Directed and diffusive contributions to urban traffic flow patterns

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Directed and diffusive contributions to urban traffic flow patterns

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Abstract. – As a result of an extensive field trial in Frankfurt/Main, the urban traffic pattern is represented by a distribution of local entropy variables. This letter aims at reproducing this distribution by use of two arguments —a geometrical and a hierarchical one. A characteristic scaling relation for urban traffic is derived.

Introduction: urban traffic patterns. – Vehicular traffic flow has for some time been in the focus of physical research. Initially, problems of the collective vehicle dynamics were tackled either by the hydrodynamic school of modelling [1–5], or by the computer-friendly school of cellular automata (CA) [6, 7]. Meant to describe predominantly motorway phenomena where long road segments without structural attributes allow for the relaxation of complex vehicle-vehicle interaction processes, the road network itself as a structural variable did not figure in these early works. In urban traffic the vehicular flow is, however, primarily shaped by properties of just this network rather than the vehicle-vehicle interactions: the (co-operative) design of traffic signals, individual route choice behaviour, the hierarchicity of different road classes and legal regulations determine the urban traffic patterns at least as much as the sheer transport volumes. Recently, several workers have addressed different facets of this spectrum of urban traffic, *e.g.*: the authors of [8] have presented a detailed analysis of the turning dynamics at non-signalled intersections; the authors of [9] and [10] investigated the interaction of geometrically interacting vehicle flows in a specially designed (urban) road network (with [11] as an important theoretical trail-blazer) and the authors of [12] introduced elements of game theory into the analysis of drivers' route choice behaviour.

The present letter makes use of a recent development in traffic monitoring: so-called *floating car data* (FCD) allow to track specially equipped vehicles continually (depending on configuration) over their trajectories in a considered road network. (From the physicists' point of view, FCD are equivalent to tracer dynamics, *i.e.* to reconstructing a flow picture from information on the behaviour of representative particles.) In particular, turning relations at intersections are immediately accessible because of the vectorial character of FCD such that local origin-destination matrices (ODM) can be directly measured. Given a sufficient spatio-temporal density of the FCD, it is thus principally possible to map the complete origin-destination flows patterns in a covered network. This is a major potential progress in comparison with the reconstruction algorithms for ODMs from counting data [13–15] as had to be used before FCD became technologically possible.

In an extended field trial in Frankfurt/Main sponsored by the German Higher Education and Research Ministry (BMBF) [16], FCD were gathered in an urban context by a representative fleet (*i.e.* no taxis or other special purpose vehicles were involved which might have biased the data). Due to the limited number of measuring vehicles, it was, however, not possible to extract validated local ODMs in a sufficient number. Therefore, an alternative statistical approach has to be devised to access the directional component contained in the FCD.

Local flow entropy distribution. – Let the local partial flows at a fixed intersection α be designed by $q_{ij}^{(\alpha)}(t)$, where i denotes the incoming road segment and j the outgoing one (the term "segment" here means the link between two intersections as given by the considered road network). With i and j running over all segments joining at intersection α, the $q_{ij}^{(\alpha)}(t)$ constitute the time-dependent local ODM. Then, obviously, at any time t the flow balance has to hold:

$$\sum_{ij} q_{ij}^{(\alpha)}(t) = \sum_{i} q_{i,\text{in}}^{(\alpha)}(t) = \sum_{j} q_{j,\text{out}}^{(\alpha)}(t) = Q^{(\alpha)}(t), \tag{1}$$

introducing *en passant* the gross segment flows for incoming ($q_{i,\text{in}}^{(\alpha)}$) and outgoing ($q_{j,\text{out}}^{(\alpha)}$) segments, respectively. Since all the partial flows are counted positive, they do not sum up to zero as is often the convention used in other physical contexts. As mentioned in the introduction, the set of ODMs $\{q_{ij}^{(\alpha)}(t)\}_\alpha$ would describe the traffic flow patterns in the considered network completely but is —due to lack of data— not accessible. Two restrictions have to be acknowledged: First, the temporal resolution of the data is insufficient, which is why any time dependence in the flow patterns is dropped from here on, and only the global average (24 hours for seven days a week) are taken into consideration. Second, the measured local ODMs are rarely complete, *i.e.* there are unmeasured elements $q_{ij}^{(\alpha)}$. As this may lead to erroneous conclusions, the introduction of reduced variables is suggested. Setting

$$\begin{aligned} s_i^{(\alpha)} &= -\sum_j p_{ij}^{(\alpha)} \ln p_{ij}^{(\alpha)}, \\ p_{ij}^{(\alpha)} &= q_{ij}^{(\alpha)} / q_{i,\text{in}}^{(\alpha)}, \end{aligned} \tag{2}$$

the dimensionality of the problem is reduced and replaced by the entropy-like variable $s_i^{(\alpha)}$. For the following discussion, the intersections α themselves and their relations with one another are not taken into consideration; it is the distributive character of the segments alone which shall be studied. Therefore, the upper index α is dropped and a new index s introduced. With s running now over all intersections and adjoined segments,

$$s_i^{(\alpha)} \Longrightarrow s_s, \tag{3}$$

a statistical analysis of the distributive behaviour of urban road networks becomes feasible. The segment output entropies s_s range between zero for segments with minimal "output disorder" (*i.e.* $p_{ij^*}^{(\alpha)} = 1$ and $p_{ij}^{(\alpha)} = 0 \ \forall j \neq j^*$; note how the usual order interpretation of entropy changes sign here) and a maximal value for equidistributed output. This maximal value depends on the number N_s of available output segments.

The distribution $P(s_s)$ of the segment output entropies over the segment ensemble can now be understood as a flow distribution "fingerprint" of any considered road network. Having

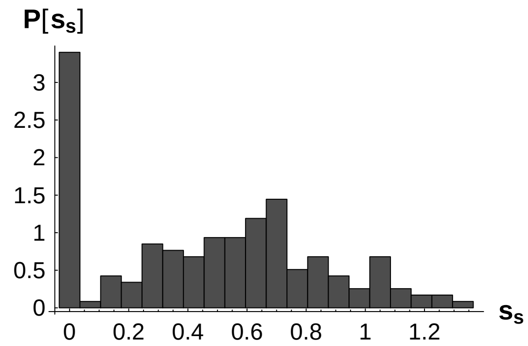

Fig. 1 – Distribution of segment output entropies $P(s_s)$ as measured in an extended field trial in Frankfurt/Main between spring and fall 2002.

stated the inaccessability of the full set of local ODMs above, $P(s_s)$, although containing fewer information, shall be regarded as an alternative statistical expression. Being a statistical quantity, it is not necessary that the segment ensemble underlying $P(s_s)$ comprises the entire road network as long as it is a representative subset. In fig. 1 the distribution $P(s_s)$ as was measured for Frankfurt/Main is depicted. The statistical basis consists of 168 distributive segments s, each having a minimum of 10 recorded turnings as the basis for the p_{ij}. The total number of measurement points underlying $P(s_s)$, having been obtained in a period between spring and fall 2002, is $\sim 1.3 \times 10^5$.

Directed and diffusive flow. – Theoretical reproduction of the distribution $P(s_s)$ is started from a simple model and based on two arguments. The model is shown in fig. 2: In a circular area thought to represent the entire urban area, two road classes with respect to their transportation properties are considered. Low-hierarchical roads sustaining relatively small flows are modelled —in maximal simplification— to form a uniform continuous background density. Due to the vectorial character of road segments, this density cannot be an

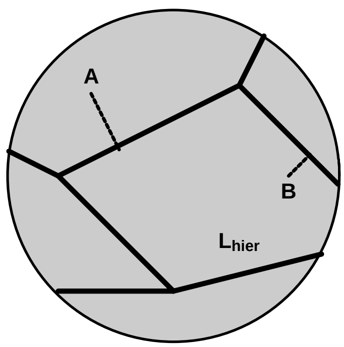

Fig. 2 – Model of an urban road network: a backbone of high-hierarchical road segments (thick lines) complements the smeared-out, uniform background of low-hierachical roads (grey shade). Typical travels use the backbone for fast long-haul portions and the background network for local delivery (in the sketch from A onto the backbone and from the backbone to B); see text.

areal density, but has to be a tupel of line densities in x- and y-dimensions:

$$\{\rho_x, \rho_y\}, \quad \text{where } \rho_x = \rho_y = \rho. \tag{4}$$

The second class, the high-hierarchical roads for large flows, form a backbone on this uniform background of total length L_{hier}.

In the following, two arguments are presented from which the distributive character $P(s_s)$ is argued to derive.

Hierarchical transport. – First, the phenomenon $s_s = 0$ is studied. The route for any trip $A \Rightarrow B$ (see fig. 2) follows from a so-called cost function minimization. In the two-class model introduced above, the total "cost" C_{AB} of the trip $A \Rightarrow B$ consists of the weighted gross distances driven on the high-hierarchical backbone (l_{hh}) and the low-hierarchical background (l_{lh}):

$$C_{AB} = \epsilon_{\text{hh}} l_{\text{hh}} + \epsilon_{\text{lh}} l_{\text{lh}}, \tag{5}$$

where the ϵ are the respective weights (they can, for instance, be thought of as travel time per path element if only a travel time optimization is sought after). Minimizing C_{AB} over all possible paths yields the favourable route. In the context of this letter, a detailed analysis of the ϵ is not required —they can, without loss of too much reality, be set $\epsilon_{\text{lh}} \gg \epsilon_{\text{hh}}$, such that the l_{lh} are just the perpendiculars onto the shortest backbone path from origin and destination (as sketched in fig. 2).

Now, the argument for $s_s = 0$ can be spelled out by studying the traversed intersections: The portion of the path $A \Rightarrow B$ that is leading over the backbone consists, according to the route choice mechanism presented above, of $\mathcal{O}(1)$ intersections where the route backbone links to the background, or branchings occur on the backbone itself. These intersections contribute to the diffusive parts of $s_s > 0$ in $P(s_s)$. On the other hand, the number of intersections of the backbone with the background where the flow stays directed on the backbone (corresponding to $s_s = 0$) is found as $2l_{\text{hh}}\rho \gg \mathcal{O}(1)$. This indicates that traffic on the backbone is predominantly directed. This argument holds for arbitrarily varying A and B. Introducing the total number of observed trips T, the number of diffusive intersection crossings on the backbone is

$$N_{\text{diff}} \sim \mathcal{O}(T). \tag{6}$$

The number of directed intersection crossings on the backbone, however, is given as

$$N_{\text{dir}} \sim 2\rho T \lambda_{\text{bb}}(L_{\text{hier}}), \tag{7}$$

where $\lambda_{\text{bb}}(L_{\text{hier}})$ denotes the trip fraction on the backbone averaged over all trips. As $N_{\text{dir}} \gg N_{\text{diff}}$, the peak at zero in fig. 1 becomes plausible.

Diffusive transport. – Having established the character of the high-hierarchical backbone in an urban road network, now the character of intersections in the low-hierarchical background shall be studied. In fig. 3 the basic idea is illustrated: traffic inflow from the negative x-direction is, assuming the three rectangular output channels along the positive x-axis and both directions of the y-axis (the most common configuration in real traffic networks), distributed according to the geometrical probability of destinations. This implies the assumption of a constant value for the areal density of destinations B over the entire urban area. Also, destinations with smaller x-values than that of the intersection —implying turning backwards— are not considered. It should be noted that, as any incoming traffic can be mapped onto the positive x-direction, the distribution model in fig. 3 is radially symmetric. The segment

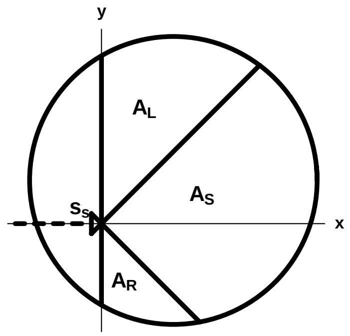

Fig. 3 – Geometrical flow distribution on the road network background. Incoming traffic is only propagated in non-negative x-direction into sectors that derive from the number of output segments (here, the most common number three). The propagation probabilities are thought to be proportional to the areas A_{L}, A_{S} and A_{R} containing all potential destinations.

output entropies are now directly accessible by simple areal integration when the turning-off probabilities p_{ij} in eq. (2) are identified as

$$\begin{aligned} p_{\mathrm{left}} &= A_{\mathrm{L}}/(A_{\mathrm{L}} + A_{\mathrm{S}} + A_{\mathrm{R}}), \\ p_{\mathrm{straight}} &= A_{\mathrm{S}}/(A_{\mathrm{L}} + A_{\mathrm{S}} + A_{\mathrm{R}}), \\ p_{\mathrm{right}} &= A_{\mathrm{R}}/(A_{\mathrm{L}} + A_{\mathrm{S}} + A_{\mathrm{R}}). \end{aligned} \tag{8}$$

Letting the intersection —and thereby the in-feeding segment s— vary over the entire circular urban range, a distribution of the s_s can be constructed by use of the above geometrical expressions and eqs. (2). It is clear that this distribution cannot produce values $s_s \gtrsim 0$, as there is no position of the intersection possible where one of the areas A_{L}, A_{S} or A_{R} will be very much larger than the others.

Figure 4 shows the theoretically obtained distribution $P^{\mathrm{t}}(s_s)$ as the sum of the δ-like peak at $s_s = 0$ and the geometrical distribution discussed above:

$$P^{\mathrm{t}}(s_s) = c_\delta \delta(0) + P^{\mathrm{geom}}(s_s). \tag{9}$$

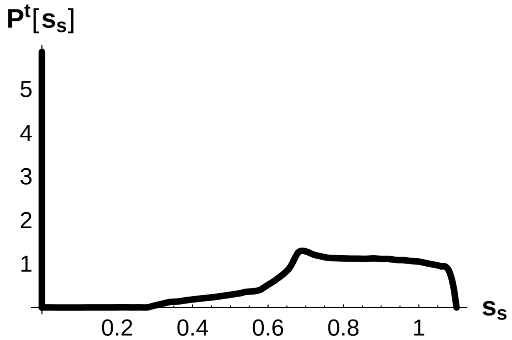

Fig. 4 – Theoretical distribution of segment output entropies. The δ-like peak at zero reflects the directed character of traffic on the high-hierarchical road network backbone, whereas the non-zero part results from a geometrical argument for the low-hierarchical background.

Table I – *Maximal values for segment output entropies depending on the number of available segments.*

N_s	$\max[s_s]$
2	0.6931
3	1.0996
4	1.386
...	...

The calibration of the δ peak at zero is based on a least-square fit for the geometrical part $P^{\rm geom}(s_s > 0)$ onto the corresponding part of the measured distribution, fig. 1, and the bar width used in the coarse-graining of $P(s_s)$:

$$c_\delta \simeq 0.41. \tag{10}$$

Discussion. – When comparing figs. 1 and 4, it can first be stated that the general shape is reproduced. Furthermore, the maxima of $P(s_s > 0)$ and $P^{\rm geom}(s_s)$ coincide remarkably well. The flanks of these partial distributions are, however, not well reproduced. This is a consequence of the applied road class model (fig. 2), which neglects finer gradation of the road class hierarchy in favour of unrealistic homogeneity and symmetry. Especially the range $0.2 \lesssim s_s \lesssim 0.6$ is underestimated in the geometrical argument (8) due to missing intermediate parts in the road class hierarchy. Also, the above geometrical argument applied to intersections with two output channels would contribute to this part of the distribution. Comparing the maximal values $\max[s_s]$ in P and $P^{\rm t}$ with table I, it becomes apparent that intersections with four output segments seem to play the leading role as traffic distributors in realistic urban road networks such as the studied one in Frankfurt/Main. The heights of the peaks at $s_s = 0$ are of comparable order; here, the error in the least-squares fit of $P(s_s > 0)$ and $P^{\rm geom}(s_s)$ enters the calibration.

Finally, a scaling relation can be derived by comparing the estimate (7) and the statistical basis for $P^{\rm geom}$ (*i.e.* the number ν of all intersection crossings in the background road network) via the calibration (10). The number ν results from summing up the product of background density and average length $\lambda_{\rm bg}$ of the perpendiculars in the background onto and away from the backbone over the total number of trips T:

$$\nu = 2\rho T \lambda_{\rm bg}. \tag{11}$$

Then, one finds the ratio of the average trip fractions on the backbone and the background of the road network as one of the key descriptive quantities for urban traffic:

$$R = \frac{\lambda_{\rm bb}}{\lambda_{\rm bg}} \simeq \frac{0.41}{1-0.41} \simeq \frac{2}{3}\,. \tag{12}$$

It should be noted that R requires the input of $\lambda_{\rm bb}(L_{\rm hier})$ which cannot be deduced from the model assumptions. Hence, the estimate (12) is based on the observation data via the least-squares fit mentioned above. When testing the relation (11) for its practical relevance, commuters may become suspicious. It should be pointed out, however, that the introduction of only two road classes leaves much room for definitory freedom, and that the presented model contains short-range trips as well. Furthermore, the problem of linking the considered urban area to its surrounding suburbs and long-haul traffic has been left untouched.

In conclusion, it can be stated that the measured distribution of segment output entropies in urban road networks can be reproduced by theoretical arguments. The important ingredients are hierarchicity of the road network and a geometrical *ansatz* for the local flow distribution. An important question for future studies is whether the experimental diagram fig. 1 possesses universal character, *i.e.* whether it can be reproduced in similar shape for different urban road networks. Also, the fundamental traffic constant R should be tested *vs.* more refined hierarchical road class models.

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