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Spectral scalability as a result of geometrical self-similarity in fractal multilayers

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Abstract. – The optical spectra of fractal multilayer dielectric structures have been shown to possess spectral scalability, which has been found to be directly related to the structure's spatial (geometrical) self-similarity. Phase and amplitude scaling relations, as well as effects of finite structure size, have been derived.

Introduction. – It is commonly known that the optical spectra of periodic dielectric materials (including periodic multilayers in particular) possess forbidden gaps, which are demonstrated to directly result from spatial periodicity [1–3]. On the other hand, disordered dielectric media have been discovered to slow down, localize, and confine light waves traveling through them [4, 5]. The same effects are known for electrons and other quantum particles in periodic and random potential, respectively (see [6] and the review [7]). So, both periodic and random structures (which represent the two extreme, and hence most studied, cases of multilayers), exhibit characteristic spectral effects that result from their topology.

Recent studies reveal that one type within the "intermediate" case (nonperiodic but deterministic structures) also displays characteristic spectral effects not present in either of extreme cases. It was found [8] that *quasiperiodic* (*e.g.*, Fibonacci) multilayers have *self-similar* spectra. Their transmission bands represent Cantor sets, a well-known example of one-dimensional (1D) fractals. It was proved that spectral self-similarity is a characteristic property of spatial quasiperiodicity. The same is equally applicable for electronic spectra.

In this letter, we would like to address another class of deterministic nonperiodic structures, namely *fractal multilayers*. We show that their geometrical self-similarity results in *spectral scalability*, earlier observed by us in numerical computations (see ref. [9]). This paper analytically shows that the origin of scalability is the self-similarity inherent to all fractal multilayers, and this is a manifestation of correlation between geometrical properties of the structures and properties of their eigenvalue spectra.

Fractal multilayers. – One of the common examples of fractal multilayers is the well-known triadic Cantor stack generated using the "middle third removal" procedure [10] (see fig. 1a). However, this procedure can be generalized. The most straightforward way to do so is to complicate the removal routine, applying it not only to the middle third, but to

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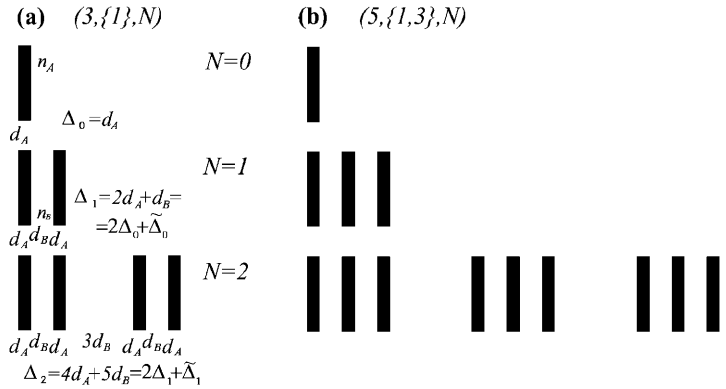

Fig. 1 – Sample fractal multilayer structures: (a) Cantor "middle third" structures $(3, \{1\}, N)$ with the relations for Δ_N and $\tilde{\Delta}_N$ shown; (b) higher-G Cantor structures $(5, \{1, 3\}, N)$ for smaller values of N.

arbitrary (yet similar from generation to generation) regions of the structure. Some variations are described in [10] and investigated in [9, 11–13].

Here we introduce a more general procedure, which encompasses most of 1D fractals that have prefractals and hence can be used in multilayer design. The algorithm starts with an *initiator*, a single dielectric layer (label it A) with refractive index n_A and thickness d_A. The initiator is stacked together G times, and the layers are numbered in base G (starting with zero). Then, those parts whose numbers belong to a given subset of digits $\boldsymbol{C} \subset \{0, 1, \ldots, G-1\}$ are replaced with layers of another dielectric (labeled B), with refractive index $n_B \neq n_A$ and thickness d_B. This replication-replacement (RR) procedure is then repeated for the resulting structure (which now consists of G layers), with the only difference that a group of G B-type layers is now used to replace the appropriate fragments. Repeating this RR procedure, multiple times yields the desired fractal multilayer.

Here, an arbitrary integer $G > 2$ together with the subset $\boldsymbol{C}$ form the *generator* of the structure, while the number N of RR procedures applied is called the *number of generations*. The whole structure can be referred to as a $(G, \boldsymbol{C}, N)$ structure. One can see that the usual N-stage middle third Cantor stack is nothing but a particular case of $(3, \{1\}, N)$. Other particular cases include higher-G Cantor structures ($G = 3, 5, 7, \ldots$; $\boldsymbol{C} = \{1, 3, \ldots, G-2\}$, N) [9], non-symmetric stacks [10], and generalized Cantor bars [12].

Sample stacks are shown in fig. 1, and the construction details can be inferred therefrom.

To conclude this section, let us list some simple but important relations concerning fractal multilayers. First of all, the total number of layers in such a structure is G^N (here and further, several adjacent layers of the same material count as separate layers). Among these layers, $(G-C)^N$ are A-type and the rest are B-type, C being the number of members in $\boldsymbol{C}$. Then, the total thickness of a $(G, \boldsymbol{C}, N)$ structure can be written as a recurrent relation:

$$\Delta_N = (G-C)^N d_A + \left(G^N - (G-C)^N\right) d_B \equiv (G-C)\Delta_{N-1} + C\tilde{\Delta}_{N-1}\,. \tag{1}$$

From eq. (1) one can obtain a scaling relation for $\tilde{\Delta}_N = G^N d_B$, and in all cases $\Delta_0 = d_A$. For all calculations, the constituent layers were chosen have equal optical thickness, *i.e.*

$$n_A d_A = n_B d_B = d^* \equiv \pi c / 2\omega_0\,. \tag{2}$$

This condition causes the spectra to be periodic with respect to frequency, the period equal to $2\omega_0$. This outcome is very convenient, since it provides a natural way to normalize the

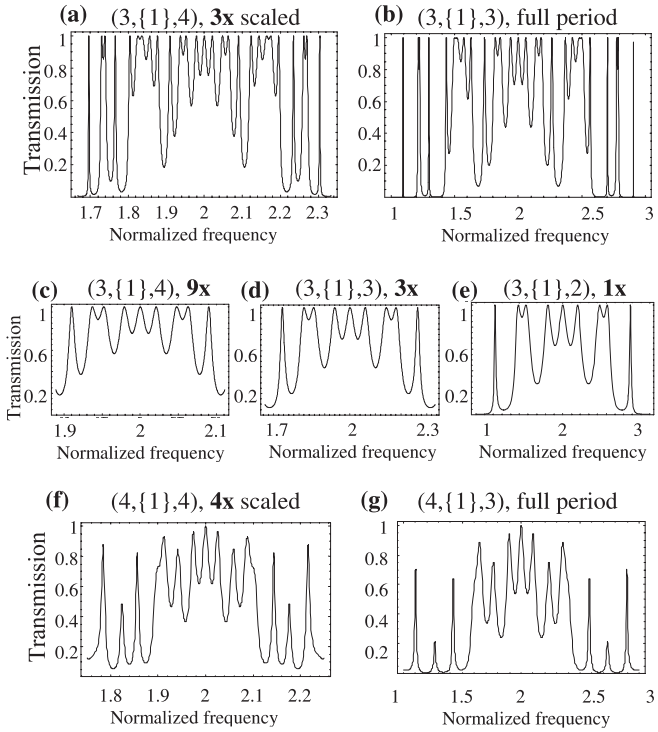

Fig. 2 – Scalability of optical spectra for fractal multilayers: (a) $(3, \{1\}, N = 4)$, the central part of the spectrum magnified in the frequency scale by 3 *vs.* (b) the full period of the $N = 3$ spectrum; (c) the central part of $N = 4$ magnified by $9 = 3^2$, (d) the central part of $N = 3$ magnified by 3, and (e) the full period of the $N = 2$ spectrum; (f) $(4, \{1\}, 4)$, the central part of the spectrum magnified by 4 *vs.* (g) full period of the $(4, \{1\}, 3)$. Compare the looks of (a) and (b); (c), (d), and (e); (f) and (g).

frequency scale introducing the *dimensionless frequency* $\eta \equiv \omega/\omega_0$. It also allows only *one* period of spectrum to be referred to as "spectrum", which is what will be done hereafter.

Spectral scalability. – Keeping this in mind, we can now present a simple definition of spectral scalability as follows. *We have found that the whole spectrum of a* $(G, \boldsymbol{C}, N)$ *stack appears as a part of a* $(G, \boldsymbol{C}, N + 1)$ *stack spectrum.* If we magnify a certain part of the latter (the area centered on $2\omega_0$, or on $\eta = 0, 2, \ldots$, to be exact) by a factor of G, its shape will coincide very well with that of the former spectrum. This property was observed and reported by us earlier [9] for $(3, \{1\}, N)$ and $(5, \{1, 3\}, N)$ structures (see fig. 2a-e). With the same method used in the calculations, subsequent research has revealed that this property holds for any G and $\boldsymbol{C}$ (see, *e.g.*, fig. 2f, g). So does the relation for the factor by which one has to magnify the central part of the $(G, \boldsymbol{C}, N_1)$ stack spectrum for matching with that of $(G, \boldsymbol{C}, N_2 < N_1)$ stack. Termed the *scaling factor* between $(G, \boldsymbol{C}, N_1)$ and $(G, \boldsymbol{C}, N_2)$ structures, it equals

$$S = G^{N_1 - N_2}\,. \tag{3}$$

The fact that the scaling factor in (3) exactly equals the geometrical factor of self-similarity, which is clearly seen from the construction procedure, alone hints at the idea that *geometrical self-similarity of fractal multilayers and scalability of their optical spectra are related.* However,

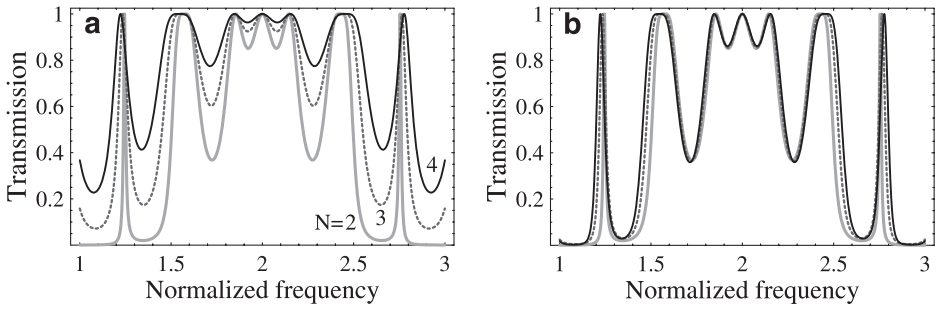

Fig. 3 – Exact comparison of the scaled spectra for $(3, \{1\}, N)$ structures: (a) unchanged, (b) raised to a power γ.

such qualitative speculations are clearly not enough to state that spectral scalability is a direct result of geometrical self-similarity.

A more convincing proof of this statement may be obtained from analytical calculations. It is worth noting, however, that spectral scalability, while visually apparent as in fig. 2, is difficult to be described mathematically because close inspection of the spectra reveals that there is no exact coincidence either in the value of the transmission coefficient or in the peak locations (see fig. 3a). However, these discrepancies do not change the shape of the spectral curve noticeably, thus not hindering the observation of scalability.

First, we consider the simplest case, the middle third Cantor stacks $(3, \{1\}, N)$. To analytically calculate the spectra of such multilayers, it is possible to use the *self-similarity method of calculation* [11,12], which is a generalization of Airy formulas based on the structure being self-similar. According to this method, the reflection and transmission coefficients for the $(3, \{1\}, N+1)$ and $(3, \{1\}, N)$ structures are related as

$$R_{N+1}(\eta) = g_r\big[R_N(\eta), T_N(\eta), \tilde{\Delta}_N, \eta\big], \qquad T_{N+1}(\eta) = g_t\big[R_N(\eta), T_N(\eta), \tilde{\Delta}_N, \eta\big], \tag{4}$$

where $\tilde{\Delta}_N$ is as defined by eq. (1). The functions

$$g_r(x, y, d, \eta) = x + \frac{xy^2\varepsilon^2(d, \eta)}{1 - x^2\varepsilon^2(d, \eta)}, \qquad g_t(x, y, d, \eta) = \frac{y^2\varepsilon(d, \eta)}{1 - x^2\varepsilon^2(d, \eta)} \tag{5}$$

are obtained using effective medium formalism in [11]. The initial conditions for these recurrent relations are derived from the normal-incidence reflection and transmission coefficients for a single layer (*i.e.*, a structure with $N = 0$), such that

$$R_0(\eta) = -r + \frac{rtt'\varepsilon_0^2(\Delta_0, \eta)}{1 - r^2\varepsilon_0^2(\Delta_0, \eta)}, \qquad T_0(\eta) = \frac{tt'\varepsilon_0(\Delta_0, \eta)}{1 - r^2\varepsilon_0^2(\Delta_0, \eta)}. \tag{6}$$

Here r, t, t' are normal-incidence Fresnel's coefficients for the layer interfaces, and

$$\varepsilon(d, \eta) \equiv \exp\left[\frac{i}{c}\eta\omega_0 n_B d\right], \qquad \varepsilon_0(d, \eta) \equiv \exp\left[\frac{i}{c}\eta\omega_0 n_A d\right] \tag{7}$$

are the phase exponents.

Now, to proceed with the analysis of scalability, we need to compare the following quantities:

$$T_{N+1}\left(\frac{\eta}{3}\right) \qquad \text{and} \qquad T_N(\eta). \tag{8}$$

Using the relation for $\tilde{\Delta}_N$ and substituting (4) and (5) into (8), one can obtain

$$T_{N+1}\left(\frac{\eta}{3}\right) = \frac{T_N^2\left(\frac{\eta}{3}\right)\exp\left[\frac{i}{c}\frac{\eta}{3}\omega_0 n_B\cdot 3^N d_B\right]}{1-R_N^2\left(\frac{\eta}{3}\right)\exp\left[\frac{2i}{c}\frac{\eta}{3}\omega_0 n_B\cdot 3^N d_B\right]}, \tag{9}$$

$$T_N(\eta) = \frac{T_{N-1}^2(\Delta_{N-1},\eta)\exp\left[\frac{i}{c}\eta\omega_0 n_B\cdot 3^{N-1} d_B\right]}{1-R_{N-1}^2(\Delta_{N-1},\eta)\exp\left[\frac{2i}{c}\eta\omega_0 n_B\cdot 3^{N-1} d_B\right]}. \tag{10}$$

We see that the phase exponents in (9) and (10) are exactly equal, and the sole difference between $T_{N+1}(\frac{\eta}{3})$ and $T_N(\eta)$ lies in the coefficients, $T_N(\frac{\eta}{3})$, $R_N(\frac{\eta}{3})$ and $T_{N-1}(\eta)$, $R_{N-1}(\eta)$, respectively. But if one expands these coefficients in the same way, using (4) and (5), one can see that the difference will again manifest itself only in the coefficients, this time, $T_{N-1}(\frac{\eta}{3})$, $R_{N-1}(\frac{\eta}{3})$ and $T_{N-2}(\eta)$, $R_{N-2}(\eta)$, respectively.

Tracing this procedure down to $N=0$ and seeing that all frequency-dependent exponents that appear along the way are equal for both terms in (8), we finally reach the point where subsequent substitution of eqs. (4) is no longer possible. At this point, the factors to be compared are $T_1(\frac{\eta}{3})$, $R_1(\frac{\eta}{3})$ and $T_0(\eta)$, $R_0(\eta)$. The corresponding phase terms are $\varepsilon(\Delta_0,\eta)$ and $\varepsilon_0(\Delta_0,\eta)$ as defined in eq. (7), and they are equal if the condition (2) is met. The difference in coefficients is smaller as r decreases, and the agreement is total if $r^2\approx 0$.

As we have seen, all frequency-dependent exponents in the expressions (8) are equal at any stage of decomposition. So it can be said that the quantities in eq. (8) have *identical phase structure*, with a minor difference in the coefficients. Since the characteristic spectral features (transmission resonances and local band gaps) are essentially phase phenomena (resulting from constructive and destructive interference, respectively), similar phase structure results in similar appearance of spectral portraits as confirmed by fig. 2.

Rigorous analytical generalization of these results to all fractal multilayers is rather straightforward and can be achieved by further generalizing the Sun-Jaggard computation procedure, *e.g.*, according to the multiple-reflection effective-medium formalism presented in [14].

Amplitude mismatch: vertical scalability. – However, as can be seen in fig. 3a, in the areas *between* characteristic features there is a significant difference in transmittance value T. This difference, which does not alter the shape of spectra, can be eliminated if, in addition to the above-mentioned frequency scaling (3), the value of $T_{N+1}(\eta/G)$ is raised to a certain power γ (see fig. 3b). Thus, the final *scalability equation* has the form

$$\left[T_{N+1}\left(\frac{\eta}{G}\right)\right]^{\gamma} = T_N(\eta). \tag{11}$$

Numerical analysis reveals that γ depends on the structure parameters and varies slightly with frequency. In the region close to $\eta=0$, where scalability is most often observed, we have found γ to equal

$$\gamma = \alpha + \frac{1-\alpha}{f^2}, \qquad f \equiv \frac{G-C}{G}, \tag{12}$$

where α was found to be small ($\alpha\cong 0.1$, see fig. 4).

It is important to note that γ *only* depends on the ratio between $G-C$ and G, called the *dielectric filling fraction*, and so, *e.g.*, the structures $(6,\{2,3\},N)$, $(6,\{3,4\},N)$, $(6,\{1,4\},N)$ have the same γ. The fact that it does not depend on the position of "removed" layers, nor on the structure's lacunarity, nor on the value of n_B/n_A, together with an observation that in the frequency region in question the propagating wave exhibits little internal reflection, suggests

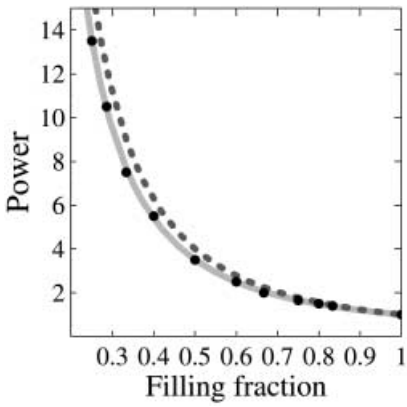

Fig. 4

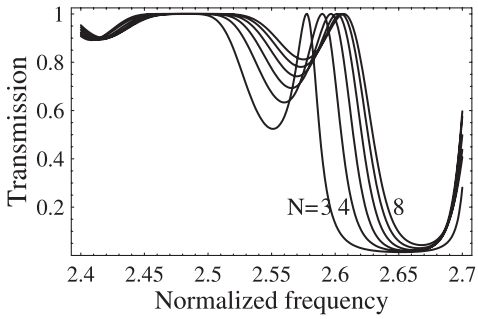

Fig. 5

Fig. 4 – The dependence $\gamma(f)$. The dots are numerical data, and the solid curve is the best-fit function. The dashed line is the analytically derived function.

Fig. 5 – An enlargement of a small part of the $(3, \{1\}, N)$ spectra. N ranges from 3 to 8. The peak mismatch is smaller as N is larger.

that *it is only the amount of dielectric "removed" during the transition $N \to N+1$ that is important for γ*. Thus, we can move forward to conclude that the only important parameter in f (see eq. (12)) is its numerator $G-C$, while the sole part of the denominator G is to allow for the frequency scaling (3).

That said, it is enough to investigate the simplest case in order to analytically establish the relation $\gamma(f)$. Instead of an arbitrary $(G, \boldsymbol{C}, N)$, consider an equivalent structure $(G, \boldsymbol{C'} = \{G-C, \ldots, G-1\}, N)$, which is effectively a single layer whose thickness scales as

$$d_N = (G-C)^N d_A \,. \tag{13}$$

Using the Airy formulas for the transmission, we can rewrite the scalability equation

$$\left|\frac{(1-r^2)\varepsilon'}{1-r^2\varepsilon'^2}\right| = \left|\frac{(1-r^2)\varepsilon'^f}{1-r^2\varepsilon'^{2f}}\right|^\gamma \Longrightarrow \frac{1+r^4-2r^2\cos\delta}{1+r^4-2r^2} = \left(\frac{1+r^4-2r^2\cos f\delta}{1+r^4-2r^2}\right)^\gamma , \tag{14}$$

where $\varepsilon' \equiv e^{i\delta} = \exp[(G-C)^N \eta\pi/2]$.

Since we are staying close to $\eta = 0$, we can assume $\eta \ll 1$ and therefore $\delta \ll 1$. At the 2nd order of Taylor series of $\cos\delta$ (the first order, obviously, leads to the identity $1 = 1^\gamma$), we finally arrive at

$$1 + \frac{2r}{(1-r)^2}\delta^2 = 1 + \gamma\frac{2r}{(1-r)^2}(f\delta)^2 \Longrightarrow \gamma = \frac{1}{f^2} \,, \tag{15}$$

which shows a good agreement with the numerically obtained eq. (12) (see fig. 4). An even better analytical agreement can be achieved using a finer approximation for $\cos\delta$. This also results in a weak dependence $\gamma(\eta)$, as was numerically confirmed and found not significant for the observation of scalability.

Peak mismatch: perturbation in characteristic effects. – So far we have shown that both phase structure matching and amplitude matching can be derived analytically. However, in fig. 3 one can observe small mismatches in the resonance peak locations for the spectral curves in (11). This agrees with the difference in the coefficients for the quantities (8) and shows

that spectral scalability is only *approximate* in real multilayers. However, we state that it results from the *finite size* of the structures under study. So, they are in fact prefractals rather than true fractals, so spatial self-similarity in them is not exact either. This disturbs the scalability effect in much the same way as it occurs in other types of media. For example, finite periodic structures cannot exhibit completely zero transmission in the band gaps, and in finite disordered media light cannot be completely trapped. In this manner, N-th generation Cantor multilayers can be compared to N-period 1D photonic crystals, while it is commonly known that band gaps are prominent at much larger N than were used for the plots in fig. 3.

However, if certain conditions are met, one can observe decent band gaps even in periodic multilayers with as many as four periods. An analogous statement is true for scalability in fractal multilayers. But the condition to be desired is opposite. As was noted earlier and confirmed in numerical calculations, the peak mismatch decreases if the refractive index contract is small, while band structure is more pronounced if the contrast is large enough [1,2].

Had it been otherwise, *i.e.*, if N approached infinity, it is our guess that spectral scalability would be exact. This can be indirectly confirmed by plotting the scaled spectra for several generations. As seen in fig. 5, the mismatch goes smaller as N grows larger.

Conclusion. – To summarize, using the method that inherently contains spatial self-similarity [11] along with scaling relations (3) and (13), we have found that fractal multilayers exhibit scalability both in phase and in the value of transmittance according to eq. (11). So, it can be concluded that *spectral scalability is actually the result of spatial self-similarity*, and moreover, it is a *characteristic relation* between a topological property of a multilayer structure and a spectral property of wave propagation. These results are also applicable for the electronic spectra in a fractal potential.

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