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Incomplete ordering of the voter model on small-world networks

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Abstract. – We investigate how the topology of small-world networks affects the dynamics of the voter model for opinion formation. We show that, contrary to what occurs on regular topologies with local interactions, the voter model on small-world networks does not display the emergence of complete order in the thermodynamic limit. The system settles in a stationary state with coexisting opinions whose lifetime diverges with the system size. Hence the nontrivial connectivity pattern leads to the counterintuitive conclusion that long-range connections inhibit the ordering process. However, for networks of finite size, for which full uniformity is reached, the ordering process takes a time shorter than on a regular lattice of the same size.

In the last decade, social sciences have started to deal with large-scale modeling of a variety of spreading and ordering phenomena that involve cooperative behavior [1]. In this context, classical models developed in statistical physics to study the onset of order in matter [2] have turned out to be useful for the investigation of the principles at the basis of social ordering. For instance, the Ising and voter models and their variations are prototypical models for a wide class of social interaction phenomena [3–9]. The voter model is possibly the minimal model for opinion spreading and the study of the onset of consensus. It is usually defined on a regular lattice of dimension d. Each site is characterized by a discrete variable s that may assume two values ($s = \pm 1$) representing two opposite opinions; for instance, the electoral choice in favor of two different candidates. Starting from a disordered initial configuration, the model follows a simple dynamical evolution in which at each time step one site is selected at random and made equal to one of its nearest neighbors (chosen at random on its turn). This dynamics mimics the homogenization of opinions through the confrontation of peers and leads to the formation and coarsening of ordered regions where individuals share the same opinion. In $d = 1$ and 2 the model eventually converges to an ordered state with all variables

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having the same value [10]. This state is *absorbing* since the system cannot escape from it once it is reached [11].

While the use of regular lattices to model the interaction between elementary objects is well justified for most physical situations, such an assumption is not obvious in the context of social sciences. Many social systems indeed show interaction patterns that find a better characterization as complex networks with distinctive connectivity properties [12, 13]. In a graph representation, where nodes identify the individuals and links their direct interactions, many social and natural networks exhibit peculiar topological properties related to the presence of highly connected individuals and long-range connections. Among these features, the most well documented is the small diameter of social networks, *i.e.* each individual can reach any other one passing through a very small number of intermediate nodes. In addition, social interactions favor the connection between common acquaintances leading to the presence of high clustering among nodes. This is quantitatively expressed by a high probability that if two nodes share a neighbor they are directly connected on their turn. This last property, along with the small diameter, defines the so-called small-world behavior [14, 15]. The use of small-world–like topologies in models of opinion spreading is then a logical step in the direction of a more realistic approach to the phenomenon. The interaction patterns involved in the process of opinion formation are very likely similar to those of social networks such as the web of sexual contacts [16] or scientific collaborations [17].

In order to investigate how complex connectivity patterns might influence opinion formation, we study the effect of the small-world topology on the evolution of the voter model. The prototypical network possessing the small-world character is the Watts-Strogatz (WS) model that has been extensively studied in several contexts [14, 15, 18]. In particular, we have considered the WS network as defined in ref. [14]. Starting from a one-dimensional lattice of N sites with periodic boundary conditions and each node connected with $2k$ nearest neighbors, a stochastic rewiring is introduced. Nodes are visited one by one sequentially and each of the k links connecting the node to its nearest neighbors in the clockwise sense is rewired with probability p to a randomly chosen node. As p is increased, the WS network interpolates between a one-dimensional lattice ($p = 0$), with only geographical neighbors in contact, and a random graph ($p = 1$), where short- and long-range connections are equally likely. The small-world behavior (small diameter, high clustering) is exhibited for values of the rewiring probability p such that $1/(kN) \ll p \ll 1$. The transition between the one-dimensional topology and the small-world one occurring for $p \approx 1/(kN)$ is governed by the value of ξ, the average distance between nodes connected with shortcuts. ξ is the only nontrivial length in the network and can be simply shown to scale as $1/(kp)$ [19, 20]. If the network size N is much smaller than ξ, that is $p \ll 1/(kN)$, the system does not have long-range connections and is a one-dimensional lattice. For $N \gg \xi$, many shortcuts are present and originate the small-world behavior.

For the study of the ordering of the voter model, the natural quantity of interest is the fraction n_A of active bonds, *i.e.* the density of links connecting sites with opposite values of s. These are the links where the dynamics takes place. Such a quantity vanishes if the system orders entirely, while it remains finite if domains coexist, and $1/n_\mathrm{A}$ is a measure of the average size of domains. In the case $p = 0$ (one-dimensional system), due to the diffusive motion of interfaces between domains, the fraction of active bonds decays with respect to time t as $n_\mathrm{A} \sim t^{-1/2}$ [21] up to a crossover time $t_0 \sim N^2$, after which n_A exhibits an exponential relaxation to the absorbing state $n_\mathrm{A} = 0$ (see fig. 1). Such a fast decay is the effect of the finite size of the system. In the thermodynamic limit one recovers a pure power law relaxation to the absorbing state.

We consider now a Watts-Strogatz network with $k = 2$, $p = 0.05$ and values of N such that the system is safely inside the small-world regime ($N \geq 200$). The behavior changes

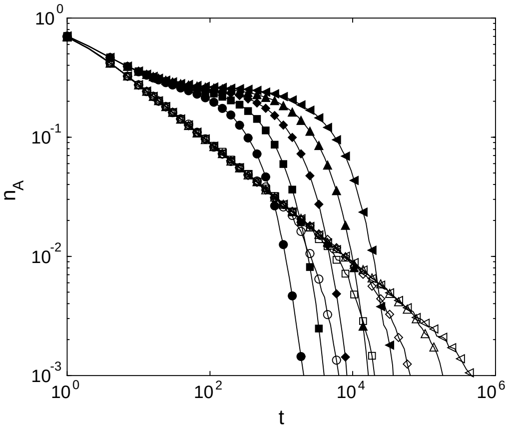

Fig. 1 – Log-log plot of the fraction n_{A} of active bonds between nodes with different opinion. Values are averaged over 1000 runs. Time is measured in Monte Carlo steps per site. Empty symbols are for the one-dimensional case ($p = 0$). Filled symbols are for rewiring probability $p = 0.05$. Data are for $N = 200$ (circles), $N = 400$ (squares), $N = 800$ (diamonds), $N = 1600$ (triangles up) and $N = 3200$ (triangles left).

dramatically (fig. 1): After a transient, the plot of n_{A} exhibits a plateau, indicating that domains remain on average of constant size. This regime is ended by an exponential approach to the absorbing state $n_{\mathrm{A}} = 0$. The duration of the plateau grows with N, but (as can be seen from fig. 1) the time to reach complete ordering is *smaller* for the WS network than for a regular lattice with the same number of nodes. This is in agreement with the naive expectation that long-range connections should speed up the homogenization process. However, this occurs in a highly nontrivial way: During most of the evolution, n_{A} is higher on the small-world network than on a regular lattice, *i.e.* the small-world network is, for a long time interval, more disordered, and orders rapidly only at the very end. We have checked that this phenomenology is not an artifact of the rewiring procedure to build the Watts-Strogatz network: analogous results are obtained when the small-world topology is produced by adding random links to a one-dimensional lattice.

The nature of the exponential approach to the absorbing state is related to a standard finite-size effect in the presence of absorbing states. Any finite system settles in a stationary state with constant activity until it hits the absorbing state because of a large spontaneous fluctuation [11]. The survival probability $P_{\mathrm{s}}(t)$ that the system is still in an active state after time t decays exponentially, $P_{\mathrm{s}}(t) \sim \exp[-t/\tau]$. Here τ is the average lifetime in the active state and is found to increase with the system size as $\tau \sim N$ (fig. 2, inset). This implies that in the thermodynamic limit ($N \to \infty$), the system remains indefinitely in the stationary state, with everlasting activity, *i.e.* incomplete ordering. Hence, the voter model in the small-world regime behaves as its mean-field version (Euclidean lattice with $d = \infty$) that does not reach an ordered state [21]. This finding is quite interesting: in the thermodynamic limit, the presence of long-range connections does not make the ordering process easier, rather it inhibits it. The small-world topology of the network represents a barrier against convergence to order.

In order to understand better the origin of this incomplete ordering, we study networks with $N = 10^5$ nodes, for which the lifetime τ is much larger than the time scales of interest and may therefore be considered in practice as infinite. In this case, as long as $p > 10^{-5}$, the network is in the small-world regime and n_{A} tends to a finite stationary value (see fig. 3). The

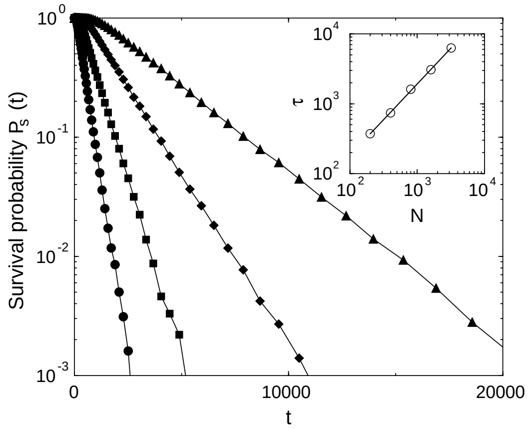

Fig. 2 – The survival probability $P_{\mathrm{s}}(t)$ as a function of time for a network with $p = 0.05$. Data are for $N = 200$ (circles), $N = 400$ (squares), $N = 800$ (diamonds) and $N = 1600$ (triangles) and are computed averaging over 1000 runs. Inset: the average lifetime in the active state τ as a function of the system size N. Circles are numerical values obtained by measuring τ as the inverse slope of the curves shown in the main part of the figure. The solid line is the best fit to the expression $\tau \propto N^{\gamma}$ giving $\gamma = 1.02 \pm 0.02$.

network settles in a dynamically active regime in which complete consensus does not emerge. The value of n_{A} in the stationary state depends on p. In the one-dimensional lattice, the ordering process takes place via free diffusion of domain boundaries (active bonds) and their annihilation upon encounter. In the small-world networks, shortcuts are an obstacle for free diffusion of active bonds and tend to pin domain boundaries. More in detail, the role played by sites connected with shortcuts underlies the following simple scaling analysis, valid for $t \gg 1$ and small p. For short times domains form and start to coarsen: $1/n_{\mathrm{A}}$, their average size, is

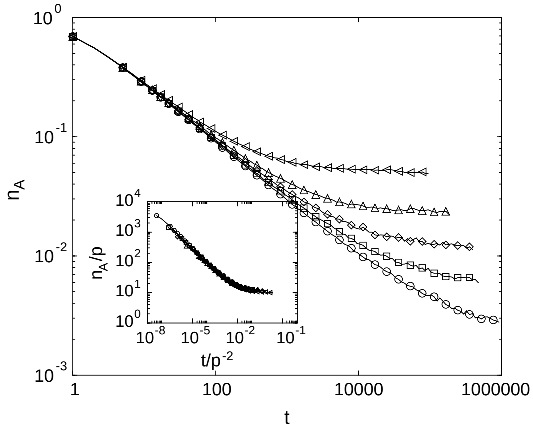

Fig. 3 – The fraction n_{A} of active links as a function of time for a large network with $N = 10^5$ nodes. Data are for $p = 0.0002$ (circles), $p = 0.0005$ (squares), $p = 0.001$ (diamonds), $p = 0.002$ (triangles up) and $p = 0.005$ (triangles left) Inset: the same data, divided by p and plotted as a function of t/p^{-2} in order to show the validity of eq. (1).

much smaller than the average shortcut distance $\xi = 2/(kp)$. In this regime the evolution is practically equal to the one-dimensional voter model, with n_{A} decreasing as $t^{-1/2}$. When the length of domains reaches the size of ξ, the only other length in the problem, the behavior changes. The crossover to the stationary state occurs therefore when $1/n_{\mathrm{A}} \sim \xi$, defining a diverging crossover time $t^* \sim p^{-2}$. Since the crossover takes place at $n_{\mathrm{A}} \sim p$, we obtain that

$$n_{\mathrm{A}}(t,p) = p\mathcal{G}\big(t/p^{-2}\big), \tag{1}$$

where the scaling function $\mathcal{G}(x)$ approaches a constant value for $x \gg 1$. Equation (1) is well obeyed by numerical results, that show a good data collapse on the predicted behavior (fig. 3, inset).

In summary, we have shown that complex topological properties of small-world networks strongly affect the behavior of the voter model, leading to incomplete ordering in the thermodynamical limit and to counterintuitive phenomena for small systems. We believe that the behavior of more general models for social influence [22] is similarly modified by the topology of the interaction network. It would be interesting to test on these models the effect of more heterogeneous topologies, such as scale-free networks, which are known to alter several dynamical processes occurring on them [23–25].

During the completion of this work, we have become aware of some recent work [26] on the ordering process of the Ising model on the WS network, presenting conclusions somewhat similar to ours.

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