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Local fluctuation dissipation relation

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Abstract. – In this letter I show that the recently proposed local version of the fluctuation dissipation relations follows from the general principle of stochastic stability in a way that is very similar to the usual proof of the fluctuation dissipation theorem for intensive quantities. Similar arguments can be used to prove that in an aging experiment, where the averages are done over many runs, all sites stay at the same effective temperature at the same time.

The fluctuation dissipation relations in off-equilibrium dynamics are a crucial tool to explore the landscape of a disordered system. These fluctuation dissipation relations are different from the predictions of the fluctuation dissipation theorem at equilibrium. The existence of these new relations has been firstly proved in mean-field theories [1–4]; however it has been later shown that they follow from the general principle of stochastic stability [5–8], that, roughly speaking, asserts that a random perturbation acts in a smooth way.

The fluctuation dissipation relations can be expressed in a rather simple form that can be interpreted using the concept of the effective temperature [9, 10]. Moreover, the main parameters entering into the fluctuation dissipation relations have a simple interpretation in equilibrium statistical mechanics [1, 2, 8]: for a given system, the fluctuation dissipation relations do not depend on how the system is put in an off-equilibrium situation (as soon as it remains slightly out of equilibrium). These fluctuation dissipation relations have been amply observed in numerical simulations [11–14] and quite recently in real experiments [15–17].

Observables that are the average over the whole sample are the mostly studied case. In this case, there is a static-dynamic relation that connects the dynamic fluctuation dissipation relations to the static average of global quantities. Recently, there have been a few investigations on fluctuation dissipation relations that involve only local variables [18, 19]. The aim of this letter is to derive these local fluctuation dissipation relations in a more general framework and to find the appropriate static-dynamic relation. In order to reach this goal, we have to introduce a new concept: the probability distribution of the single spin overlap, a generalization of the probability distribution of the total overlap of the system.

Let us recall the usual equilibrium fluctuation theorem. If we consider a pair of conjugated variables, the response function and the spontaneous fluctuations are deeply related. Indeed, if $R_{\mathrm{eq}}(t)$ is the integrated response (*i.e.* the variation of the magnetization at time t when we add a magnetic field from time 0 on) and $C_{\mathrm{eq}}(t)$ is the correlation among the magnetization at time zero and at time t, we have that $R_{\mathrm{eq}}(t) = \beta(C_{\mathrm{eq}}(0) - C_{\mathrm{eq}}(t))$, where $\beta = (kT)^{-1}$. If we eliminate the time and we plot parametrically R_{eq} as function of C_{eq}, we have that $-\mathrm{d}R_{\mathrm{eq}}/\mathrm{d}C_{\mathrm{eq}} = \beta$.

In an aging system the generalized fluctuation dissipation relations can be formulated as follows. Let us suppose that the system is carried from high temperature to low temperature at time 0 and is in an aging regime. We can define a response function $R(t_\mathrm{w}, t)$ as the variation of the magnetization at time t when we add a magnetic field from time t_w on. In a similar way $C(t_\mathrm{w}, t)$ is the correlation among the magnetization at time t_w and at time t. We can define a function $R_{t_\mathrm{w}}(C)$: we must plot $R(t_\mathrm{w}, t)$ *vs.* $C(t_\mathrm{w}, t)$ by eliminating the time t in the region $t > t_\mathrm{w}$, where the response function is different from zero.

The fluctuation dissipation relations state that for large t_w the function $R_{t_\mathrm{w}}(C)$ converges to a limiting function $R(C)$. We can define

$$-\frac{\mathrm{d}R}{\mathrm{d}C} = \beta X(C), \tag{1}$$

where $X(C) = 1$, for $C > C_\infty$, and $X(C) < 1$ for $C < C_\infty$, where we have defined $C_\infty = \lim_{t\to\infty} C_\mathrm{eq}(t)$.

The shape of the function $X(C)$ gives important information on the free-energy landscape of the problem [1–8]. It has been shown that in stochastically stable systems the function $X(C)$ is related to equilibrium properties of the system. Let us illustrate this point by considering for definitiveness the case of a spin glass. Given two equilibrium configurations σ and τ, we define the global overlap as

$$q(\sigma, \tau) = N^{-1} \sum_{i=1,N} \sigma_i \tau_i, \tag{2}$$

where N is the total number of spins. The function $P_J(q)$ is the probability distribution of the overlap for a given sample and the function $P(q)$ is defined as the average of $P_J(q)$ over the samples. We introduce the function $x(q)$ defined by $\mathrm{d}x(q)/\mathrm{d}q = P(q)$. The relation among the dynamic fluctuation dissipation relations and the statics quantities is simply

$$X(C) = x(C). \tag{3}$$

Recent numerical results [18,19] indicate that the fluctuation dissipation relations and the static-dynamics connection can be generalized to local quantities in systems where a quenched disorder is present and aging is heterogeneous [20]. For *one given sample* we can consider the local integrated response function $R_i(t_\mathrm{w}, t)$, that is the variation of the magnetization at the point i at time t when we add a magnetic field at the point i starting at the time t_w. The local correlation function $C_i(t_\mathrm{w}, t)$ is the correlation between the spin at the point i at two different times (*i.e.* t_w and t). Quite often in a system with quenched disorder aging is very heterogenous: the functions C_i and R_i change dramatically from one point to another [20].

It has been observed in simulations [18,19] that local fluctuation dissipation relations seem to hold:

$$-\frac{\mathrm{d}R_i}{\mathrm{d}C_i} = \beta X_i(C_i), \tag{4}$$

where $X_i(C_i)$ has quite strong variations with the site i. In the framework of an explicit model for the dynamics [18], it has been analytically shown that, in spite of these strong heterogeneities, if we define the effective β_i^eff at time t at the site i as

$$-\frac{\mathrm{d}R_i(t_\mathrm{w}, t)}{\mathrm{d}C_i(t_\mathrm{w}, t)} = \beta X_i(t_\mathrm{w}, t) \equiv \beta_i^\mathrm{eff}(t_\mathrm{w}, t), \tag{5}$$

the quantity $\beta_i^\mathrm{eff}(t_\mathrm{w}, t)$ does not depend on the site [18, 19]. In other words, a thermometer coupled to a given site would measure (at a given time) the same effective temperature independently of the site: different sites are thermometrically indistinguishable.

The aim of this paper is to show that these results are a consequence of stochastic stability in an appropriate contest and that there is a local relation among static and dynamics. The crucial step consists in defining a local probability distribution of the overlap for a given system at point i (*i.e.* $P_i(q)$). If we define $x_i(q) = \int_0^q P_i(q')\mathrm{d}q'$, the static-dynamic connection for local variables is very similar to the one for global variables and is given by $X_i(C) = x_i(C)$.

The definition of the local overlap should be different from the usual one. Indeed, the local overlap of two equilibrium configurations (*i.e.* $\sigma_i \tau_i$) is always equal to ± 1. Naively, the probability distribution of the local overlap would be the sum of two delta-functions at ± 1. In the same way, it is impossible to define a meaningful two-times correlation of a give quantity in a single experimental run.

A useful definition of the local overlap can be obtained if we consider M identical copies (or clones) of our sample: we introduce $N \times M$ σ_i^a variables, where $a = 1, M$ (eventually, we send M to infinity) and N is the (large) size of our sample ($i = 1, N$). The Hamiltonian in this Gibbs ensemble is just given by

$$H_K(\sigma) = \sum_{a=1,M} H(\sigma^a) + \epsilon H_{\mathrm{R}}[\sigma], \tag{6}$$

where $H(\sigma^a)$ is the Hamiltonian for a fixed choice of the couplings J and $H_{\mathrm{R}}[\sigma]$ is a random Hamiltonian that couples the different copies of the system. A possible choice is $H_{\mathrm{R}}[\sigma] = \sum_{a=1,M;i=1,N} K_i^a \sigma_i^a \sigma_i^{a+1}$, where the variables K_i^a are identically distributed independent random Gaussian variables with zero average and variance 1 (periodic boundary conditions in clone space are implicitly used and $a+1$ for $a = M$ should be read as 1). We can consider other ways to couple the systems (*e.g.*, $H_{\mathrm{R}}[\sigma] = \sum_{a,b=1,M;i=1,N} K_i^{a,b} \sigma_i^a \sigma_i^b$), but let us stick to the above one.

Our central hypothesis is that in the limit $M \to \infty$ all intensive self-averaging quantities (with respect to the different choices of K at fixed J) are smooth function of ϵ for small ϵ. This hypothesis is a kind of generalization of stochastic stability. According to this hypothesis, when $\epsilon \to 0$, the site-dependent correlation function and response function tend to the result for $\epsilon = 0$ uniformly in time. From now on the site-dependent correlation function is redefined as $C_i(t,t') = M^{-1} \sum_{a=1,M} \langle \sigma_i(t) \sigma_i(t') \rangle$ and a similar redefinition is done for the response function.

In the case of small, but non-zero ϵ, let us take two equilibrium configurations σ and τ. For given K, we consider the site-dependent overlap

$$q_i(\sigma \cdot \tau) = M^{-1} \sum_{a=1,M} \sigma_i^a \tau_i^a. \tag{7}$$

We define the K-dependent probability distribution $P_i^K(q)$ as the probability distribution of the above overlap. We finally obtain

$$P_i(q) = \lim_{\epsilon \to 0} P_i^\epsilon(q), \qquad P_i^\epsilon(q) = \overline{P_i^K(q)}, \tag{8}$$

where the bar denotes the average over K, the limit $\epsilon \to 0$ is done *after* the limits $M \to \infty$ and $N \to \infty$ (alternatively, we keep $\epsilon^2 M$ and $\epsilon^2 N$ much larger than 1).

The proof of the local fluctuation dissipation relations can be obtained by copying, *mutatis mutandis*, the proof of the usual fluctuation dissipation relations [8]. For example, we can consider the following perturbation:

$$\Delta H_i^{(2)} \equiv \sum_{a=1,M;b=1,M} h^{a,b} \sigma_i^a \sigma_i^b, \tag{9}$$

where the variables h are Gaussian random variables with zero average and variance δ/M. The Hamiltonian is modified by adding to it the term $\delta\Delta H_i^{(2)}$ at all times greater than t_w. If we assume, for simplicity, a Langevin type of time evolution, the same steps of [8] give

$$\chi_i^{(2)}(t_\mathrm{w},t) \equiv \frac{\partial \Delta H_i^{(2)}}{\delta} = 2\int_{t_\mathrm{w}}^{t} \mathrm{d}t' C_i(t',t)\frac{\partial R_i(t',t)}{\partial t'} \,. \tag{10}$$

The above equation can be rewritten as

$$\chi_i^{(2)}(t_\mathrm{w},t) = 2\beta \int_{C_i(t_\mathrm{w},t)}^{1} \mathrm{d}C C X_i(t,C), \qquad \beta X_i\big(t, C(t',t)\big) \equiv \frac{\partial R_i(t',t)}{\partial t'}\left(\frac{\partial C_i(t',t)}{\partial t'}\right)^{-1}. \tag{11}$$

Equivalently, $X_i(t,C) = \frac{\partial R_i(t',t)}{\partial t'}(\frac{\partial C_i(t',t)}{\partial t'})^{-1}$, evaluated at the value of t' such that $C(t',t) = C$.

Let us assume, for simplicity, that we stay in the region where $C_i(t_\mathrm{w},t)$ goes to zero when $t\to\infty$, so that the lower limit of integration can be set to zero. For finite δ, an integration by parts tells us that $\langle \Delta H_i^{(2)}\rangle = \beta\int \mathrm{d}q P_i(q)(1-q^2)$. If we make the crucial hypothesis that the limits $t\to\infty$ and $\delta\to 0$ may be exchanged, we get

$$\lim_{t\to\infty} \chi_i^{(2)}(t_\mathrm{w},t) \equiv \chi_i^{(2)}(t_\mathrm{w}) = 2\int_0^1 \mathrm{d}C C X_i(C) = \beta\int \mathrm{d}q P_i(q)\big(1-q^2\big), \tag{12}$$

where we have also supposed the existence of the limit $\lim_{t\to\infty} X_i(t,C) \equiv X_i(C)$; this last hypothesis implies that $\chi_i^{(2)}(t_\mathrm{w})$ does not depend on the time t_w as it should be: the infinite time limit of the response should be independent of the time at which the perturbation is added.

Generalizing the above arguments by introducing an s-spin perturbation $\Delta H_i^{(s)}$ (*e.g.*, $\Delta H_i^{(3)} = \sum_{a,b,c=1,M} h^{a,b}\sigma_i^a\sigma_i^b\sigma_i^c$) we get

$$\begin{aligned}\chi_i^{(s)}(t_\mathrm{w}t) &= s\int_{t_\mathrm{w}}^{t} \mathrm{d}t' C_i(t',t)^{s-1}\frac{\partial R_i(t',t)}{\partial t'} = \\ &= \beta s\int \mathrm{d}C C^{s-1} X_i(t,C) \longrightarrow_{t\to\infty} \beta\int \mathrm{d}q P_i(q)(1-q^s) = s\beta\int \mathrm{d}q x_i(q) q^{s-1}.\end{aligned} \tag{13}$$

Up to now we have assumed that $\lim_{t\to\infty} X_i(t,C)$ exists. However, the same arguments of [8] imply that using the above equations for all values of s, we can arrive at the conclusion that the quantity $X_i(t,C)$ has a limit when the time goes to infinity (*i.e.* to a formulation of the local fluctuation dissipation relations). The limit is given by

$$X_i(C) = \lim_{t\to\infty} X_i(t,C) = x_i(C) \equiv \int_0^C \mathrm{d}q P_i(q)\mathrm{d}q. \tag{14}$$

The above equations give the connection among static quantities and the local fluctuation dissipation relations.

The thermometric indistinguishability of the sites can be proved by considering two far away sites i and k and by introducing a perturbation that depends on both the spins at i and the spins at k. A typical example is

$$\Delta H_{i,k}^{(3,2)} \equiv \sum_{a_1,a_2,a_3,b_1,b_2=1,M} h_i^{a_1,a_2,a_3,b_1,b_2}\sigma_i^{a_1}\sigma_i^{a_2}\sigma_i^{a_3}\sigma_k^{b_1}\sigma_k^{b_2}, \tag{15}$$

where the variables h are Gaussian random variables with zero average and variance δ/M^4. If we proceed as before, we find that the probability distribution $P_{i,k}(q_i,q_k)$ can be written in terms of the infinite time limits of $X_i(t,C_i)$ and $X_k(t,C_k)$ and of the function $f^{t_\mathrm{w}}_{k,i}$ defined by

$$C_k(t_\mathrm{w},t) = f^{t_\mathrm{w}}_{k,i}\big(C_i(t_\mathrm{w},t)\big), \tag{16}$$

when t_w and t are both large. If we disregard the possibility of oscillations, the function $f^{t_\mathrm{w}}_{k,i}$ should have a limit when the time t_w goes to infinity (this limit may be discontinuous if C_i and C_k age on a different time scale). The absence of oscillations is a crucial hypothesis that is automatically satisfied in any model where the local correlation at different times satisfies simple scaling relations.

Let us assume that these extra hypotheses are satisfied. After some computation, we find that

$$\begin{aligned}\int \big(1-q_i^{s_i}q_k^{s_k}\big)\mathrm{d}P_{i,k}(q_i,q_k) \;&=\; \int_{t_\mathrm{w}}^{t}\mathrm{d}t\Big[X_i\big(t,C_i(t)\big)\frac{\partial C_i(t)^{s_i}}{\partial t}C_k(t)^{s_k} + k \longleftrightarrow i\Big]\\ &\longrightarrow \int_0^1 X_i(C_i)\big(f_{k,i}(C_i)\big)^{s_k}\,\mathrm{d}(C_i)^{s_i} + k \longleftrightarrow i.\end{aligned} \tag{17}$$

If the functions $X_i(C_i)$ and $X_k(C_k)$ are equal for corresponding arguments, *i.e.*

$$X_i(C_i) = X_k\big(f_{k,i}(C_i)\big), \tag{18}$$

we obtain that

$$P_{i,k}(q_i,q_k) = P(q_i)\delta\big(q_k - f_{k,i}(q_i)\big) = \int_0^1 \mathrm{d}x\delta\big(q_i - q_i(x)\big)\delta\big(q_k - q_k(x)\big), \tag{19}$$

where the last equality makes sense only in the case where the function $q_i(x)$ (*i.e.* the inverse of $x_i(q)$) is well defined.

If $X_i(C_i)$ were not equal to $X_k f_{k,i}(C_i)$, $P_{i,k}(q_i,q_k)$ would contain a new term proportion $\delta'(q_k - f_{k,i}(q_i))$. The probability $P_{i,k}(q_i,q_k)$ must be a positive distribution, a derivative of delta-functions cannot be present in a probability and consistency implies that eq. (18) holds. It is easy to see that this condition is just what is needed to impose the thermometric indistinguishability of the sites during aging,

$$X_i\big(t,C_i(t)\big) = X_k\big(t,C_k(t)\big). \tag{20}$$

The above equation is non-trivial for $C_i(t) < q_i^{\mathrm{EA}}$, where the local Edwards-Anderson order parameter is determined by the condition $P_i(q) = 0$ (or, equivalently, $x_i(q) = 1$) for $q > q_i^{\mathrm{EA}}$. The site dependence of the local Edwards-Anderson order parameter is a very important effect and it is encoded in the local variations of the function $P_i(q)$.

The probability distribution of the local overlap $P_i(q)$, is related to a dynamical quantity: it must depend only on the local environment around the point i and therefore it must have a straightforward limit when the volume of the system goes to infinity (*e.g.*, it should be independent of the boundary conditions). It is remarkable that for far away points (i,k) the two probability distributions $P_i(q)$ and $P_k(q)$ are independent one from the other, but the joint probability distribution of q_i and q_k does not factorize as shown by eq. (19).

Summarizing, for a given sample it is possible to give a definition of a local probability distribution of the overlap that depends on the site. The properties of this local probability distribution are related to the local fluctuation dissipation relations. The condition of thermometric indistinguishability of the sites turns out to be a byproduct of our approach: during the aging regime, all the sites are characterized by the same effective temperature. Here, we have made no explicit assumption concerning the validity of static or dynamical ultrametricity or on the existence of a large separation of the time scales. Although the generalized fluctuation dissipation relations have been firstly derived in ultrametric mean-field models [1, 2], their validity relies only on more general properties like stochastic stability.

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