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Extraordinarily superpersistent chaotic transients

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Abstract. – We uncovered a class of transient chaos for which the average lifetime obeys the following scaling law: $\tau \sim \exp[C_0 \exp[C_1 \varepsilon^{-\gamma}]]$, where C_0, C_1, and γ are positive constants and ε is a scaling parameter. This occurs in dynamical systems preceding an unstable-unstable pair bifurcation, subject to noise of amplitude ε. The extreme longevity of the transient lifetime for small ε is striking, which has not been reported previously. We formulate a theory to explain this type of *extraordinarily superpersistent chaotic transients*, and point out physical relevance and implications.

Superpersistent chaotic transients (in the normal sense) are characterized by the following scaling law for their lifetime [1, 2]:

$$\tau \sim \exp[C\epsilon^{-\alpha}], \tag{1}$$

where ϵ is a parameter variation, $C > 0$ and $\alpha > 0$ are constants. As ϵ approaches the critical value zero, the transient lifetime τ becomes superpersistent in the sense that the exponent in the exponential dependence diverges. The fundamental dynamical mechanism for this type of transient chaos was argued to be the unstable-unstable pair bifurcation, in which an unstable periodic orbit on the boundary of a chaotic invariant set collides with another unstable periodic orbit pre-existing outside the set [1,2]. The same mechanism was believed to be responsible for the bifurcation that creates a riddled basin [3]. Earlier the transients were also identified in a class of coupled-map lattices, leading to the speculation that asymptotic attractors may not be relevant for turbulence [4]. Superpersistent chaotic transients were demonstrated in laboratory experiments using electronic circuits [5]. Signatures of noise-induced superpersistent chaotic transients were recently identified [6] in the advective dynamics of inertial particles in open chaotic flows.

In this letter, we report a new class of transient chaos that is more persistent than superpersistent chaotic transient in the following sense of scaling:

$$\tau \sim \exp[C_0 \exp[C_1 \varepsilon^{-\gamma}]], \quad \text{for small } \varepsilon, \tag{2}$$

where $C_0 > 0$, $C_1 > 0$, and $\gamma > 0$ are constants. The transient is induced by noise in the parameter regime preceding an unstable-unstable pair bifurcation, where ε is the noise

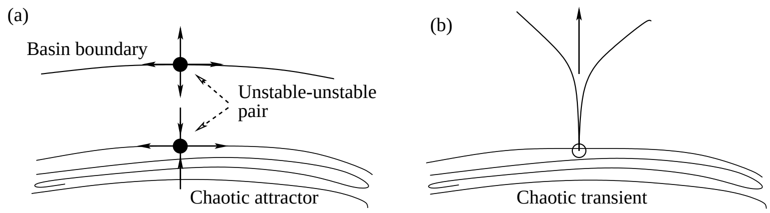

Fig. 1 – (a) In the absence of noise, a chaotic attractor, the basin boundary, and the pair of unstable periodic orbits. (b) Escaping channel induced by noise through the stochastic collision between the unstable-unstable pair.

amplitude. Because of the double exponential dependence and the algebraic divergence for small noise, the average lifetime of the transients can be significantly longer than that given by the normal superpersistent scaling law (1). We thus call this type of transient chaos *extraordinarily superpersistent chaotic transients*. To establish the scaling law (2), we construct an analyzable model that captures the fundamental dynamics near an unstable-unstable pair bifurcation, under the influence of noise. In particular, let p be the system parameter and as it increases through p_c, an unstable-unstable pair bifurcation occurs. For p in the vicinity of p_c, the model can be converted to a class of one-dimensional stochastic differential equations, the solution to which yields the scaling law (2) for $p < p_c$ (subcritical regime). More precisely, the scaling (2) is valid for $0 < \varepsilon < \varepsilon_c$, where $\varepsilon_c \to 0$ as $p \to p_c$. For $\varepsilon > \varepsilon_c$, the transient is normally superpersistent in the sense of the scaling law (1). That is, as p approaches the bifurcation point p_c from below, the regime in which the extraordinarily superpersistent chaotic transients can be observed shrinks and eventually vanishes. For $p = p_c$, the regular scaling law (1) holds. For $p > p_c$ (supercritical regime), the average transient lifetime follows the regular scaling law (1) for large noise but it approaches a constant for small noise. Thus, in this supercritical parameter regime where there is already a chaotic transient, the lifetime has no dependence on the noise amplitude if it is small but the lifetime decreases as the noise is increased. This means that the transient lifetime in the presence of noise can be significantly shorter than that in the absence of noise. This is in contrast to a speculation in ref. [2] that noise tends to increase the transient lifetime in the supercritical regime. We believe these findings are important not only for the fundamentals of nonlinear systems, but also for a better understanding of the interplay between noise and chaotic dynamics.

The dynamical mechanism for superpersistent chaotic transients is the unstable-unstable pair bifurcation [1–3], which can occur in high-dimensional chaotic systems that are described by invertible maps of at least three dimensions (corresponding to flows of at least four dimensions) or noninvertible maps of at least two dimensions. For simplicity we consider a two-dimensional noninvertible map. Imagine two unstable periodic orbits of the same period, one on the chaotic attractor and another on the basin boundary, as shown in fig. 1(a). In a noiseless situation, as p is increased through p_c, the two orbits *coalesce* and disappear simultaneously, leaving behind a "channel" in the phase space through which trajectories on the chaotic attractor can escape to another attractor, as shown in fig. 1(b). The chaotic attractor is thus converted into a chaotic transient, but the channel created by this mechanism is extremely narrow, leading to the regular superpersistent scaling law (1). Our main interest in this letter is transient chaos induced by noise for $p < p_c$, where there is an attractor in the absence of noise. For small noise, there can be small but nonzero probability that a chaotic trajectory moves to the location of the channel and stays there for a finite amount of time to escape through the channel while it is open.

Suppose the largest Lyapunov exponent of the chaotic attractor is $\lambda > 0$. Due to noise, the unstable-unstable pair can undergo collisions at random times, giving rise to a set of channels (at the location of the orbit and all its preimages) that open stochastically in time. During their opening, the channels are locally transverse to the attractor. In order to escape, a trajectory must spend at least time T at the openings. The trajectory must then come to within distance of about $\exp[-\lambda T(\varepsilon)]$ of the periodic orbit on the attractor or any of its preimages. The probability for this to occur is proportional to $\exp[-\lambda T(\varepsilon)]$. The average time for the trajectory to remain on the attractor, or the average transient lifetime, is thus

$$\tau \sim \exp[\lambda T(\varepsilon)]. \tag{3}$$

We see that the dependence of $T(\varepsilon)$ on ε, which is the average time that the trajectories spend in the escaping channel, or the *tunneling time*, is the key quantity determining the scaling of the average chaotic transient lifetime τ with noise.

To obtain the scaling dependence of the tunneling time $T(\epsilon)$ on ϵ, we note that, since the escaping channel is extremely narrow, for typical situations where $\lambda > 0$ and $T(\epsilon)$ large, the dynamics in the channel is approximately one-dimensional along which the periodic orbit on the attractor is stable but the orbit on the basin boundary is unstable for $p < p_{\mathrm{c}}$ (fig. 1(a)). This feature can thus be captured through the following simple one-dimensional map:

$$x_{n+1} = x_n^2 + x_n + p + \epsilon\xi(n), \tag{4}$$

where x denotes the dynamical variable in the channel, p is a bifurcation parameter with critical point $p_{\mathrm{c}} = 0$, ϵ is the noise amplitude, and $\xi(n)$ is a Gaussian random variable of zero mean and unit variance. For $p < p_{\mathrm{c}} = 0$, the map has a stable fixed point $x_{\mathrm{s}} = -\sqrt{-p}$ and an unstable fixed point $x_{\mathrm{u}} = \sqrt{-p}$. These two collide at p_{c} and disappear for $p > p_{\mathrm{c}}$, mimicking an unstable-unstable pair bifurcation. Assuming the tunneling time $T \gg 1$, we can approximate the map by the following one-dimensional stochastic differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 + p + \varepsilon\xi(t). \tag{5}$$

For convenience we write $x_{\mathrm{r}} = x_{\mathrm{s}}$ for $p < p_{\mathrm{c}}$ and $x_{\mathrm{r}} = 0$ for $p \geq p_{\mathrm{c}}$. Let $P(x,t)$ be a probability density function of the stochastic process (5), which obeys the Fokker-Planck equation [7],

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(x^2+p\right)P(x,t)\right] + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2}\,. \tag{6}$$

Let l be the effective length of the channel in the sense that a trajectory with $x > l$ is considered to have exited the channel. The average time required for a trajectory to travel through the channel is effectively the mean first-passage time from x_{r} to l. Our interest is in the trajectories that do escape. For such a trajectory, we assume that once it falls into the channel through x_{r}, it will eventually exit the channel at $x = l$ without going back to the original chaotic attractor. This is reasonable considering that the probability for a trajectory to fall in the channel and then to escape is already exponentially small (eq. (3)) and, hence, the probability for any "second-order" process to occur, where a trajectory falls in the channel, moves back to the original attractor, and falls back in the channel again, is negligible. For trajectories in the channel there is thus a reflecting boundary condition at $x = x_{\mathrm{r}}$: $[P(x,t) - \partial P/\partial x]|_{x=x_{\mathrm{r}}} = 0$. That trajectories exit the channel at $x = l$ indicates an absorbing boundary condition at $x = l$: $P(l,t) = 0$. Assuming that trajectories initially are near the opening of the channel

TABLE I – *Scaling laws of τ in different parameter regimes.*

τ	$0 < \varepsilon < \varepsilon_c$	$\varepsilon > \varepsilon_c$
$p < p_c$	$\exp[C_0 e^{C_1 \varepsilon^{-2}}]$	$\exp[C_2 \varepsilon^{-2/3}]$
$p = p_c$	$\exp[C\varepsilon^{-2/3}]$	$\exp[C\varepsilon^{-2/3}]$
$p > p_c$	constant	$\exp[C_3 \varepsilon^{-2/3}]$

(but in the channel), we have the initial condition $P(x, x_r) = \delta(x - x_r^+)$. Under these boundary and initial conditions, the Fokker-Planck equation can be solved, yielding the following mean first-passage time [7]:

$$T_p(\varepsilon) = \frac{2}{\varepsilon^2} \int_{x_r}^{l} dy \exp[-bH(y)] \int_{x_r}^{y} \exp\left[bH(y')\right] dy', \tag{7}$$

where $H(x) = x^3 + 3px$ and $b = 2/3\varepsilon^2$.

To evaluate the double integral in eq. (7), we make use of the approximate forms of $H(x)$ near x_s and x_u: $H(x) \approx 2|p|^{3/2} - 3\sqrt{|p|}(x - x_s)^2 \equiv H_1(x)$ for $x \approx x_s$ and $H(x) \approx -2|p|^{3/2} + 3\sqrt{|p|}(x - x_u)^2 \equiv H_2(x)$ for $x \approx x_u$. For $p < p_c$, the integrand $\exp[-bH(x)]$ in the first integral exhibits both increasing and decreasing behaviors on $[x_r, l]$ and the second integral $\int_{x_s}^{y} \exp[bH(w)]dw$ is an increasing function of y. It is thus convenient to break the double integral into five integrals on various subintervals:

$$\begin{aligned}
\int_{x_s}^{l} dy \exp[-bH(y)] \int_{x_s}^{y} \exp[bH(w)]dw &= \int_{x_s}^{0} dy \exp[-bH(y)] \int_{x_s}^{y} \exp[bH(w)]dw + \\
&+ \int_{0}^{l} dy \exp[-bH(y)] \int_{x_s}^{0} \exp[bH(w)]dw + \int_{0}^{x_u} dy \exp[-bH(y)] \int_{0}^{y} \exp[bH(w)]dw + \\
&+ \int_{x_u}^{l} dy \exp[-bH(y)] \int_{0}^{x_u} \exp[bH(w)]dw + \int_{x_u}^{l} dy \exp[-bH(y)] \int_{x_u}^{y} \exp[bH(w)]dw,
\end{aligned}$$

which can be individually estimated using the approximate functions $H_1(x)$ and $H_2(x)$. We obtain([1]): $T_p(\varepsilon) \sim \frac{1}{\sqrt{-p}} \exp[\frac{(-p)^{3/2}}{\varepsilon^2}]$ for $\varepsilon < \varepsilon_c = |p|^{3/4}$ and $T_p(\varepsilon) \sim \varepsilon^{-2/3}$ for $\varepsilon > \varepsilon_c$. Substituting $T_p(\varepsilon)$ in eq. (3) for the small-noise regime ($\varepsilon < \varepsilon_c$) gives the scaling law (2) for extraordinarily superpersistent chaotic transients. For large noise ($\varepsilon > \varepsilon_c$), we see that the transient is normally superpersistent.

For $p \geq p_c$, the function $H(x)$ is increasing so that the double integral in eq. (7) can be estimated in terms of an infinite power series which can be proved to be convergent. Our detailed analysis yields the following: (1) for $p = p_c = 0$, $T_p(\varepsilon) \sim \varepsilon^{-2/3}$, and (2) for $p > p_c$, $T_p(\varepsilon) \sim 1/\sqrt{p} = \text{const}$ for $\varepsilon < \varepsilon_c$ and $T_p(\varepsilon) \sim \varepsilon^{-2/3}$ for $\varepsilon > \varepsilon_c$.

Our theoretically predicted scaling laws for the chaotic-transient lifetime in the three regimes are summarized in table I, where constants C, C_1, C_2, and C_3 are all positive. From the right column of the table, we see that the critical noise amplitude ε_c is the noise level beyond which the transients are normally superpersistent for both the subcritical and supercritical cases. The conceptual meaning of ε_c is as follows. For the subcritical case, there is a distance between the unstable-unstable pair and there is no open channel in the deterministic case. Noise induces random opening and closing of the channel. As such, ε_c is proportional

([1])Y. Do and Y.-C. Lai, preprint (2004).

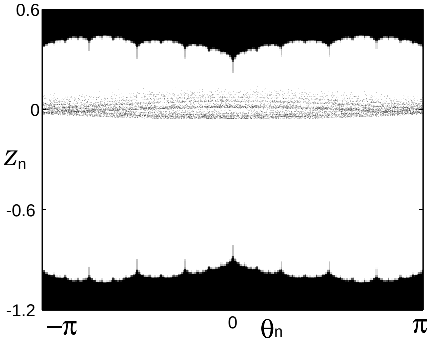

Fig. 2 – For the two-dimensional map model, in the absence of noise, a chaotic attractor near $z = 0$, its basin of attraction (blank), and the basin of the attraction of the attractor at $z = +\infty$ (black).

to the distance between the pair of unstable periodic orbits. In the supercritical regime, an escaping channel is already open. If the noise level is too small, it will not have any appreciable effect on the channel dynamics. The critical noise level ε_c thus depends on the width of the open channel. In both cases, ε_c depends on the parameter variation from the critical point.

We now provide numerical support for the scaling laws in table I. We utilize the following two-dimensional, noninvertible map which was first used by Grebogi *et al.* [1, 2] to illustrate superpersistent chaotic transients in the deterministic case: $\theta_{n+1} = 2\theta_n \bmod 2\pi$ and $z_{n+1} = az_n + z_n^2 + \beta \cos\theta_n$, where a and β are parameters. Because of the z_n^2-term in the z-equation, for large $|z_n|$ we have $z_{n+1} > z_n$. There is thus an attractor at $z = +\infty$. Near $z = 0$, depending on the choice of the parameters, there can be either a chaotic attractor or none. For instance, for $0 < \beta \ll 1$, there is a chaotic attractor near $z = 0$ for $a < a_c = 1 - 2\sqrt{\beta}$ and the attractor becomes a superpersistent chaotic transient for $a > a_c$ [1,2]. Figure 2 shows, for $\beta = 0.04$ and $a = a_c = 0.6$, the chaotic attractor near $z = 0$, its basin of attraction (blank), and the basin of the attraction of the attractors at $z = +\infty$ (black). Signature that an unstable-unstable pair bifurcation is about to occur can be seen by the vertical, downward tip at $\theta = 0$, which is close to the chaotic attractor. To describe a channel dynamics at $\theta = 0$, we can rewrite the z-mapping in the model as $x_{n+1} = x_n + x_n^2 + p$, where $x = z - z_*$, z_* is the minimum of the right-hand side of the z-mapping, and $p = \sqrt{\beta}(a - a_c) - [(a - a_c)/2]^2$, *i.e.* $p \approx \sqrt{\beta}(a - a_c)$ for small $|a - a_c|$.

Figure 3(a) shows the behaviors of the tunneling time for five cases in the subcritical regime ($p = -0.0001$, $p = -0.001$, $p = -0.01$, $p = -0.1$, and $p = -0.2$, and the five vertical dashed lines from right to left indicate the values of $\ln 1/\varepsilon_c$ for these five cases, respectively), for the critical case ($p = p_c = 0$), and for the supercritical case ($p = 0.0001$). We observe a robust algebraic scaling with the 2/3 exponent for the critical case. For large noise $\varepsilon > \varepsilon_c$, the scalings for the subcritical and supercritical cases coincide with that for the critical case. For the supercritical case, however, the tunneling time plateaus for $\varepsilon < \varepsilon_c$. Since the plateaued value of T is approximately the tunneling time in the absence of noise, we see that as the noise amplitude is increased, the tunneling time decreases, as predicted by our theory. (Note that this contrasts the speculation in ref. [2] that noise tends to prolong the chaotic transient lifetime.) For the subcritical cases, the tunneling time increases substantially for $\varepsilon < \varepsilon_c$ as compared with that in the critical case. Figure 3(b) shows, on a double-logarithmic *vs.* logarithmic scale, the behaviors of the tunneling time for the five cases in the subcritical regime. The approximately linear fits in the small-noise range indicate that the *tunneling times themselves* for those cases obey the superpersistent scaling law. This can be considered as *indirect* evidence for extraordinarily superpersistent chaotic transient. Note that, because

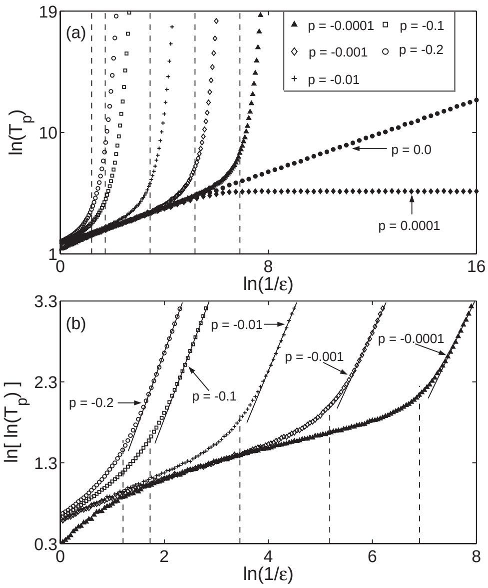

Fig. 3 – (a) Scaling behaviors of the average tunneling times $T_p(\varepsilon)$ with noise on a logarithmic scale for five cases in the subcritical regime ($p = -0.0001$, $p = -0.001$, $p = -0.01$, $p = -0.1$, and $p = -0.2$), for the critical case ($p = p_c = 0$), and for the supercritical case ($p = 0.0001$). (b) Replots of the tunneling times on a double-logarithmic *vs.* logarithmic scale for the five cases in the subcritical regime in (a). Because of the extremely rapid increase in the tunneling time as ε is decreased from ε_c, it is difficult to extend the scaling range for the noise variation.

of the extremely rapid increase in the tunneling time as ε is decreased from ε_c, it is difficult to extend the range of the noise variation in fig. 3(b). (For instance, decreasing ε by one order of magnitude increases $T_p(\varepsilon)$ by a factor of e^{100}.)

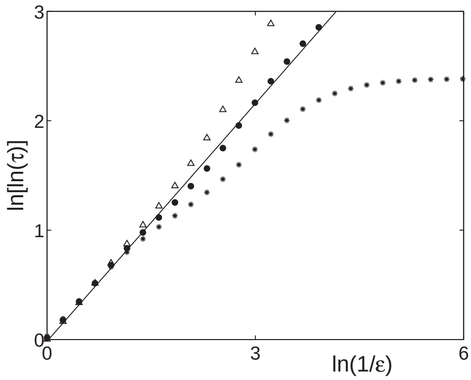

Fig. 4 – For our numerical model, noisy scaling laws of the average chaotic transient lifetime for the subcritical ($a = 0.5 < a_c$, triangles), critical ($a = a_c$, filled circles), and supercritical ($a = 0.7 > a_c$, asterisks) cases. The straight line indicates normally superpersistent chaotic transient, which holds for $a = a_c$ and for relatively large noise range in both the subcritical ($a < a_c$, the upper data set) and the supercritical ($a > a_c$, the lower data set) cases.

Figure 4 shows the scaling of the average chaotic transient lifetime with the noise amplitude ε on a proper scale. The scaling in the critical case follows the normal superpersistent law (1). For the subcritical and supercritical cases, the scalings coincide with the normal superpersistent law only for large noise ($\varepsilon > \varepsilon_c \approx (0.1\sqrt{\beta})^{3/4}$). In the supercritical regime, the transient lifetime deviates from the normal superpersistent scaling law for $\varepsilon < \varepsilon_c$ and then levels off, approaching a constant as $\varepsilon \to 0$. In the subcritical regime, for small noise, the chaotic transient lifetime increases much faster as compared with the critical case. This behavior, in combination with fig. 3(b), represents numerical evidence for our predicted extraordinarily superpersistent chaotic transients. Note that the ordinate of fig. 4 is already on a double-logarithmic scale. Because of the scaling constants in (2), a triple-logarithmic scale plot would be inappropriate because such a plot still would not yield a linear behavior.

A feature of our theoretical predictions is that for relatively large noise, the scaling laws for the average transient lifetime for the subcritical, critical, and supercritical parameter regimes all collapse onto a single curve characterizing the normal superpersistent transient behavior that has indeed been observed experimentally [5]. Possible challenges to experimental detection of an extraordinarily superpersistent chaotic transient include: 1) its distinction from a normally superpersistent transient occurs only in the small-noise regime, and 2) its lifetime can be extremely long. However, we are hopeful that these difficulties can be overcome, allowing for experimental test of the fundamental phenomenon of extraordinarily superpersistent chaotic transient.

In summary, we have addressed the phenomenon of noise-induced superpersistent chaotic transients and derived scaling laws governing the dependence of the average transient lifetime on noise amplitude. Our main contribution is the discovery of extraordinarily superpersistent chaotic transients, which is induced by noise in a parameter regime preceding an unstable-unstable pair bifurcation. These results can be relevant to physical situations such as the advection of inertial particles in open chaotic flows(2) [6, 8].

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(2)In ref. [6], superpersistent chaotic transients were identified for noise-induced escape from chaotic attractors formed by inertial particles in open fluid flows. The range of noise variation there corresponds to, for instance in fig. 4, the noise range on the left-hand side where all three data sets for the subcritical, critical, and supercritical cases, respectively, collapse onto a single line characterizing normally superpersistent chaotic transient. Extraordinarily superpersistent chaotic transients in the subcritical regime occur for noise amplitude smaller than those in this range.