A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

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To cite this article: A. De Martino *et al* 2003 *EPL* **62** 14

EUROPHYSICS LETTERS 1 April 2003

Europhys. Lett., **62** (1), pp. 14–20 (2003)

Hierarchy of the equilibrium states of self-gravitating systems

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(received 14 November 2002; accepted in final form 28 January 2003)

PACS. 05.20.-y – Classical statistical mechanics.
PACS. 04.40.-b – Self-gravitating systems; continuous media and classical fields in curved spacetime.

Abstract. – We introduce a set of order parameters to classify the possible equilibrium states of rotating self-gravitating systems. As an example, the formation and thermodynamics of asymmetric binaries is discussed.

Gravity-dominated systems such as classical self-gravitating gases are known to possess highly non-standard thermodynamic properties [1–3]. The long-range nature of the Newtonian potential prevents extensivity [4], causing such features as the inequivalence of statistical ensembles, the occurrence of negative specific heats and of collapse transitions, and the existence of inhomogeneous ground states [5–9]. These attributes are not peculiar to gravitating gases alone. Similar traits mark the behaviour of systems with long-range, non-integrable potentials [10], of finite systems or of systems of a size comparable to the interaction range, such as atomic clusters or highly excited ("hot") nuclei [11, 12], are even present in certain short-ranged models [13] and in general at phase separations [14]. Besides the relevance to astrophysics and cosmology strictly speaking, understanding the equilibrium state of self-gravitating systems is hence important also from a broader, fundamental viewpoint [14].

Recently [15, 16] (to be later referred to as I), we investigated the possible equilibrium shapes of three-dimensional rotating self-gravitating gases via a microcanonical mean-field theory in which the system is confined in a finite spherical volume V with the total energy E and the total angular momentum $\boldsymbol{L}$ conserved, the short-distance singularity of the Newtonian potential being cured by the use of Lynden-Bell statistics [5]. At high E (kinetic energy dominates), the system lives in a homogeneous "gas"-like state, eventually deformed by rotation to a disk. At low E, instead, gravity induces a collapse to either a single dense "star"-like cluster (low $\boldsymbol{L}$), or to symmetric "binaries" (two identical dense clusters, high $\boldsymbol{L}$). These three phases are separated, at intermediate E and $\boldsymbol{L}$, by a large phase-coexistence region with negative specific heat, where dense structures (either a single cluster or a binary) embedded in a vapor are found.

Clearly, the above picture does not account for all possible equilibrium states. In particular, at high angular momenta it is to be expected that several maximum-entropy states exist, as a result of multiple bifurcations of the solutions of the equation describing entropy maxima. Such new solutions can be discovered only through the application of appropriate "fields", *i.e.* by devising appropriate order parameters. In this paper we introduce a convenient family of order parameters which allows to construct all such solutions, including "exotic" ones with negligible entropy, and classify them hierarchically. As an example, we describe the formation and thermodynamics of asymmetric binaries (which are far more common in the universe than symmetric ones and were excluded from I), along with some rare equilibria.

We begin by recalling the key steps in the theory of I. The Hamiltonian is $(i, j = 1, \dots, N)$

$$H = \frac{1}{2m} \sum_i p_i^2 - Gm^2 \sum_{i<j} |\boldsymbol{r}_i - \boldsymbol{r}_j|^{-1} \tag{1}$$

with $\boldsymbol{r}_i \in V \subset \mathbb{R}^3$ denoting the position and $\boldsymbol{p}_i \in \mathbb{R}^3$ the momentum of the i-th particle, m being its mass. G is the gravitational constant. The task is to find the particle density profiles that maximize Boltzmann's entropy $S = k \log W$, where the state sum W is given by

$$W = \frac{\alpha}{N!} \int \delta(H - E) \delta\Big(\boldsymbol{L} - \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i \Big) \, \mathrm{d}^{3N}\boldsymbol{r} \, \mathrm{d}^{3N}\boldsymbol{p} \tag{2}$$

with α a constant. W is a function of E and $\boldsymbol{L}$, as are S and the desired density profiles. The radius R of the box is taken to be fixed. Adhering to the notation of I, we shift to the reduced variable $\boldsymbol{x} = \boldsymbol{r}/R$, which we shall indicate as $\boldsymbol{x} = (x_1, x_2, x_3)$ (Cartesian representation) or $\boldsymbol{x} = (x, \theta, \phi)$ (spherical representation). Particles are assumed to have "hard cores" [5] and to fill a finite, not too large (to avoid jamming) volume fraction of the container. A simple way to implement the latter assumption is to choose the density profile c to satisfy $0 \le c \le 1$ and to be normalized as $\int c(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} = \Theta$. Θ is a parameter $(0 \le \Theta \le 4\pi/3)$ related to the fraction of V occupied by particles. Throughout this study, as in I, we take it to be $\Theta = 0.02$, a value that turns out to be sufficiently large to avoid dilution and sufficiently small to avoid overfilling. Finally, we take $m = 1$ and measure energy and angular momentum in units of GN^2/R and $\sqrt{RGN^3}$, respectively.

One can calculate W explicitly by combining a Laplace transform method (integrals over momenta) with an appropriate coarse-graining in real space (integrals over positions, see I for details). At stationary points of the entropy surface $S/k = \log W$, *i.e.* points where $\delta S/\delta c = 0$, it is found that the above-defined density profile c satisfies the condition

$$\log \frac{c(\boldsymbol{x})}{1 - c(\boldsymbol{x})} = \frac{\beta}{\Theta} \int \frac{c(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x}' + \frac{1}{2} \beta (\boldsymbol{\omega} \times \boldsymbol{x})^2 - \mu \,, \tag{3}$$

where

$$\beta = \frac{3/2}{\big[E - \frac{1}{2} \boldsymbol{L}^T \mathbb{I}^{-1} \boldsymbol{L} - \Phi[c] \big]} \,, \tag{4}$$

$\boldsymbol{\omega} = \mathbb{I}^{-1} \boldsymbol{L}$ is the angular velocity, and μ is a Lagrange multiplier implementing the constraint on Θ. $\mathbb{I}$ and Φ stand for the inertia tensor $\mathbb{I} = (I_{ab})_{a,b=1}^3$ in units of NR^2 and the gravitational potential in units of GN^2/R, respectively:

$$I_{ab}[c] = \frac{1}{\Theta} \int c(\boldsymbol{x}) \big(x^2 \delta_{ab} - x_a x_b \big) \, \mathrm{d}\boldsymbol{x}, \qquad \Phi[c] = -\frac{1}{2\Theta^2} \int \frac{c(\boldsymbol{x}) c(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{x}'. \tag{5}$$

In general, the solutions of eq. (3) can be written in spherical coordinates as

$$c(\boldsymbol{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm}(x) Y_{lm}(\theta, \phi), \tag{6}$$

where the "weights" $b_{lm}(x)$ of the spherical harmonics $Y_{l,m}$ have to be determined by solving (numerically) a system of integral equations whose precise form, discussed in I, is not important for our present purposes. If $\boldsymbol{L}$ lies along the 3-axis and if the center of mass is fixed at $\boldsymbol{x} = \boldsymbol{0}$, entropy-maximizing solutions of (3) can be divided into two main classes:

a) Axial-rotationally (around the 3-axis) symmetric ones, *e.g.* gas-cloud (high E, low L), distorted gas clouds or disks (high E, high L), and single clusters (low E, low L). These are independent of the ϕ angle, and thus for them the series (6) must involve only the *zonal* harmonics, with $m = 0$.

b) Axial-rotationally asymmetric ones, like double clusters or binaries. These, instead, must depend on ϕ and hence on harmonics with $m \neq 0$. They can be either *spatially symmetric* (like symmetric binaries, SBs) or *spatially asymmetric* (asymmetric binaries, ABs).

Each of these should be signaled by a non-zero value of a precise order parameter. Notice that, fixing the center of mass requires exclusion of $l = 1$ harmonics from (6), since defining $\langle \ldots \rangle := \int \ldots c(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ one has, using (6) and carrying out the appropriate integrations,

$$\begin{aligned} x_1^{\mathrm{CM}} &\equiv \langle x_1 \rangle = \sqrt{\frac{4\pi}{3}} \int x^3 b_{1,1}(x)\,\mathrm{d}x, \\ x_2^{\mathrm{CM}} &\equiv \langle x_2 \rangle = \sqrt{\frac{4\pi}{3}} \int x^3 b_{1,-1}(x)\,\mathrm{d}x, \\ x_3^{\mathrm{CM}} &\equiv \langle x_3 \rangle = \sqrt{\frac{4\pi}{3}} \int x^3 b_{1,0}(x)\,\mathrm{d}x. \end{aligned} \tag{7}$$

A convenient quantifier for homogeneous solutions and single clusters is

$$D_{0,0} = \left| \int x^2 b_{0,0}(x)\,\mathrm{d}x \right|. \tag{8}$$

It is simply a constant proportional to Θ, since, recalling that $\int c(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \equiv \langle 1 \rangle = \Theta$, one has $\int c(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 2\sqrt{\pi} \int x^2 b_{0,0}(x)\,\mathrm{d}x$.

For disk solutions, it is convenient to consider the quantity

$$\langle x^2 - 3x_3^2 \rangle = -\sqrt{\frac{16\pi}{5}} \int x^4 b_{2,0}(x)\,\mathrm{d}x. \tag{9}$$

In fact, for a homogeneous gas or, in general, for a perfect spherical cloud one would have $\langle x^2 - 3x_3^2 \rangle = 0$ evidently. When rotation deforms the cloud by shrinking it along the 3-axis, one has $\langle x^2 - 3x_3^2 \rangle \neq 0$. Hence for disks one can use the order parameter

$$D_{2,0} = \left| \int x^4 b_{2,0}\,\mathrm{d}x \right|. \tag{10}$$

The order parameter discerning axial-rotationally symmetric structures from axial-rotationally asymmetric ones must account for some ϕ-dependence. With the center of mass fixed at $\boldsymbol{x} = \boldsymbol{0}$, the lowest-order term in (6) that can provide such a feature is the *sectoral* harmonics $l = 2$. Indeed, the most convenient order parameter is

$$D_{2,2} = \left| \int x^4 b_{2,2}(x)\,\mathrm{d}x \right|, \tag{11}$$

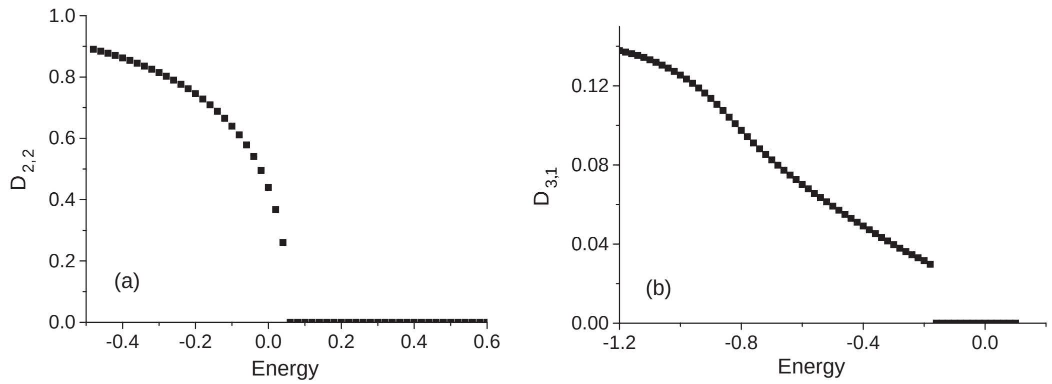

Fig. 1 – (a) Behaviour of D_{22} as a function of energy at $L = 0.6$. (b) Behaviour of $D_{3,1}$ as a function of energy at $L = 0.46$.

which is proportional to the difference between the diagonal components of the inertia tensor in the 1 and 2 directions. Using (5) and (6) it is in fact easy to see that

$$D_{2,2} = \Theta\sqrt{\frac{15}{16\pi}}\left|I_{11} - I_{22}\right|. \tag{12}$$

In I, $D_{2,2}$ was used as an order parameter for SBs, but it is evidently non-zero also for ABs. In oder to distinguish these two states, one needs a further order parameter. On physical grounds, the natural choice is $\langle x_1^3 \rangle$ (if the two clusters are aligned along the x_1-axis), since in spatially asymmetric structures this quantity must be non-zero because of evident symmetry reasons, at odds with what must happen in spatially symmetric structures. Now using (6) one can see by working out explicitly the integrals involving spherical harmonics that

$$\langle x_1^3 \rangle = \sqrt{\frac{12\pi}{25}}\int x^5 b_{1,1}(x)\,\mathrm{d}x - \sqrt{\frac{6\pi}{175}}\int x^5 b_{3,1}(x)\,\mathrm{d}x + \sqrt{\frac{2\pi}{35}}\int x^5 b_{3,3}(x)\,\mathrm{d}x. \tag{13}$$

In principle, all of the above terms should be considered. However, in any spatially symmetric structure (gas, single cluster, SBs) all of them are zero, because odd harmonics cannot contribute due to their symmetry properties. Each of them is non-zero only in the presence of spatial asymmetries, as in ABs; hence any of them can be employed as an order parameter. In particular, we find it convenient to use the quantity

$$D_{3,1} = \left|\int x^5 b_{3,1}(x)\,\mathrm{d}x\right|, \tag{14}$$

whose behaviour at $L = 0.46$ is shown in fig. 1b, together with the behaviour of $D_{2,2}$ (fig. 1a). At odds with SBs, ABs were not discussed in I, where odd harmonics were excluded from (6). It is therefore interesting to discuss their formation and thermodynamics at this point.

AB solutions of (3) appear suddenly at $E = -0.18$ for $L = 0.46$, where the order parameter jumps discontinuously from zero to a finite value (at odds with SBs which emerge continuously from class (a) solutions, see I). They exist and are stable in a small range of values of L up to $L \simeq 0.5$. This whole region lies in the mixed phase with negative specific heat of I, hence asymmetric binaries at equilibrium are embedded in a vapor and they do not exist as a

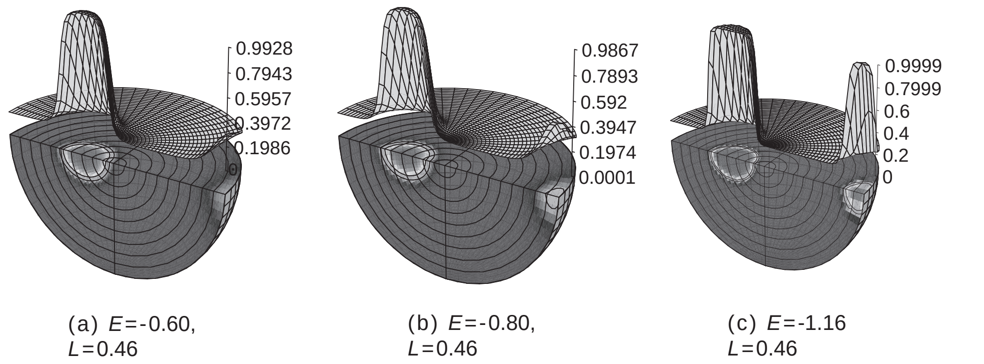

Fig. 2 – Three ABs obtained from eq. (3). The contour plot is shown together with the density profile. (a) Incipient binary, with a broad low-density body detached from the central cluster. (b) Intermediate stage. (c) Well-formed binary (very low energy).

"pure" phase (for a discussion about pure phases and phase separation in the microcanonical ensemble, see [14]). A sample of asymmetric binaries obtained in this way is shown in fig. 2. In their range of existence (except for their beginning) ABs are significantly more probable than SBs, as shown in fig. 3a, where the entropy of the different stable solutions at $L = 0.46$ is shown. A simple physical argument suffices to explain why at high angular momenta ABs must have a much lower entropy than SBs, if they exist at all. SBs have a larger moment of inertia than ABs, hence a smaller rotational energy. As a consequence, for SBs the "random" kinetic contribution $E - L^2/(2I)$ is larger than for ABs. Thus they have a larger entropy. So ABs are expected to be strongly suppressed or significantly less probable at high angular momenta. Indeed, we were unable to find stable ABs for $L > 0.5$. The fact that ABs tend to form the smaller cluster in proximity of the boundary of the box is due to the fact that they originate by the evaporation of small masses from a single large, dense cluster. The conservation of the center of mass relegates them to the border. In well-formed binaries, the peak of

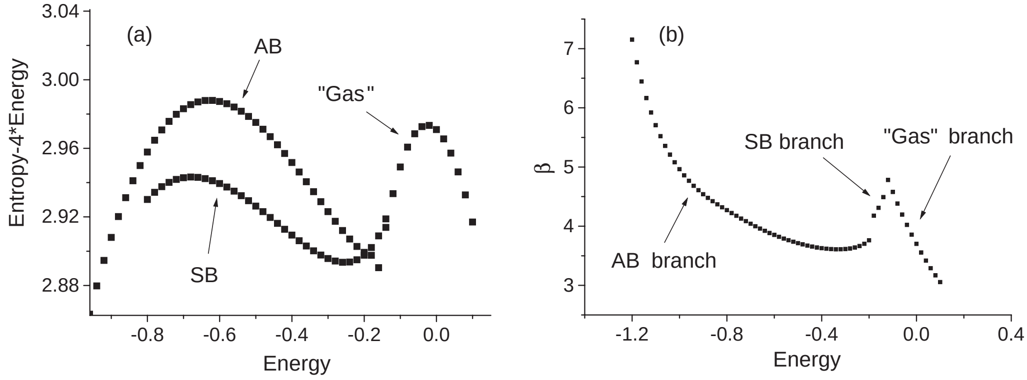

Fig. 3 – (a) Entropy of the different stable solutions of eq. (3) at $L = 0.46$. The quantity $4E$ has been subtracted from the entropy to make the separation and the curvature more evident. (b) Caloric curve β *vs.* E at $L = 0.46$. Notice that in spite of the fact that $\partial_E \beta < 0$ along most of the AB branch, the curvature of the entropy surface as a function of E and L, *i.e.* $S_{EE}S_{LL} - S_{EL}^2$, is negative along all of the AB branch.

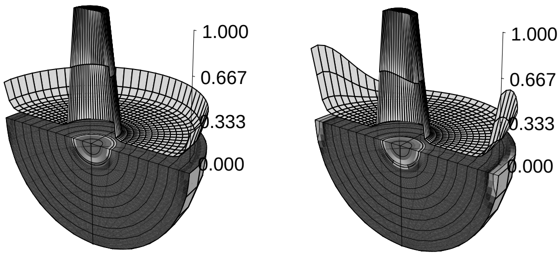

Fig. 4 – Two "exotic" solutions of (3) obtained via the order parameters (15). (a) Single cluster surrounded by a ring. (b) Single cluster with two "satellites". These solutions coexist at $E = -0.5$ and $L = 0.7$.

density in the smaller cluster is separated from the boundary. Interestingly, however, the AB entropy branch does not merge with the SB branch, implying that SBs cannot be formed by evaporating increasing amounts of particles from a central cluster. In fig. 3b the caloric curve $\beta = \partial S/\partial E$ *vs.* E is shown at $L = 0.46$. The jump occurring at $E = -0.2$, where ABs become more favorable than SBs, is an artifact of the asymptotics employed in deriving (3). (Being S a smooth and multiply differentiable function, such a feature is clearly spurious.)

On a general level, it is reasonable to argue from the above discussion that one can classify the different equilibrium states by introducing the family of order parameters defined by

$$D_{l,m} = \left| \int x^{l+2} b_{l,m}(x)\, \mathrm{d}x \right| \tag{15}$$

with integers $l = 0, 1, 2, \ldots$ and $m = -l, -l+1, \ldots, l$. As we have seen, for small l, each of these functions has a simple physical interpretation. (As for the remaining $D_{2,m}$'s, they can be easily seen to be related to the off-diagonal components of the inertia tensor (5).) It is clear that higher-order D's are all connected to homogeneous polynomials in x_1, x_2, x_3, which are in turn connected to the Cartesian representation of spherical harmonics [17]. For example, for $l = 3$, $D_{3,0}$ is connected to the quantity $\langle 2x_3^3 - 3x_1^2 x_3 - 3x_2^2 x_3 \rangle$. Their physical interpretation for larger l is however less straightforward.

Using these order parameters, solutions can be classified hierarchically. In our setting, with the center of mass fixed at $\boldsymbol{x} = \mathbf{0}$ one has the following picture. Homogeneous gas clouds have all D_{lm}'s equal to zero except $D_{0,0}$. For deformed clouds (disks) only $D_{l,0}$ with l even are non-zero. For symmetric binaries (which are connected to "gas"-like solutions continuously, see fig. 3a), also $D_{2,2} \neq 0$ (and, in general, all $D_{l,m}$ with l and m even), all other D's being zero, while for asymmetric binaries $D_{3,1}$ is also not zero. This classification can be continued further, for the order parameters (1) allow to construct all possible solutions of (3), by adding a proper field that selects the solution for which the chosen order parameter is not zero. As an example, we show in fig. 4 two of several equilibria that can be obtained in this way at very high angular momentum. Of course, such "exotic" solutions of (3), if stable, have much lower entropy than those discussed in I or than ABs (also, as they are stitching to the walls of the container, they are mostly of academic interest).

To summarize, we have introduced a family of order parameters that allow to characterize the different equilibrium shapes that can be encountered (homogeneous gas, disk, symmetric binary, asymmetric binary, along with more exotic states) in self-gravitating systems, and discussed their physical meaning in simple cases. In particular, we have used such order parameters to describe the formation and thermodynamics of asymmetric binaries, which require odd terms in (6) and were excluded in I. The present study does not substantially alter the global phase diagram presented there, since asymmetric binaries do not exist as a "pure" thermodynamic state. Among the problems that remain open we would like to mention the following two. First is the dependence on Θ. Our results have been obtained at fixed $\Theta = 0.02$. Different Θ's may lead to different results. By letting $\Theta \to 0$ (in which limit Lynden-Bell statistics goes to Boltzmann statistics and the l.h.s. of eq. (3) goes to the Boltzmann form $\log c(\boldsymbol{x})$), one obtains the case of overlapping particles (no hard cores), which is a classical problem in cosmology dating back to [18]. The rotating case is dealt with in [19]. For such "Boltzmann" particles we have not found any evidence of asymmetric binaries. On the other hand, when Θ grows ($\Theta \leq 4\pi/3$) the system may become too closely packed and the interesting features one observes for small Θ might disappear at some point. The second problem is the dependence on R. Throughout this paper we have assumed that R is fixed, but it would be interesting to analyze these questions in a context in which R varies, for instance to mimic an expanding-universe scenario as suggested, *e.g.*, in [20].

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