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Nonreciprocity and the second law of thermodynamics: An exact relation for nonlinear media

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Abstract. – Applying monochromatic filters to the ends of a one-dimensional nonlinear medium, we find an exact identity relating the nonreciprocity —the difference in transmission from the left or right— to the power flowing with equal sources on both sides. Thus, nonreciprocity has an unanticipated link to an apparent violation of the second law of thermodynamics; this violation is only apparent because of a subtle dependence on the (arbitrarily small) filter bandwidth. At high intensities, a nonperturbative transition to a noisy state destroys the identity. Apart from the theoretical insight provided by filters, not contaminating other frequency channels has obvious advantages in communications.

Introduction. – The reciprocity theorem has a long history in acoustics and optics [1, 2], and is valid for all linear time-reversal invariant media. When time-reversal invariance is broken, nonreciprocity is obtained. This is achieved in optics by a magnetic field or with magnetic materials [3]. Alternatively, moving objects can be used, as in the Sagnac effect [4] or acoustic tomography [5, 6]. Nonlinearity is also sufficient to cause nonreciprocity: diode-like photonic structures have been proposed using this fact [7–9].

Wave propagation in nonlinear media is complicated by the fact that even for monochromatic radiation, the medium generates all higher harmonics, coupling an infinite number of frequency channels together. For a linear system, different frequencies act as independent channels. In addition to simplifying the analysis, this decoupling produces constraints from the second law of thermodynamics. For instance, for a one-dimensional nondissipative medium, if monochromatic sources of equal intensity are switched on at both ends, there is no net flow of energy from one side to the other. (In order to accurately represent a single frequency channel of two black bodies, the relative phase of the two sources should be random, and a phase average performed.) This is true even without time-reversal invariance. Equivalently, with a single source the transmitted power is independent of whether the source is to the left or the right. These results do not apply to a nonlinear medium.

Here we use a new method to analyze wave propagation in one-dimensional nonlinear media, by applying monochromatic filters at the two ends. All the reflected and transmitted power is thus at the same frequency as the incident waves, with higher harmonics confined to the scattering medium. Apart from the fact that the analysis is considerably simplified by the need to consider only one input and output channel at each end, one can try to apply the second law of thermodynamics: since both the input and the output on each side are at the same frequency, it would seem that the incoming and outgoing waves could be viewed as being emitted and absorbed by a black body behind the filter.

Through numerical and analytical calculations on various physical models, with this single-channel configuration, we find a remarkable nonlinear identity relating the nonreciprocity in the transmitted power to the net power flowing from one side to the other with sources of equal intensity (and random relative phase) on both sides. In fact, from the discussion in the previous paragraph, one would expect the latter quantity, the net power flowing from one side to the other with sources of equal intensity on both sides, to be zero, so that nonreciprocity and this "pseudo-violation" of the second law are linked. (We show at the end of this paper why the second law is not really violated.) At high intensities, a nonperturbative transition to a noisy state destroys the identity.

We define $J_{\mathrm{LR}}(a)$ as the transmitted power from left to right when a wave of amplitude a impinges on the left of the medium, and $J_{\mathrm{RL}}(a)$ similarly. We also define $J_{\mathrm{Th}}(a)$ as the net power flowing from right to left in the "thermodynamic" configuration: when sources to the left and right are simultaneously switched on, with amplitude a and $ae^{i\phi}$, with the relative phase ϕ being averaged over. Our main result is then

$$J_{\mathrm{LR}}(a) - J_{\mathrm{RL}}(a) = 4J_{\mathrm{Th}}\big(a/\sqrt{2}\big) \tag{1}$$

for any a up to a maximum where a phase transition occurs (see the next section of this paper). The factor of $\sqrt{2}$ on the right-hand side causes the total incident power to be equal in all three cases: when the source is to the left, when the source is to the right, and with sources on both sides. (In the first two cases, the power emitted by the source is proportional to $|a|^2$, while in the last case, the power emitted by each source is proportional to $|a|^2/2$.) Without the factor of $\sqrt{2}$, no identity resembling eq. (1) is found. As the incident amplitude a is increased, the medium undergoes a phase transition from a periodic to an aperiodic steady state, and eq. (1) is no longer satisfied. We conjecture that eq. (1) is general for any scattering process with two input and output channels at the same frequency —whether one-dimensional or not— that is nondissipative, invariant under time translation and time reversal, and (in view of the phase transition) is perturbatively accessible.

For simplicity, we consider longitudinal (acoustic) waves. This has the additional motivation that all examples of nonreciprocity through nonlinearity considered so far have been optical.

Numerical results. – The first model we examine is a chain of N particles connected by anharmonic springs. If y_i are the displacements of the particles from their equilibrium positions,

$$m_i\ddot{y}_i = -\partial_{y_i}\big[V(y_i - y_{i-1}) + V(y_{i+1} - y_i)\big] \tag{2}$$

for all the particles except the first and last one, with

$$V(y) = \frac{1}{2}y^2 + \frac{\epsilon}{4}y^4. \tag{3}$$

The first and last particles have attached filters and are coupled to the environment. If $f_{\mathrm{i}}(x-vt) + f_{\mathrm{o}}(x+vt)$ is the displacement due to the incoming and outgoing waves beyond the

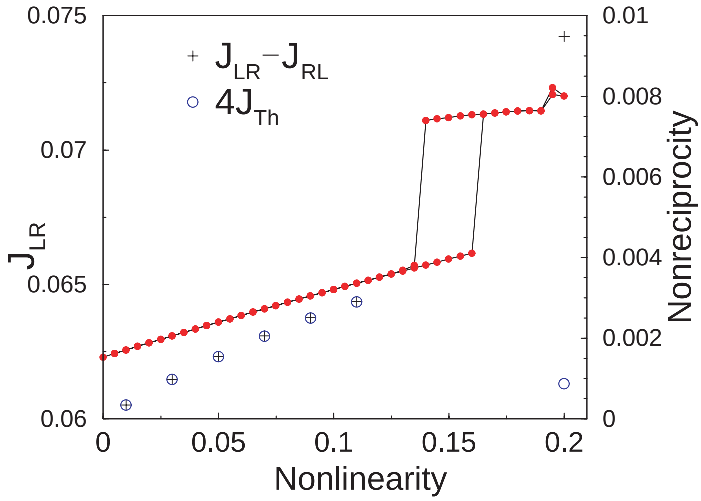

Fig. 1 – Numerical results for a $4+2$ particle chain with masses 100, 1.7, 1.4, 1.9, 1.3 and 100. The left-to-right current J_{LR} is plotted as a function of the nonlinearity ϵ (left vertical scale). The nonreciprocity, $J_{\mathrm{LR}} - J_{\mathrm{RL}}$, is compared with the phase-averaged current with both sources turned on at half intensity, J_{Th} (right vertical scale). The two quantities are equal in the perturbative state, but not in the nonperturbative state.

left boundary of the medium, the force exerted by these waves on the boundary is proportional to $-\partial_x[f_{\mathrm{i}} + f_{\mathrm{o}}]$, while the velocity of the boundary is equal to the velocity just outside the medium, $\partial_t[f_{\mathrm{i}} + f_{\mathrm{o}}]$. Since $-v\partial_x[f_{\mathrm{i}} + f_{\mathrm{o}}] = -\partial_t[f_{\mathrm{i}} + f_{\mathrm{o}}] + 2\partial_t f_{\mathrm{i}}$, the external force on the boundary is the sum of a "viscous" damping term and a term specified by the incoming waves [10]. For monochromatic waves,

$$\begin{aligned} m_1\ddot{y}_1 &= -m_1\omega_0^2 y_1 - V'(y_1 - y_2) - \kappa\dot{y}_1 + A_L\cos(\omega t), \\ m_N\ddot{y}_N &= -m_N\omega_0^2 y_N - V'(y_N - y_{N-1}) - \\ &\quad - \kappa\dot{y}_N + A_R\cos(\omega t + \phi). \end{aligned} \tag{4}$$

The $-m\omega_0^2$ term "tethers" the end particles and makes them act as filters if $m_{1,N} \to \infty$ and $\omega = \omega_0$: they are transparent at ω, since then the forces from the interior $-V'$ and from the exterior $-\kappa\dot{y} + A\cos(\omega t)$ are equal, while at higher harmonics since $m_{1,N}(n^2\omega_0^2 - \omega_0^2) \to \infty$, $y_{1,N}(n\omega_0) = 0$, and the boundaries are fixed. The filter bandwidths can be made arbitrarily small by increasing $m_{1,N}$.

These equations were simulated for a chain of $4+2$ particles with $\omega_0 = 1$, in units where $\kappa = a_{\mathrm{L,R}} = 1$ (with both sources on, $a_{\mathrm{L}} = a_{\mathrm{R}} = 1/\sqrt{2}$). Various values of $m_2 \ldots m_5$ were used; the results shown in fig. 1 are representative. The nonlinearity parameter ϵ in eq. (3) was increased at fixed $a_{\mathrm{L,R}}$, equivalent to increasing $a_{\mathrm{L,R}}$ at fixed ϵ. The system undergoes a first-order transition (with hysteresis) as ϵ is increased, from a periodic state with frequency $\omega = \omega_0 = 1$, to a noisy state. The existence of two states is somewhat similar to that in ref. [11] for optics. As seen in fig. 1, the perturbatively accessible periodic state satisfies eq. (1), whereas the nonperturbative state does not.

In the noisy state, preliminary results when the incident wave is entirely from the left show broad peaks in the transmitted power at $\omega \approx 0.4\omega_0$ and $\approx 0.15\omega_0$, in addition to the peaks at harmonics of ω_0. As ϵ is increased further, there is another transition at $\epsilon \approx 0.4$, with a jump in the average transmitted power and a broad-band component to the power spectrum. (As $m_{1,N} \to \infty$, the noise in the transmitted power vanishes outside the nonlinear medium. At finite $m_{1,N}$ it is a suppressed version of the power spectrum inside the medium.) The jumps from the perturbative state to the first noisy state and thence to the second noisy state are at different values of ϵ when the incident wave comes from the left instead of the right. A detailed dynamical analysis would be required to characterize the various noisy states and the transitions between them. However, this is not the focus of this paper.

Analytical results. – The second model we consider is a continuum one: two adjacent nonlinear layers, with the linear wave equation satisfied outside. Thus

$$n_i^2 \ddot{y} = B_i \partial_x^2 y + \mu_i \partial_x (\partial_x y)^2, \tag{5}$$

where y is the displacement of the wave, and n_i, B_i, μ_i vary from region to region. Inside the scattering medium, the two layers have parameters (n_1, B_1, μ_1) and (n_2, B_2, μ_2). Outside, $n = B = 1$ and $\mu = 0$. The scatterer covers the region $-1 < x < 1$, with the boundary between the two layers at $x = 0$. Equation (5) retains the leading nonlinearity in the elasticity of the medium; the energy density of the wave is $\frac{1}{2} n^2 \dot{y}^2 + \frac{1}{2} (\partial_x y)^2 + \mu (\partial_x y)^3 / 3$. At the boundary at $x = 0$, y and $B \partial_x y + \mu (\partial_x y)^2$ (the force exerted on the boundary from the two regions it separates) are continuous. We also consider an alternative to eq. (5):

$$n_i^2 \ddot{y} = B_i \partial_x^2 y + \mu_i \partial_x (\partial_x y)^3. \tag{6}$$

Equations (5) and (6) are in the class of Fermi-Pasta-Ulam (FPU) equations [12].

Equations (5) and (6) were solved using perturbation theory. Higher harmonics of the fundamental frequency ω are confined to the scattering medium $-1 < x < 1$ as standing waves. For ω, the filters are transparent, and the boundary conditions at $x = \pm 1$ are the same as at $x = 0$. The equations were solved to third order using Mathematica$^{\mathrm{TM}}$, with $\mu = 1$ and various values for $n_{1,2}$, $B_{1,2}$ and ω. This yielded the leading $\mathrm{O}(|a|^4)$ correction to the outgoing power to the left (and right) for eq. (5), and *two* nonlinear contributions, to $\mathrm{O}(|a|^6)$, for eq. (6). In both cases, eq. (1) was verified. Note that this perturbative expansion was incapable of seeing the nonperturbative noisy phase observed numerically. Removing the filters caused eq. (1) to break down.

The nonzero right-hand side of eq. (1) might seem to contradict the second law of thermodynamics. If identical black bodies (at the same temperature) are placed on both sides of the medium, behind the filters, the filters should let equal-intensity monochromatic waves into the medium. Since a nonzero right-hand side to eq. (1) implies a net flow of power from one side to another with equal intensity monochromatic sources, this seeming equivalence between black-body and monochromatic sources suggests that with identical black-body sources, one would heat up and the other cool down. (This is different from ref. [13], where the sources had to be maintained out of equilibrium, so that second law arguments were inapplicable.) The fallacy in this argument is that, as illustrated by our discrete model, any filter has a nonzero (albeit arbitrarily small) bandwidth; in our discrete model, $m_{1,N}$ must be finite. There are, therefore, two different regimes. In the first, the filter bandwidth is small compared to the source bandwidth, in which case the response of the medium does not change if the source bandwidth is increased, and thermodynamic arguments applicable to black bodies can indeed be invoked. The other regime is when the source bandwidth is much smaller than the filter bandwidth, in which case the response of the medium does not change if the source bandwidth is decreased till it is perfectly monochromatic. It is impermissible to use thermodynamic arguments from the first regime, for the second regime of monochromatic sources (the focus of this paper) [14]. Therefore, the second law is not violated by the net flow of power for equal-strength monochromatic sources on both sides that we have obtained here. From the discussion before eqs. (4), black-body sources at the ends correspond to white noise being applied to the terminal particles (which is then filtered by them). With the damping term in eq. (4), eqs. (2) and (4) are then generalized Langevin equations, which reach thermal equilibrium [15], in accordance with the second law.

Discussion. – In this paper, we have obtained an exact identity for wave propagation in nonlinear media, relating the nonreciprocity in transmission to a pseudo-violation of the second law of thermodynamics. This has been done with a setup with monochromatic filters at the boundaries of the scatterer, enabling different frequency channels to be kept separate. Through a mapping to a Langevin equation, we have shown that the second law is satisfied, and that the apparent violation occurs because of a singular dependence on the filter bandwidths. We have also obtained a nonperturbative transition to a noisy state when the nonlinearity (equivalently, the external driving) is strong. The results obtained have been with sound waves instead of light, for which, to our knowledge, nonlinearity-induced nonreciprocity has not been demonstrated so far.

As mentioned earlier, we conjecture that eq. (1) is valid for any nondissipative time-translation and time-reversal invariant scattering process with two input and output channels at the same frequency, whether one-dimensional or not, provided that one is in the weak driving (perturbative) regime. Apart from the phase transition shown in this paper, the necessity of the last condition can be seen mathematically: explicit counterexamples to eq. (1) can be constructed without it [17].

It would be interesting to examine how far eq. (1) extends if the filters were at different frequencies (one a multiple of the other), if there were more than two input/output ports, or with light instead of sound. A physical explanation of eq. (1) would help resolve such questions. It should be possible to have multiple channels in parallel in the perturbative phase if their frequency ratios were irrational, but it is not clear how strong the interference between them would be otherwise. Second-law constraints could be probed further: with almost monochromatic sources, the dependence of both sides of eq. (1) on source bandwidth could be investigated.

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