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Stochastic resonance in a linear system with signal-modulated noise

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Abstract. – The stochastic resonance characteristics of a linear system driven by a signal-modulated noise and an unmodulated noise are studied. Both the signal-modulated noise and the unmodulated noise are additive in nature and they are correlated. Our study shows that the correlation between the two noises leads to stochastic resonance. An exact analytic expression of the signal-to-noise ratio R is obtained. Based on it, some characteristic features of stochastic resonance are revealed. In the final analysis, the existence of a resonance peak of R *vs.* noise intensity is a noise correlation effect. On the other hand, the existence of a resonance peak of R *vs.* noise correlation time τ exhibits a signal-modulated effect of noise. It should be emphasized that the resonance peak of the R-τ curve exists even for uncorrelated noises. This is very different from the case of dependence on the noise strengths.

Introduction. – In the early stage of stochastic resonance (SR) study, nonlinearity of a system was regarded as one of the three necessary conditions for SR to exist. In 1996, Berdichevsky *et al.* investigated a linear system driven with multiplicative noise and discovered that SR can exist in a linear system when the multiplicative noise is dichotomous [1]. In 1997, Barzykin *et al.* studied a linear system driven with colored Gaussian multiplicative noise and also found SR in it [2]. Later on, they took into account the initial phase of the input signal [3]. In 1999, Berdichevsky and Gitterman further studied linear systems with multiplicative noise and additive noise that are correlated [4]. All these and other studies in the current literature show that a linear system exhibits SR if it is driven with colored multiplicative noise. Naturally, a question should be asked: Does SR exist in a linear system driven with colored additive noise? In this letter, we answer this question by studying a linear system with correlated additive colored noises one of which modulated by signal.

In the research of stochastic resonance, the relation between signal and noise is generally taken as additive. In 1992, Dykman *et al.* [5] studied the case where the signal is multiplied to noise. Namely, noise is modulated by the signal. When an asymmetric bistable system is driven with it, stochastic resonance appears. In contrast to the additive case, new characteristics emerge [5]. What we study in this letter are the stochastic resonance characteristics of a linear system driven by a signal-modulated noise and an unmodulated noise. Both the signal-modulated noise and the unmodulated noise are additive in nature and they are correlated. The physical motivations of this paper are: 1) In view of the importance of stochastic linear systems in physics, chemistry, and biology [6] and due to the periodically modulated noise arising at the output of the amplifier of the optics device and radio astronomy device [5], to establish a physical model in which the SR can contain the effects of the two factors, the linearity of the system and the periodical modulation of the noise. 2) To give a theoretical foundation for the study of the SR characteristic features of a single-mode laser. Our study shows that this model leads to stochastic resonance. An exact analytic expression of the signal-to-noise ratio is obtained. Based on it, some characteristic features of SR are revealed.

The model and the signal-to-noise ratio. – We consider a linear system driven by correlated Gaussian colored noises described by the following Langevin equation:

$$\dot{x} = -\alpha x + \xi(t) A_0 \cos \Omega t + \zeta(t), \tag{1}$$

in which

$$\zeta(t) = \xi(t) + \eta(t). \tag{2}$$

$\xi(t)$ and $\eta(t)$ are O-U noises with

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \tag{3}$$

$$\langle \xi(t)\xi(t') \rangle = \frac{Q}{\tau} e^{-\frac{|t-t'|}{\tau}}, \tag{4}$$

$$\langle \eta(t)\eta(t') \rangle = \frac{D}{\tau} e^{-\frac{|t-t'|}{\tau}}, \tag{5}$$

and

$$\langle \xi(t)\eta(t') \rangle = \langle \eta(t)\xi(t') \rangle = 2\lambda \frac{\sqrt{QD}}{\tau} e^{-\frac{|t-t'|}{\tau}}. \tag{6}$$

Equations (2)-(6) give

$$\begin{aligned} \langle \zeta(t) \rangle &= 0, & (7)\\ \langle \zeta(t)\zeta(t') \rangle &= \frac{Q + D + 2\lambda\sqrt{QD}}{\tau} e^{-\frac{|t-t'|}{\tau}}, & (8)\\ \langle \xi(t)\zeta(t') \rangle &= \frac{Q + \lambda\sqrt{QD}}{\tau} e^{-\frac{|t-t'|}{\tau}}, & (9) \end{aligned}$$

and

$$\langle \eta(t)\zeta(t') \rangle = \frac{D + \lambda\sqrt{QD}}{\tau} e^{-\frac{|t-t'|}{\tau}}. \tag{10}$$

The solution of eq. (1) with eqs. (2)-(6) and initial value x_0 is given by

$$x(t') = x_0 e^{-\alpha t'} + e^{-\alpha t'} \int_0^{t'} e^{\alpha s} \xi(s) A_0 \cos \Omega s \, \mathrm{d}s + e^{-\alpha t'} \int_0^{t'} e^{\alpha s} \zeta(s) \, \mathrm{d}s. \tag{11}$$

Then the correlation function reads

$$\begin{aligned}
&\langle x(t+t')x(t')\rangle = \\
&= e^{-\alpha(t+t')}e^{-\alpha t'}\left\langle x_0^2 + \int_0^{t+t'} e^{\alpha s}\xi(s)A_0\cos\Omega s\,\mathrm{d}s\int_0^{t'} e^{\alpha s'}\xi(s')A_0\cos\Omega s'\,\mathrm{d}s' + \right.\\
&\qquad + \int_0^{t+t'} e^{\alpha s}\xi(s)A_0\cos\Omega s\,\mathrm{d}s\int_0^{t'} e^{\alpha s'}\zeta(s')\,\mathrm{d}s' + \\
&\qquad + \int_0^{t+t'} e^{\alpha s}\zeta(s)\,\mathrm{d}s\int_0^{t'} e^{\alpha s'}\xi(s')A_0\cos\Omega s'\,\mathrm{d}s' + \\
&\qquad \left. + \int_0^{t+t'} e^{\alpha s}\zeta(s)\,\mathrm{d}s\int_0^{t'} e^{\alpha s'}\zeta(s')\,\mathrm{d}s'\right\rangle. \qquad (12)
\end{aligned}$$

By completing the integration of eq. (12) and letting $t' \to \infty$, finally making the Fourier transform, we obtain the power spectrum

$$\begin{aligned}
S(\omega) &= \int_{-\infty}^{\infty} e^{i\omega t}\langle x(t+t')x(t')\rangle\big|_{t'\to\infty}\,\mathrm{d}t \\
&= S_1(\omega) + S_2(\omega). \qquad (13)
\end{aligned}$$

The spectrum $S(\omega)$ divides naturally into two parts, the signal output $S_1(\omega)$ and the noise output $S_2(\omega)$. Finally, we get an exact analytic expression of the signal-to-noise ratio R:

$$R = \frac{S_{\mathrm{signal}}}{S_{\mathrm{noise}}} \qquad (14)$$

with total signal output power S_{signal} and total noise output power S_{noise} [7]:

$$\begin{aligned}
S_{\mathrm{signal}} &= \int_{-\infty}^{\infty} S_1(\omega)\,\mathrm{d}\omega = \\
&= (2\pi)A_0^2\frac{Q}{\tau}\left\{\frac{(a-1/\tau)(a+1/\tau)+(a+1/\tau)\Omega}{2\big((a-1/\tau)^2+\Omega^2\big)\big((a+1/\tau)^2+\Omega^2\big)} + \right.\\
&\qquad \left. + \frac{a+1/\tau}{4a\big((a+1/\tau)^2+\Omega^2\big)} - \frac{a-1/\tau}{4a\big((a-1/\tau)^2+\Omega^2\big)}\right\}, \qquad (15)
\end{aligned}$$

$$S_{\mathrm{noise}} = \int_{-\infty}^{\infty} S_2(\omega)\,\mathrm{d}\omega = (2\pi)\frac{\big(Q+D+2\lambda\sqrt{QD}\big)}{a(\tau a+1)}. \qquad (16)$$

Conclusion. – The following stochastic resonance phenomena induced by the noises correlation and the signal modulation of noise are drawn from eqs. (14)-(16):

i) On the R-D curve.

We now describe the dependence of the signal-to-noise ratio R upon the unmodulated noise intensity D. In the positive correlation region ($0 \prec \lambda \preceq 1$), no SR exist. R decreases monotonously with increasing D. However, in the negative correlation region ($-1 \preceq \lambda \prec 0$) SR appears. Its characteristic features are shown in figs. 1 and 2. In fig. 1, we see that the resonance peak height increases as the absolute value of the noise correlation coefficient $|\lambda|$ is increased. The peak position shifts to the right with increasing $|\lambda|$. Figure 2 shows that the

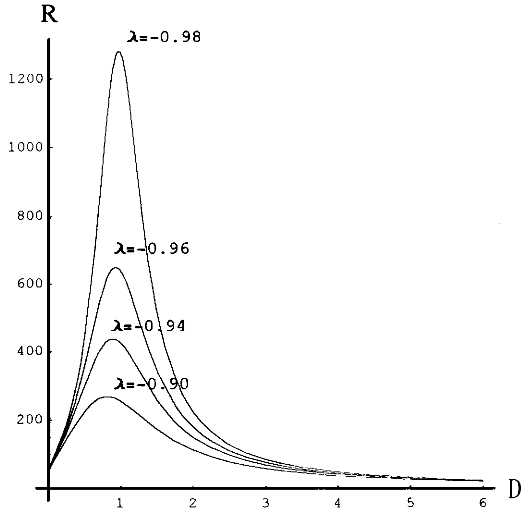

Fig. 1

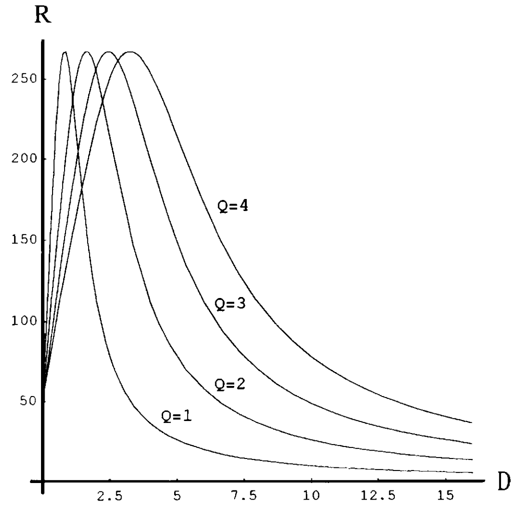

Fig. 2

Fig. 1 – The signal-to-noise ratio R as a function of noise intensity D (eqs. (14)-(16)) for different values of the noise correlation coefficient λ. The values of the other parameters are $a = 20$, $Q = 1$, $\tau = 2$, $A_0 = 10$, and $\Omega = 0.3$.

Fig. 2 – The signal-to-noise ratio R as a function of noise intensity D (eqs. (14)-(16)) for different values of the noise intensity Q. The values of the other parameters are $a = 20$, $\lambda = -0.9$, $\tau = 2$, $A_0 = 10$, and $\Omega = 0.3$.

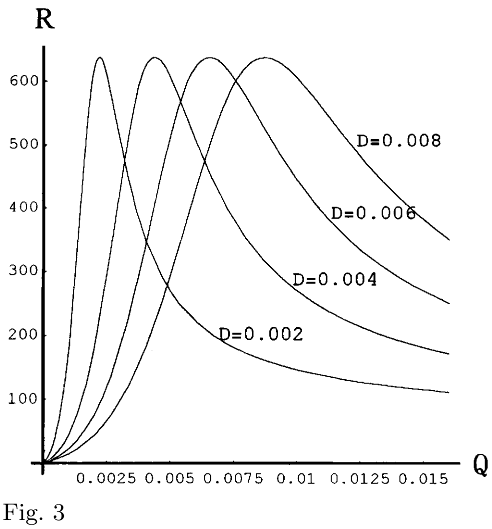

Fig. 3

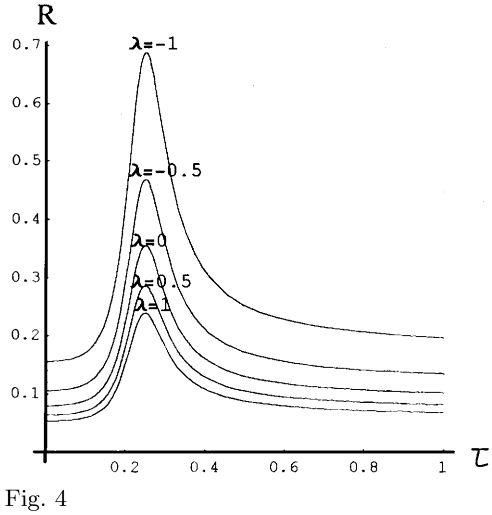

Fig. 4

Fig. 3 – The signal-to-noise ratio R as a function of noise intensity Q (eqs. (14)-(16)) for different values of the unmodulated noise intensity D. The values of the other parameters are $a = 20$, $\lambda = -0.96$, $\tau = 0.001$, $A_0 = 10$, and $\Omega = 0.3$.

Fig. 4 – The signal-to-noise ratio R as a function of noise correlation time τ (eqs. (14)-(16)) for different values of the noise correlation coefficient λ. The values of the other parameters are $a = 4$, $D = 15$, $Q = 1$, $A_0 = 1.6$, and $\Omega = 1$.

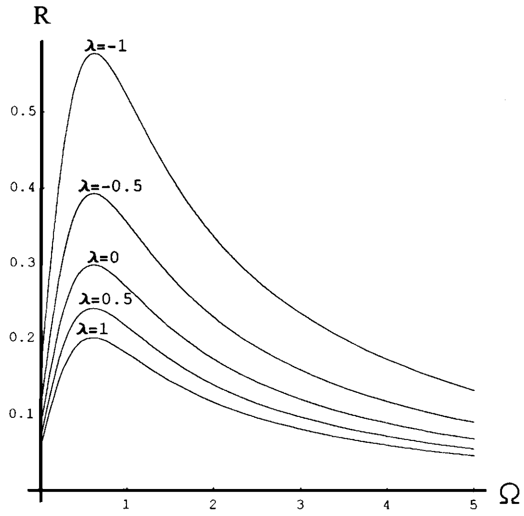

Fig. 5 – The signal-to-noise ratio R as a function of signal frequency Ω (eqs. (14)-(16)) for different values of the noise correlation coefficient λ. The values of the other parameters are $a = 4$, $D = 15$, $Q = 1$, $A_0 = 1.6$, and $\tau = 0.3$.

peak height does not change with Q but the peak position shifts to the right with increasing Q. Finally, we point out the fact that both the resonance peak height and the peak position of the R-D curve do not change with noise correlation time τ. And in the white-noise limit ($\tau \longrightarrow 0$), the resonance peak still exists.

ii) On the R-Q curve.

Below we describe the dependence of the signal-to-noise ratio R upon the noise intensity Q. Figure 3 shows that when we take the noise intensity D as a parameter, the peak height does not change with D but the peak position shifts to the left with decreasing D. When we take the noise correlation coefficient λ and the noise correlation time τ as a parameter, respectively, the behaviour of the R-Q curve is as follows. In the positive-correlation region ($0 \prec \lambda \preceq 1$), no SR exist. R decreases monotonously with increasing Q. However, in the negative-correlation region ($-1 \preceq \lambda \prec 0$) SR appears. And both the resonance peak height and the peak position of the R-Q curve do not change with noise correlation time τ. The resonance peak still exists in the white-noise limit ($\tau \longrightarrow 0$).

Comparing R-D curves and R-Q curves (figs. 2 and 3), one can see that when the noise intensity $Q = 0$, one has $R = 0$. But when the noise intensity $D = 0$, one has $R \neq 0$. This is required by our model and reflects the fact that the signal is multiplied to the noise $\xi(t)$ but not to $\eta(t)$.

iii) On the R-τ curve.

In fig. 4, we depict the dependence of the signal-to-noise ratio R upon the noise correlation time τ for different values of λ. It is clear that the resonance peak of the R-τ curve appears in the whole region of λ. The peak height increases as the noise correlation coefficient λ is decreased. But the peak position does not change with λ. The meaning of fig. 4 is as follows. The response of our linear system not only can be maximized by increasing the noise intensity D (or Q) up to an optimal value, but also can be maximized by increasing the noise correlation time τ up to an optimal value.

iv) On the R-Ω curve.

The dependence of the signal-to-noise ratio R upon the signal frequency Ω for different values of λ is described in fig. 5. Similar to the case of R-τ curve, the resonance peak of the R-Ω curve appears in the whole region of λ. The peak height increases as the noise correlation coefficient λ is decreased. The peak position does not change with λ.

In summary, the existence of a resonance peak of the R-D and R-Q curves in the negative-correlation region is a noise correlation effect. It must be emphasized that in the R-Q curve of fig. 3, the SR occurs even with very small values of the parameters D and τ. On the other hand, the existence of a resonance peak of the R-τ and R-Ω curves exhibits a signal-modulated effect of noise. As a matter of fact, when the signal is added to the noise $\acute{\xi}(t)$ but not multiplied to it, the resonance peak of the R-τ and R-Ω curves does not exist. It should be emphasized that the resonance peak of the R-τ and R-Ω curves exists even for uncorrelated noises, that is, $\lambda = 0$. This is very different from the case of dependence on the noise strengths D and Q. Finally, as an important application of the theoretical results of this paper, in another paper we have investigated the SR of a single-mode laser system that is driven by periodically modulated additive noises which are cross-correlated and find new SR characteristic features of the laser intensity output. This is based on a recent work in which the SR for a single-mode laser system driven by correlation additive noises that are unmodulated is studied [8]. The key point of ref. [8] is that in the regime of linearization [9], we get a linear equation of laser intensity and new characters of SR are revealed.

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