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Iterated random walk

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Abstract. – The iterated random walk is a random process in which a random walker moves on a one-dimensional random walk which is itself taking place on a one-dimensional random walk, and so on. This process is investigated in the continuum limit using the method of moments. When the number of iterations $n \to \infty$, a time-independent asymptotic density is obtained. It has a simple symmetric exponential form which is stable against the modification of a finite number of iterations. When n is large, the deviation from the stationary density is exponentially small in n. The continuum results are compared to Monte Carlo data for the discrete iterated random walk.

Introduction. – When a walker moves at random along a straight support, its typical displacement at time t grows like $l \sim t^{1/2}$. If the support of the walk is itself a random walk (RW), at time t the walker is typically located at the l-th step of the walk on which it is moving. Thus the mean-square displacement of the walker behaves as $\langle X^2(t)\rangle \sim l \sim t^{1/2}$ and his actual typical displacement is reduced, growing as $t^{1/4}$ instead of $t^{1/2}$ for the conventional RW.

This model of a RW on a RW has been studied for a one-dimensional support [1] and also in higher dimensions [2,3]. It has been used to discuss the diffusion of the stored length along a polymer chain entangled in a network [4–6] and the Brownian motion of charged particles in a turbulent plasma [7]. In this latter case, the particles are constrained to diffuse along the random magnetic-field lines in the limit $B \to \infty$.

In this work we study the properties of the one-dimensional iterated random walk (IRW), *i.e.*, the RW on a RW which is itself a RW on a RW, and so on. The first three steps of this iteration process are shown in fig. 1. Working in the continuum (Gaussian) limit, we first study the moments $\langle X^p(t)\rangle$ of the probability density $P^{(n)}(X,t)$ to find the walker at (X,t) after the n-th iteration. In the next section, we examine the limit $n \to \infty$ for which the asymptotic probability density $P^{(\infty)}(X,t)$ can be obtained explicitly. We also look at the way this asymptotic density is approached when n is large. Then the continuum result is compared to Monte Carlo data for the discrete IRW. The universality of the asymptotic probability density and possible generalizations are discussed in the final section.

Moments of the iterated probability density. – We assume that our random walker starts from $X = 0$ at $t = 0$. All other one-dimensional RWs, on which he is moving, extend in time from $-\infty$ to $+\infty$ and have a common origin. In the continuum limit, to the symmetric RW is associated the Gaussian density

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{x^2}{2t}\right], \quad t > 0, \tag{1}$$

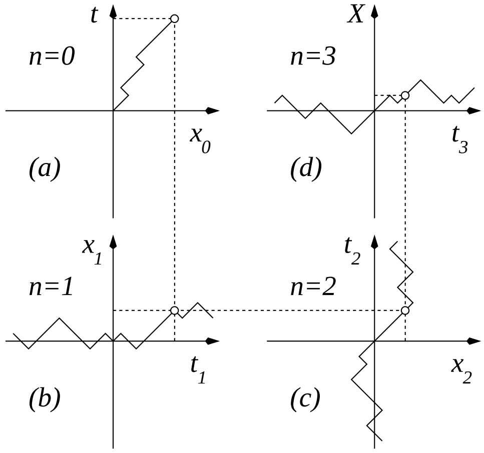

Fig. 1 – Construction of the position X of the random walker at time t for the IRW: the walker in (a) starts at $x_0(0) = X(0) = 0$ and moves at random along the sequence of steps of the RW in (b). Thus the coordinate $x_0(t)$ is also the time coordinate t_1 for the support in (b). In the same way, x_1 gives the number of steps t_2 performed by the RW in (c) on which the RW in (b) takes place. Finally, at $t_3 = x_2$, the last RW in (d) is located at X, which gives the actual position $X(t)$ of the random walker.

and the probabilty density to find the walker at (X, t) after $n + 1$ iterations is obtained by integrating the product of the successive probability densities $p(x_i, |t_i|)$ with $t_i = x_{i-1}$, $t_0 = t$ for the random walker and $x_{n+1} = X$ for the last support, over all the intermediate coordinates x_i for $i = 0$ to $i = n$ (see fig. 1):

$$P^{(n+1)}(X,t)=\int_{-\infty}^{+\infty} \mathrm{d}x_n p(X,|x_n|) \int_{-\infty}^{+\infty} \mathrm{d}x_{n-1} p(x_n,|x_{n-1}|) \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_0 p(x_1,|x_0|) p(x_0,t). \qquad (2)$$

As a consequence, the iterated probability density satisfies the recursion relation

$$P^{(n+1)}(X,t) = \int_{-\infty}^{+\infty} \mathrm{d}x p(X,|x|) P^{(n)}(x,t). \qquad (3)$$

In the following we leave the initial probability density $P^{(0)}(x,t)$ unspecified. Since the Gaussian density $p(X, |x|)$ in (3) is even in X, the same is true of all the iterated densities $P^{(n)}(x,t)$, $n > 0$. Their odd moments vanish while their even moments are given by

$$M_{2p}^{(n)}(t) = \int_{-\infty}^{+\infty} \mathrm{d}X X^{2p} P^{(n)}(X,t) = \mathcal{M}_{2p}^{(n)}(t), \qquad \mathcal{M}_p^{(n)}(t) = \int_{-\infty}^{+\infty} \mathrm{d}X |X|^p P^{(n)}(X,t), \quad (4)$$

where the moments $\mathcal{M}_p^{(n)}$ of $|X|$ have been introduced. According to (3), these moments may be written as

$$\begin{aligned} \mathcal{M}_p^{(n+1)}(t) &= \int_{-\infty}^{+\infty} \mathrm{d}x P^{(n)}(x,t) \int_{-\infty}^{+\infty} \mathrm{d}X |X|^p p(X,|x|) \\ &= 2^{p/2} \frac{\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \int_{-\infty}^{+\infty} \mathrm{d}x |x|^{p/2} P^{(n)}(x,t), \end{aligned} \qquad (5)$$

where the last relation follows from the form of $\langle |X|^p \rangle$ for the Gaussian density,

$$\int_{-\infty}^{+\infty} \mathrm{d}X |X|^p \frac{\exp\left[-\frac{X^2}{2t}\right]}{\sqrt{2\pi t}} = \frac{\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} (2t)^{p/2}. \qquad (6)$$

Equation (5) leads to the recursion relation

$$\mathcal{M}_p^{(n+1)}(t) = 2^{p/2}\frac{\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\mathcal{M}_{p/2}^{(n)}(t), \tag{7}$$

so that

$$\mathcal{M}_p^{(n)}(t) = \mathcal{M}_{p/2^n}^{(0)}(t)\prod_{k=1}^{n} 2^{p/2^k}\frac{\Gamma\left(\frac{p}{2^k}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,. \tag{8}$$

Making use of the identity [8]

$$\Pi(z) = \prod_{k=1}^{n}\frac{\Gamma\left(\frac{z}{2^k}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma(z+1)}{2^{2z(1-1/2^n)}\Gamma\left(\frac{z}{2^n}+1\right)}\,, \tag{9}$$

the moments of the iterated probability density may be written as

$$M_{2p}^{(n)}(t) = \frac{(2p)!}{2^{2p(1-1/2^n)}\Gamma\left(\frac{2p}{2^n}+1\right)}\mathcal{M}_{2p/2^n}^{(0)}(t), \qquad M_{2p+1}^{(n)}(t) = 0. \tag{10}$$

Asymptotic probability density and how it is approached. – Since the walker and all the underlying RWs have a common origin in space and time,

$$P^{(n)}(X,0) = \delta(X). \tag{11}$$

When $t > 0$ and $n \to \infty$, the even moments in (10) considerably simplify,

$$M_{2p}^{(\infty)}(t>0) = \frac{(2p)!}{4^p}\,, \tag{12}$$

and become time independent since $\mathcal{M}_0^{(0)}(t) = 1$ for any initial density function. The Fourier transform of the asymptotic probability density is related to its moments through a Taylor expansion:

$$\begin{aligned}\mathcal{P}^{(\infty)}(k,t>0) &= \int_{-\infty}^{+\infty} \mathrm{d}X \mathrm{e}^{\mathrm{i}kX} P^{(\infty)}(x,t>0) = \sum_{q=0}^{\infty}\frac{(\mathrm{i}k)^q}{q!}M_q^{(\infty)}(t>0)\\ &= \sum_{p=0}^{\infty}\left(-\frac{k^2}{4}\right)^p = \frac{4}{k^2+4}\,.\end{aligned} \tag{13}$$

The inverse Fourier transform,

$$P^{(\infty)}(X,t>0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\mathrm{e}^{-\mathrm{i}kX}\mathcal{P}^{(\infty)}(k,t>0) = \frac{2}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}k\frac{\mathrm{e}^{-\mathrm{i}kX}}{k^2+4}\,, \tag{14}$$

is easily evaluated using the method of residues. Finally, the asymtotic density takes the simple form

$$P^{(\infty)}(X,t) = \begin{cases}\delta(X) & (t=0),\\ \mathrm{e}^{-2|X|} & (t>0),\end{cases} \tag{15}$$

which is *time independent* as soon as $t > 0$ and *universal*, *i.e.*, independent of the form of the initial density $P^{(0)}$. This asymptotic density is invariant under the Gaussian transformation (3).

Let us now examine how this stationary density is approached when n is large. For this, the initial density $P^{(0)}$ must be specified. Coming back to the random-walk problem, we use the Gaussian density for which

$$\mathcal{M}^{(0)}_{2p/2^n}(t) = \frac{\Gamma\left(\frac{p}{2^n}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}(2t)^{p/2^n} \tag{16}$$

according to (6). The even moments in (10) take the following form:

$$M^{(n)}_{2p}(t) = \frac{(2p)!}{4^p}\frac{2^{2p/2^n}\Gamma\left(\frac{p}{2^n}+\frac{1}{2}\right)}{\Gamma\left(\frac{2p}{2^n}+1\right)\Gamma\left(\frac{1}{2}\right)}(2t)^{p/2^n}. \tag{17}$$

Using the duplication formula [9],

$$\frac{\Gamma(2z)}{\Gamma(z)} = 2^{2z-1}\frac{\Gamma\left(z+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{18}$$

this expression can be rewritten as

$$M^{(n)}_{2p}(t) = \frac{(2p)!}{4^p}\frac{(2t)^{p/2^n}}{\Gamma\left(1+\frac{p}{2^n}\right)}. \tag{19}$$

Thus $M^{(n)}_{2p}(0) = \delta_{p,0}$ in agreement with (11) and the length scales as $t^{1/2^{n+1}}$ after n iterations.

When $n \gg 1$, a first-order expansion in powers of 2^{-n} gives

$$\begin{aligned}
(2t)^{p/2^n} &= 1 + \ln(2t)\frac{p}{2^n} + \mathrm{O}(4^{-n}),\\
\Gamma\left(1+\frac{p}{2^n}\right) &= 1 + \Psi(1)\frac{p}{2^n} + \mathrm{O}(4^{-n}) = 1 - \gamma\frac{p}{2^n} + \mathrm{O}(4^{-n}),\\
M^{(n)}_{2p}(t) &= \frac{(2p)!}{4^p}\left\{1 + [\ln(2t)+\gamma]\frac{p}{2^n}\right\} + \mathrm{O}(4^{-n}),
\end{aligned} \tag{20}$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\gamma = 0.577\,215\,664\ldots$ is Euler's constant. Thus, the leading correction to the Fourier transform of the asymptotic density reads

$$\delta\mathcal{P}^{(n)}(k, t>0) = \frac{\ln(2t)+\gamma}{2^n}\sum_{p=1}^{\infty} p\left(-\frac{k^2}{4}\right)^p = -\frac{\ln(2t)+\gamma}{2^n}\frac{4k^2}{(k^2+4)^2} \tag{21}$$

and the inverse Fourier transform leads to

$$\delta P^{(n)}(X, t>0) = \frac{\ln(2t)+\gamma}{2^n}(|X|-1)\mathrm{e}^{-2|X|}. \tag{22}$$

The deviation from the asymptotic density is exponentially small in n, a behaviour which is independent of the initial density function according to (10). It has a weak (logarithmic) time dependence for the Gaussian density.

Scaling behaviour and discrete IRW. – The Gaussian density in (1) can be written under the more general scaling form

$$p(x,t) = t^{-1/z} f\left(\frac{x}{t^{1/z}}\right), \tag{23}$$

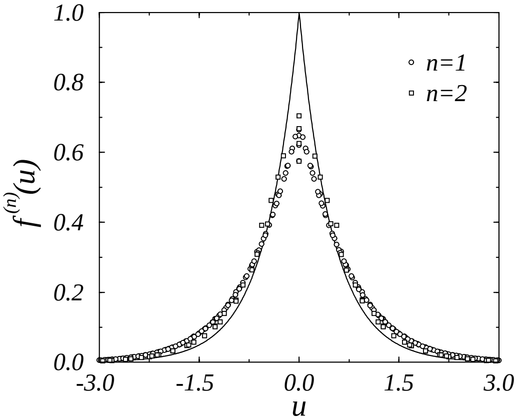

Fig. 2 – Scaling function for the iterated probability distributions of the discrete RW compared to the stationary density (solid line) obtained in the continuum limit. The histograms of the probability distributions have been obtained through Monte Carlo simulations for $t = 10^2$ to 10^5 and $t+1$, working with 10^7 samples. The statistical fluctuations are much smaller than the symbols, as indicated by the symmetry of the data.

where the dynamical exponent z, which is equal to 2 for the Gaussian density, governs the transformation of the time, $t' = t/b^z$, under a change of the length scale, $x' = x/b$. The scaling behaviour of the iterated density can be obtained by induction. Assuming that

$$P^{(n)}(X,t) = t^{-1/z_n} f^{(n)}\left(\frac{X}{t^{1/z_n}}\right), \tag{24}$$

which is true with $z_0 = z$ and $n = 0$ for the initial density, and making use of (3), we obtain

$$\begin{aligned} P^{(n+1)}(X,t) &= t^{-1/z_n} \int_{-\infty}^{+\infty} \mathrm{d}x |x|^{-1/z} f\left(\frac{X}{|x|^{1/z}}\right) f^{(n)}\left(\frac{x}{t^{1/z_n}}\right) \\ &= t^{-1/zz_n} \int_{-\infty}^{+\infty} \mathrm{d}v f\left(\frac{X}{|v|^{1/z} t^{1/zz_n}}\right) \frac{f^{(n)}(v)}{|v|^{1/z}}\,, \end{aligned} \tag{25}$$

so that $P^{(n+1)}$ scales like $P^{(n)}$ in (24). The dynamical exponent evolves according to $z_n = zz_{n-1} = z^{n+1}$ and the scaling function satisfies the recursion relation

$$f^{(n)}(u) = \int_{-\infty}^{+\infty} \mathrm{d}v f\left(\frac{u}{|v|^{1/z}}\right) \frac{f^{(n-1)}(v)}{|v|^{1/z}}\,, \quad u = \frac{X}{t^{1/z^{n+1}}}\,. \tag{26}$$

Thus, with $z > 1$, the asymptotic density $P^{(\infty)}(X,t)$ is time independent and reduces to its scaling function $f^{(\infty)}(X)$.

We now make use of this scaling behaviour to analyse Monte Carlo data for the discrete IRW. The continuum result is expected to give a good description of these data for not too small values of X and t. The histograms for the discrete probability distributions have been obtained as follows: First a RW with t steps is generated, leading the walker to x_0 (see fig. 1). Then a second walk with $|x_0|$ steps, corresponding to the first support, is constructed. When it ends at x_1, the value of $N^{(1)}(x_1)$, giving the number of walks ending at x_1 at the first iteration, is updated and the same process is repeated for the following iterations. $N_{\mathrm{s}} = 10^7$ samples have been generated in this way for $t = 10^2, 10^3, 10^4, 10^5$. Since all the x_n have the parity of t, the simulations were repeated for $t+1$ in order to double the number of points along the X-axis. The properly normalized histograms for a comparison with the density functions are obtained by dividing $N^{(n)}$ by $2N_{\mathrm{s}}$.

The scaling functions for the discrete iterated walk are compared to the invariant density (15) in fig. 2. A good data collapse is obtained for each value of n. As expected, there is a rapid convergence to the asymptotic density. The convergence is from above when $|u|$ is greater than a value close to 1, in agreement with (22).

Final remarks. – We have seen that the Gaussian iteration process leads to an invariant probability density (15) which is independent of the initial density $P^{(0)}$. Actually, the universality of the invariant density is more extended since $P^{(0)}$ may be itself considered as resulting from a finite number of iterations involving arbitrary densities $p_k(x,t)$ $(k=0,m)$. The initial density being now $p_0(x_0,t)$, we have

$$\begin{aligned} P^{(0)}(x,t) \;=\; & \int_{-\infty}^{+\infty} \mathrm{d}x_{m-1} p_m(x,|x_{m-1}|) \int_{-\infty}^{+\infty} \mathrm{d}x_{m-2} p_{m-1}(x_{m-1},|x_{m-2}|) \cdots \\ & \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_0 p_1(x_1,|x_0|) p_0(x_0,t). \end{aligned} \tag{27}$$

After an infinite number of Gaussian iterations this density will evolve, as before, to the universal time-independent invariant density. The arbitrary densities may be, for example, shifted Gaussian densities,

$$p_k(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(x-v_k t)^2}{2t}\right] \tag{28}$$

corresponding to RWs with arbitrary drifts v_k.

When, in the iteration process, the Gaussian density is replaced by a density which is even in x, scales as in (23) and has finite moments, the nonvanishing even moments of the iterated density are easily obtained in the same way as above. They involve a product similar to that appearing in (8) and are time independent when $n \to \infty$. Unfortunately, we were unable to find another instance where these moments could be simplified as in (10), allowing for an explicit calculation of the time-independent asymptotic density.

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