\,\mathrm{d}u, \tag{6}$$

while the heat flux absorbed (per unit time) by the engine from the thermal reservoir kept at the hot temperature T_1, $\dot{Q}_{T_1}$, is given by

$$\dot{Q}_{T_1} = -\int u\left[\frac{kT_1}{R_1(u)C}\frac{\partial p(u)}{\partial u} + \frac{u}{R_1(u)}p(u)\right]\mathrm{d}u. \tag{7}$$

The efficiency η is then obtained from $\eta = P/\dot{Q}_{T_1}$. When no current flows (stalling condition) and the temperatures are equal, $p(u)$ becomes an (equilibrium) Boltzmann distribution for the capacitor's energy, independent of $R_j(u)$ ($j = 1, 2$). Under these conditions the power output is zero and the net heat flux from the hot thermal reservoir vanishes. Here we assume two diodes switched in opposite directions (see fig. 4) with the following expressions for the voltage-dependent resistances: $R_1(u) = R^+\theta(u) + R^-\theta(-u)$ and $R_2(u) = R^-\theta(u) + R^+\theta(-u)$ ($\theta(u)$ is the Heaviside step function). Inserting these expressions in eqs. (6) and (7), we obtain

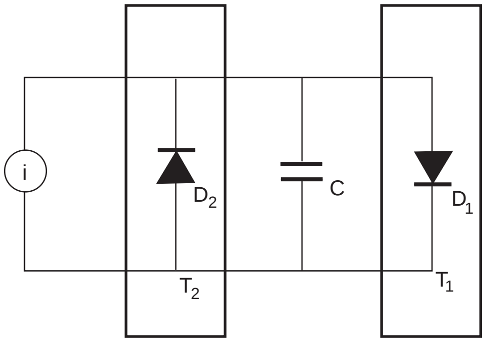

Fig. 4 – Sketch of the diode engine: two diodes D_1 and D_2 at different temperatures T_1 and T_2 and a capacitor C working against an outer current generator. From ref. [8].

the power output, the heat flux from the hot thermal bath and then the efficiency. In ref. [14] analytical expressions for these functions can be found in terms of a dimensionless current, $\bar{i}$, (the natural independent variable) and $\overline{R}^+, \overline{R}^-, \overline{T}_1$ and $\tau = \overline{T}_2/\overline{T}_1$ (the set of dimensionless parameters or *controls*) with $\overline{T}_j = kT_j/C$ $(j = 1, 2)$. Again, we calculate Ω from eq. (1) and the efficiency and the power from eqs. (6) and (7). In fig. 5 we have plotted $\overline{P}$, η and $\overline{\Omega}$ *vs.* $\bar{i}$ for $\overline{T}_1 = 10$, $\tau = 0.1$, $\overline{R}^- = 100$, $\overline{R}^+ = 1$. The corresponding optimized behaviors of the current, power output, and the efficiency are plotted against τ in figs. 6a, b and c, respectively. These plots show that the above-mentioned key features of the mechanical ratchet are also displayed for the electrical ratchet. The situation with a linear resistance R and a nonlinear diode considered in ref. [7] is a particular case when $R_1(u) = R$ is constant for all values of u. In this case we have also checked that the Ω-criterion predicts a similar intermediate working regime.

Finally, it is interesting to analyze the behavior of the efficiency when the parameters reach some limit values. For the linear regime of Feynman's ratchet ($\lambda = 0$, $\alpha \ll \tau$) it is not too difficult to find that $\eta_{\max\Omega} = (1 + \tau - 2\tau^2)/(1 + 3\tau)$, which is located between the maximum Carnot efficiency and the efficiency at maximum-power conditions given by $(1-\tau)/(1+\tau)$ [13]. For the electrical ratchet when one of the resistances increases up to infinity (for example, $\overline{R}^- \to \infty$ and $\overline{R}^+ = 1$) we obtain that the efficiency under maximum-Ω conditions is given by $(1 + 2\sqrt{\tau} - \tau - 2\tau^{3/2})/(1 + 3\sqrt{\tau})$ which, again, is below the maximum Carnot value

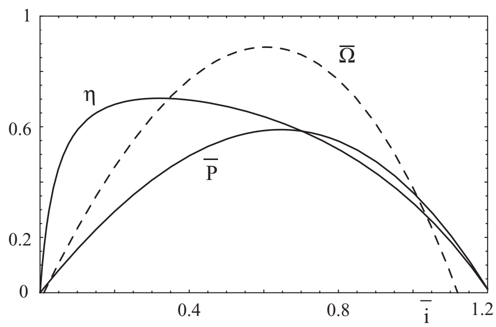

Fig. 5 – Dimensionless $\overline{\Omega}$, dimensionless power output $\overline{P}$, and efficiency η *vs.* (dimensionless) $\bar{i}$ for the Sokolov diode engine with $\tau = 0.1$, $\overline{R}^+ = 1$, $\overline{R}^- = 100$.

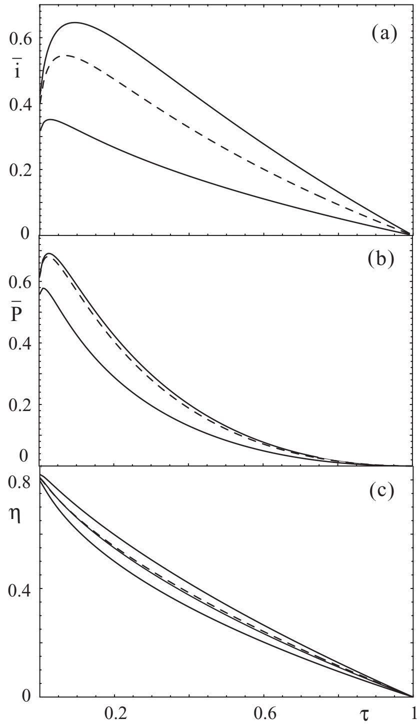

Fig. 6 – Diode engine. τ-behavior of: (a) dimensionless current at maximum power, $\bar{i}_{\max \overline{P}}$ (upper solid line), maximum efficiency, $\bar{i}_{\max \eta}$ (lower solid line), and maximum $\overline{\Omega}$, $\bar{i}_{\max \overline{\Omega}}$ (dashed line); (b) maximum power, $\overline{P}_{\max}$ (upper solid line), power at maximum $\overline{\Omega}$, $\overline{P}_{\max \overline{\Omega}}$ (dashed line) and power at maximum efficiency, $\overline{P}_{\max \eta}$ (lower solid line); (c) maximum efficiency, $\eta_{\max}$ (upper solid line), efficiency at maximum $\overline{\Omega}$, $\overline{P}_{\max \overline{\Omega}}$ (dashed line), and efficiency at maximum power, $\eta_{\max \overline{P}}$ (lower solid line). In all cases $\overline{R}^{+} = 1$, $\overline{R}^{-} = 100$. The intermediate solid line in (c) is the semisum $[\eta_{\max \overline{P}}(\tau) + \eta_{\max}(\tau)]/2$.

and above the efficiency under maximum-power conditions, given by the celebrated Curzon-Ahlborn [3–5, 14] value $1 - \sqrt{\tau}$. It is easy to check that the efficiency at maximum Ω is approximately the semisum of the maximum efficiency and the efficiency at maximum power in both cases. This semisum property also applies to endoreversible models optimized under the ecological criterion [5].

In summary, it has been found that some mesoscopic nonlinear rectifying thermal fluctuations heat engines, when optimized with the Ω-criterion, work in an intermediate regime between those of the maximum power and maximum efficiency. These results together with those already reported for macroscopic and some molecular motors endorse the Ω-criterion as a unified, optimum working regime for heat engines, independent of their size and nature.

* * *

Financial support from CICYT of Spain (Grant PB98-0261) and CyL-FSE (Grant SA097/01) are acknowledged. NS acknowledges a grant from Agencia Española de Cooperación Internacional (AECI).

REFERENCES

[1] Calvo Hernández A., Medina A., Roco J. M. M., White J. A. and Velasco S., *Phys. Rev. E*, **63** (2001) 037102-1.

[2] Sánchez Salas N., Velasco S. and Calvo Hernández A., *Energy Convers. Mgmt.*, **43** (2002) 2341.

[3] See, for example, Chen J., Yan Z., Lin G. and Andresen B., *Energy Convers. Mgmt.*, **42** (2001) 173 and references therein.

[4] Angulo-Brown F., *J. Appl. Phys.*, **69** (1991) 7465; Yan Z., *J. Appl. Phys.*, **73** (1993) 3583.

[5] Arias-Hernández L. and Angulo-Brown F., *J. Appl. Phys.*, **81** (1997) 2973; Yan Z., *J. Appl. Phys.*, **89** (2001) 1518; Angulo-Brown F. and Arias-Hernández L., *J. Appl. Phys.*, **89** (2001) 1520.

[6] Feynman R. P., Leighton R. B. and Sands M., in *The Feynman Lectures on Physics*, Vol. **I** (Addison-Wesley, Reading) 1966, pp. 46.1-46.9.

[7] Sokolov I. M., *Europhys. Lett.*, **44** (1998) 278.

[8] Sokolov I. M., *Phys. Rev. E.*, **60** (1999) 4946.

[9] Julicher F., Ajdari A. and Prost J., *Rev. Mod. Phys.*, **69** (1997) 1269; Astumian R. D., *Science*, **276** (1997) 917; Sekimoto K., *J. Phys. Soc. Jpn.*, **66** (1997) 1234; Reimann P., *Phys. Rep.*, **361** (2002) 57.

[10] Parrondo J. M. R. and Español P., *Am. J. Phys.*, **64** (1996) 1225.

[11] Parrondo J. M. R. and de Cisneros B. J., *Appl. Phys. A*, **75** (2002) 179.

[12] Sieniutycz S. and Salomon P. (Editors), *Finite-Time Thermodynamycs and Thermoeconomics* (Taylor and Francis, New York) 1990; Berry R. S., Kazakov V. A., Sieniutycz S., Szwast Z. and Tsirling A. M., *Thermodynamics Optimization of Finite-Time Processes* (Wiley, Chichester) 2000.

[13] Velasco S., Roco J. M. M., Medina A. and Calvo Hernández A., *J. Phys. D*, **34** (2001) 1000.

[14] Sánchez Salas N. and Calvo Hernández A., *J. Phys. D*, **35** (2002) 1442.