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Quantum correction in exact quantization rules

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Abstract. – An exact quantization rule for the Schrödinger equation is presented. In the exact quantization rule, in addition to $N\pi$, there is an integral term, called the quantum correction. For the exactly solvable systems we find that the quantum correction is an invariant, independent of the number of nodes in the wave function. In those systems, the energy levels of all the bound states can be easily calculated from the exact quantization rule and the solution for the ground state, which can be obtained by solving the Riccati equation. With this new method, we re-calculate the energy levels for the one-dimensional systems with a finite square well, with the Morse potential, with the symmetric and asymmetric Rosen-Morse potentials, and with the first and the second Pöschl-Teller potentials, for the harmonic oscillators both in one dimension and in three dimensions, and for the hydrogen atom.

In the development of quantum mechanics, the Bohr-Sommerfeld quantization rules of the old quantum theory [1] occupy a position intermediate between classical and quantum mechanics. The WKB approximation [2–4] is a method for the approximate treatment of the Schrödinger wave function with another quantization rule [1]. Both quantization rules are approximate. We are going to derive an exact quantization rule for the one-dimensional Schrödinger equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) = -\frac{2M}{\hbar^2}\left[E - V(x)\right]\psi(x), \tag{1}$$

where M is the mass of the particle, and the potential $V(x)$ is a piecewise continuous real function of x satisfying, for definiteness,

$$\begin{aligned} V(x) &> E, \qquad -\infty < x < x_A \quad \text{or} \quad x_B < x < \infty, \\ V(x) &= E, \qquad x = x_A \quad \text{or} \quad x = x_B\,, \\ V(x) &< E, \qquad x_A < x < x_B\,, \end{aligned} \tag{2}$$

where x_A and x_B are two turning points. Between two turning points, the momentum is $k(x) = \sqrt{2M[E - V(x)]}/\hbar$.

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Yang pointed out in a talk on monopole: "For the Sturm-Liouville problem, the fundamental trick is the definition of a phase angle which is monotonic with respect to the energy" [5]. This phase angle is the logarithmic derivative $\phi(x) = \psi(x)^{-1}\mathrm{d}\psi(x)/\mathrm{d}x$ of the wave function $\psi(x)$. Due to the Sturm-Liouville theorem, $\phi(x)$ at any given point $x = x_0$ is monotonic with respect to the energy. From the Schrödinger equation (1), $\phi(x)$ satisfies the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) = -\frac{2M}{\hbar^2}[E - V(x)] - \phi(x)^2. \tag{3}$$

It shows that $\phi(x)$ decreases monotonically with respect to x between two turning points where $E \geq V(x)$. Note that as x increases across a node of the wave function $\psi(x)$ where $E \geq V(x)$, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. The Riccati equation is a differential equation of the first order so that it is much easier to find a special solution from the Riccati equation than from the Schrödinger equation.

Letting $\tan\theta(x) = k(x)/\phi(x)$, we have $\theta(x) = \mathrm{Arctan}[k(x)/\phi(x)] + n\pi$, where $\mathrm{Arctan}\,\beta$ denotes the principle value of the inverse tangent function, $-\pi/2 < \mathrm{Arctan}\,\beta \leq \pi/2$, and n increases by one as x increases across a node of $\phi(x)$, where $E \geq V(x)$. Then, we have

$$\int_{x_A}^{x_B} \frac{\mathrm{d}\theta(x)}{\mathrm{d}x}\mathrm{d}x = N\pi - \lim_{x\to x_A+} \mathrm{Arctan}\left(\frac{k(x)}{\phi(x_A)}\right) + \lim_{x\to x_B-} \mathrm{Arctan}\left(\frac{k(x)}{\phi(x_B)}\right), \tag{4}$$

where N is the number of nodes of $\phi(x)$, and two terms with limit are vanishing if the potential $V(x)$ is continuous at the turning points. From eq. (3) we have

$$\frac{\mathrm{d}\theta(x)}{\mathrm{d}x} = k(x) - \phi(x)\left[\frac{\mathrm{d}k(x)}{\mathrm{d}x}\right]\left[\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right]^{-1}, \qquad E \geq V(x). \tag{5}$$

Integrating both sides of eq. (5) over the interval between two turning points, we obtain the quantization rule without any approximation:

$$\int_{x_A}^{x_B} k(x)\mathrm{d}x = N\pi + \int_{x_A}^{x_B} \phi(x)\left[\frac{\mathrm{d}k(x)}{\mathrm{d}x}\right]\left[\frac{\mathrm{d}\phi(x)}{\mathrm{d}x}\right]^{-1}\mathrm{d}x. \tag{6}$$

Since $\phi(x_A) > 0$, $\phi(x_B) < 0$, and $\phi(x)$ decreases monotonically in the region $x_A < x < x_B$, the number N of nodes of $\phi(x)$ in that region is larger by one than the number of nodes of the wave solution $\psi(x)$. Due to the Sturm theorem, the number $(N-1)$ of nodes of $\psi(x)$ increases as the energy E increases.

Generalize the quantization rule to the three-dimensional Schrödinger equation with a spherically symmetric potential. After separation of the angular part of the wave function, $\psi(\boldsymbol{r}) = r^{-1}R(r)Y_m^\ell(\theta,\varphi)$, the radial equation of the Schrödinger equation is

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} = -\frac{2M}{\hbar^2}\left[E - U(r)\right]R(r), \quad U(r) = \frac{\ell(\ell+1)\hbar^2}{2Mr^2} + V(r). \tag{7}$$

Since eq. (7) is similar to eq. (1), the quantization rule (6) is generalized to the three-dimensional Schrödinger equation with a spherically symmetric potential,

$$\int_{r_A}^{r_B} k(r)\mathrm{d}r = N\pi + \int_{r_A}^{r_B} \phi(r)\left[\frac{\mathrm{d}k(r)}{\mathrm{d}r}\right]\left[\frac{\mathrm{d}\phi(r)}{\mathrm{d}r}\right]^{-1}\mathrm{d}r. \tag{8}$$

The quantization rule (6) (or (8)) is proved without any approximation, so that it is exact. The first term $N\pi$ in the quantization rule relates to the contribution from the nodes

of the wave function, and the second term is called the quantum correction. We find that the quantum correction is independent of the number of nodes of the wave function for the exactly solvable systems. In those systems, one is able to calculate easily the energy levels of all bound states from the exact quantization rule and the solution of the ground state, which can be calculated directly from the Riccati equation. With this new method, we are going to re-calculate the energy levels of those systems.

The one-dimensional system with a finite square well, where $V(x) = V_A$ when $x \leq -\pi$, $V(x) = V_B$ when $x \geq \pi$, and $V(x) = 0$ when $-\pi < x < \pi$, is a typical example in the course of quantum mechanics. However, the solutions can be obtained with the exact quantization rule much more simply than with the standard method. Since $k(x)$ is constant between two turning points, the quantum correction in the quantization rule is vanishing. For the finite square well, two terms with limit in eq. (4) have to be included in the quantization rule (6) because the potential jumps at the turning points. The energy levels of the system is directly given in the quantization rule (6) and (4) [6]:

$$2\pi k_N = N\pi - \mathrm{Arctan}\left(\frac{k_N}{\phi_N(x_A)}\right) + \mathrm{Arctan}\left(\frac{k_N}{\phi_N(x_B)}\right), \tag{9}$$

where $\phi_N(x_A) = \sqrt{2M(V_A - E_N)}/\hbar$, $\phi_N(x_B) = -\sqrt{2M(V_B - E_N)}/\hbar$. When $V_A = V_B \to \infty$, we have $k_N = N/2$ [1].

The potential for a one-dimensional harmonic oscillator is $V(x) = M\omega^2 x^2/2$. The turning points are solved to be $x_B = -x_A = \alpha^{-1}\sqrt{2E_n/(\hbar\omega)}$, where $\alpha = \sqrt{M\omega/\hbar}$, and $(n+1)$ is the number of nodes of $\phi_n(x)$. The momentum between two turning points is $k_n(x) = \alpha^2[(x - x_A)(x_B - x)]^{1/2}$. The solution with one node and no pole only has the form $\phi_0(x) = -Cx$, where $C > 0$ due to the monotonic property. Substituting it into the Riccati equation (3) with the potential of a harmonic oscillator, we find $\phi_0(x) = -\alpha^2 x$ with $E_0 = \hbar\omega/2$. Evidently, $\phi_0(x)$ is negative when $x \to \infty$ and is positive when $x \to -\infty$, so that the solution satisfies the physically admissible boundary conditions. All the solutions in the following examples have similar behavior. Two integrals in the exact quantization rule (6) are calculated to be

$$\int_{x_A}^{x_B} \phi_0(x) \left[\frac{\mathrm{d}k_0(x)}{\mathrm{d}x}\right] \left[\frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x}\right]^{-1} \mathrm{d}x = -\pi/2, \tag{10}$$

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = E_n\pi/(\hbar\omega). \tag{11}$$

The quantization rule (6) coincides with the quantization rule in WKB approximation:

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = (n + 1/2)\pi. \tag{12}$$

The energy levels for the one-dimensional harmonic oscillator are [1]

$$E_n = (n + 1/2)\hbar\omega. \tag{13}$$

The one-dimensional Morse potential is $V(x) = D(e^{-2x/a} - 2e^{-x/a}) = Dy(y-2)$, where $y = e^{-x/a}$. The turning points are $y_A = e^{-x_A/a} = 1 - \sqrt{1 + E_n/D}$ and $y_B = e^{-x_B/a} = 1 + \sqrt{1 + E_n/D}$, where $(n+1)$ denotes the number of nodes of the logarithmic derivative $\phi_n(x)$. The momentum between two turning points is $k_n(x) = [(2MD/\hbar^2)(y - y_A)(y_B - y)]^{1/2}$. The solution with only one node has to be $\phi_0(x) = C_1 y + C_2$ with $C_1 > 0$. Substituting it into the Riccati equation (3) with the Morse potential, we find $\phi_0(x) = \sqrt{2MD}(y-1)/\hbar + 1/(2a)$ with

$E_0 = -[\sqrt{D} - \hbar/(a\sqrt{8M})]^2$. Two integrals in the exact quantization rule (6) are calculated to be

$$\int_{x_A}^{x_B} \phi_0(x) \left[\frac{\mathrm{d}k_0(x)}{\mathrm{d}x}\right] \left[\frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x}\right]^{-1} \mathrm{d}x = -\pi/2, \tag{14}$$

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = \frac{a\sqrt{2M}}{\hbar} \left[\sqrt{D} - \sqrt{-E_n}\right] \pi. \tag{15}$$

The quantization rule is $\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = (n + 1/2)\pi$. Thus, the energy levels for the one-dimensional system with the Morse potential are [7]

$$E_n = -\left[\sqrt{D} - \frac{(2n+1)\hbar}{2a\sqrt{2M}}\right]^2. \tag{16}$$

The asymmetric Rosen-Morse potential [7] in one dimension is $V(x) = -U_0 \operatorname{sech}^2(x/a) + U_1 \tanh(x/a)$, where $0 \le U_1 < 2U_0$. If $U_1 = 0$, $V(x)$ is called the symmetric Rosen-Morse potential. Let $y = \tanh(x/a)$, $y_A = \tanh(x_A/a)$, and $y_B = \tanh(x_B/a)$, where x_A and x_B are two turning points satisfying $V(x_A) = V(x_B) = E_n$. We have

$$y_A = -\frac{U_1}{2U_0} - \sqrt{\left(\frac{U_1}{2U_0}\right)^2 + \frac{E_n}{U_0} + 1}, \qquad y_B = -\frac{U_1}{2U_0} + \sqrt{\left(\frac{U_1}{2U_0}\right)^2 + \frac{E_n}{U_0} + 1},$$

where $(n + 1)$ is the number of nodes of the logarithm derivative $\phi_n(x)$. The momentum between two turning points is $k_n(x) = \sqrt{2MU_0/\hbar^2}[(y - y_A)(y_B - y)]^{1/2}$. From the Riccati equation (3) with the asymmetric Rosen-Morse potential we obtain the solution with one node,

$$\phi_0(x) = -\frac{1}{2a}\left\{\left(1 + \frac{8a^2MU_0}{\hbar^2}\right)^{1/2} - 1\right\} y - \frac{MU_1}{\hbar^2 C} \quad \text{with } E_0 = -\frac{\hbar^2 C^2}{2M} - \frac{MU_1^2}{2\hbar^2 C^2}.$$

Now, two integrals in the quantization rule (6) are calculated to be

$$\int_{x_A}^{x_B} \phi_0(x) \left[\frac{\mathrm{d}k_0(x)}{\mathrm{d}x}\right] \left[\frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x}\right]^{-1} \mathrm{d}x = \frac{a\pi\sqrt{2MU_0}}{\hbar} \left[1 - \frac{\sqrt{2MU_0}}{\hbar C}\right], \tag{17}$$

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = \frac{a\pi}{2}\frac{\sqrt{2MU_0}}{\hbar} \left[2 - \sqrt{\frac{-E_n - U_1}{U_0}} - \sqrt{\frac{-E_n + U_1}{U_0}}\right]. \tag{18}$$

The quantization rule (6) becomes

$$\frac{1}{2}\sqrt{\frac{-E_n - U_1}{U_0}} + \frac{1}{2}\sqrt{\frac{-E_n + U_1}{U_0}} = -\frac{(n+1)\hbar}{a\sqrt{2MU_0}} + \frac{\sqrt{2MU_0}}{\hbar C} = \frac{(Ca - n)\hbar}{a\sqrt{2MU_0}}. \tag{19}$$

Thus, the energy E_n is [7]

$$-E_n = \frac{\hbar^2(C - n/a)^2}{2M} + \frac{MU_1^2}{2\hbar^2(C - n/a)^2}. \tag{20}$$

The condition of existence for the bound state whose wave function has n nodes is $U_1 < \hbar^2(C - n/a)^2/M < 2U_0$. When $U_1 = 0$, the asymmetric Rosen-Morse potential becomes the

symmetric one. The energy levels (20) with $U_1 = 0$ hold for the symmetric Rosen-Morse potential [7].

The first Pöschl-Teller potential [7] in one dimension is

$$V(x) = \frac{\hbar^2}{2Ma^2}\left[\frac{\mu(\mu-1)}{\sin^2(x/a)} + \frac{\lambda(\lambda-1)}{\cos^2(x/a)}\right], \quad 0 < x < \frac{a\pi}{2}, \tag{21}$$

where μ and λ are constant greater than one. The potential $V(x)$ tends to infinity as x tends to 0 or $a\pi/2$. Let $y = \tan^2(x/a)$, $y_A = \tan^2(x_A/a)$, and $y_B = \tan^2(x_B/a)$, where x_A and x_B are two turning points satisfying $V(x_A) = V(x_B) = E_n$. We have

$$y_A + y_B = \frac{2Ma^2E_n/\hbar^2 - \mu(\mu-1) - \lambda(\lambda-1)}{\lambda(\lambda-1)}, \qquad y_Ay_B = \frac{\mu(\mu-1)}{\lambda(\lambda-1)},$$

where $(n+1)$ denotes the number of nodes of the logarithm derivative $\phi_n(x)$. The momentum $k_n(x)$ between two turning points is $k_n(x) = [\lambda(\lambda-1)(y-y_A)(y_B-y)/y]^{1/2}/a$ when $E \geq V(x)$. The solution of the Riccati equation with only one node is $\phi_0(x) = -\lambda y^{1/2}/a + \mu y^{-1/2}/a$ with $E_0 = \hbar^2(\mu+\lambda)^2/(2Ma^2)$.

Two integrals in the quantization rule (6) are calculated to be

$$\int_{x_A}^{x_B} \phi_0(x)\left[\frac{\mathrm{d}k_0(x)}{\mathrm{d}x}\right]\left[\frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x}\right]^{-1}\mathrm{d}x = \frac{\pi}{2}\left[(\mu+\lambda-2) - \sqrt{\mu(\mu-1)} - \sqrt{\lambda(\lambda-1)}\right], \tag{22}$$

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = \frac{\pi}{2}\left[\frac{a\sqrt{2ME_n}}{\hbar} - \sqrt{\mu(\mu-1)} - \sqrt{\lambda(\lambda-1)}\right]. \tag{23}$$

The quantization rule (6) becomes $a\sqrt{2ME_n}/\hbar = 2(n+1) + (\mu+\lambda-2)$, from which we obtain the energy E_n [7],

$$E_n = \frac{\hbar^2(\mu+\lambda+2n)^2}{2Ma^2}. \tag{24}$$

The second Pöschl-Teller potential [7] in one dimension is

$$V(x) = \frac{\hbar^2}{2Ma^2}\left[\frac{\mu(\mu-1)}{\sinh^2(x/a)} - \frac{\lambda(\lambda+1)}{\cosh^2(x/a)}\right], \tag{25}$$

where $\lambda > \mu - 1 > 0$. The potential $V(x)$ tends to infinity at $x = 0$. Let $y = \tanh^2(x/a)$, $y_A = \tanh^2(x_A/a)$, and $y_B = \tanh^2(x_B/a)$, where x_A and x_B are two turning points where $V(x_A) = V(x_B) = E_n$. We have

$$y_A + y_B = \frac{2Ma^2E_n/\hbar^2 + \mu(\mu-1) + \lambda(\lambda+1)}{\lambda(\lambda+1)}, \qquad y_Ay_B = \frac{\mu(\mu-1)}{\lambda(\lambda+1)},$$

where $(n+1)$ denotes the number of nodes of the logarithm derivative $\phi_n(x)$. The momentum $k_n(x)$ between two turning points is $k_n(x) = [\lambda(\lambda+1)(y-y_A)(y_B-y)/y]^{1/2}/a$, when $E \geq V(x)$. The solution of the Riccati equation with only one node is $\phi_0(x) = -\lambda y^{1/2}/a + \mu y^{-1/2}/a$ with $E_0 = -\hbar^2(\lambda-\mu)^2/(2Ma^2)$.

Two integrals in the quantization rule (6) are calculated to be

$$\int_{x_A}^{x_B} \phi_0(x)\left[\frac{\mathrm{d}k_0(x)}{\mathrm{d}x}\right]\left[\frac{\mathrm{d}\phi_0(x)}{\mathrm{d}x}\right]^{-1}\mathrm{d}x = \frac{\pi}{2}\left[(\mu-\lambda-2) - \sqrt{\mu(\mu-1)} + \sqrt{\lambda(\lambda+1)}\right], \tag{26}$$

$$\int_{x_A}^{x_B} k_n(x)\mathrm{d}x = \frac{\pi}{2}\left[\frac{a\sqrt{-2ME_n}}{\hbar} - \sqrt{\mu(\mu-1)} + \sqrt{\lambda(\lambda+1)}\right]. \tag{27}$$

The quantization rule (6) reads $a\sqrt{-2ME_n}/\hbar = 2(n+1)+(\mu-\lambda-2)$, and the energy E_n is [7]

$$E_n = -\frac{\hbar^2(\lambda-\mu-2n)^2}{2Ma^2}\,, \quad 0 \le n < (\lambda-\mu)/2. \tag{28}$$

The effective potential for the three-dimensional harmonic oscillator is $U_\ell(r) = \ell(\ell+1)\hbar^2/(2Mr^2) + M\omega^2 r^2/2$. The turning points are $r_A = \alpha^{-1}\{[E_{n\ell}/(\hbar\omega)] - [(E_{n\ell}/(\hbar\omega))^2 - \ell(\ell+1)]^{1/2}\}^{1/2}$ and $r_B = \alpha^{-1}\{[E_{n\ell}/(\hbar\omega)] + [(E_{n\ell}/(\hbar\omega))^2 - \ell(\ell+1)]^{1/2}\}^{1/2}$, where $\alpha = \sqrt{M\omega/\hbar}$, $(n-\ell)$ is a non-negative even integer, and $(n-\ell+2)/2$ is the number of nodes of the logarithmic derivative $\phi_{n\ell}(r)$. The momentum between two turning points is $k_{n\ell}(r) = (\alpha^2/r)[(r^2-r_A^2)(r_B^2-r^2)]^{1/2}$. The solution with only one node has to be $\phi_{n\ell}(r) = C_1 r^{-1} + C_2 r$ where $C_1 > 0$ and $C_2 < 0$. Substituting it into the Riccati equation (3) with the harmonic-oscillator potential, we find $\phi_{n\ell}(r) = (\ell+1)r^{-1} - \alpha^2 r$ with $E_{n\ell} = \hbar\omega(\ell+3/2)$ and $n = \ell$.

Two integrals in the quantization rule (8) are calculated to be

$$\int_{r_A}^{r_B} \phi_{n\ell}(r)\,(\mathrm{d}k_{n\ell}(r)/\mathrm{d}r)\left[\frac{\mathrm{d}\phi_{n\ell}(r)}{\mathrm{d}r}\right]^{-1}\mathrm{d}r = \left[\ell - \sqrt{\ell(\ell+1)} - 1/2\right]\pi/2, \quad n = \ell, \tag{29}$$

$$\int_{r_A}^{r_B} k_{n\ell}(r)\mathrm{d}r = \left[\frac{E_{n\ell}}{\hbar\omega} - \sqrt{\ell(\ell+1)}\right]\pi/2. \tag{30}$$

The quantization rule (8) becomes $\int_{r_A}^{r_B} k_{n\ell}(r)\mathrm{d}r = [n - \sqrt{\ell(\ell+1)} + 3/2]\pi/2$. Thus, the energy levels for the three-dimensional harmonic oscillator are [1]

$$E_{n\ell} = (n+3/2)\hbar\omega. \tag{31}$$

The effective potential for the hydrogen atom is $U_\ell(r) = \ell(\ell+1)\hbar^2/(2Mr^2) - e^2/r$. When $\ell > 0$, the turning points r_A and r_B satisfying $U_\ell(r_A) = U_\ell(r_B) = E_{n\ell}$ are

$$\begin{aligned} r_A &= (-2E_{n\ell})^{-1}\left\{e^2 - \left[e^4 + 2\ell(\ell+1)\hbar^2 E_{n\ell}/M\right]^{1/2}\right\}, \\ r_B &= (-2E_{n\ell})^{-1}\left\{e^2 + \left[e^4 + 2\ell(\ell+1)\hbar^2 E_{n\ell}/M\right]^{1/2}\right\}, \end{aligned} \tag{32}$$

where we denote by $(n-\ell)$ the number of nodes of the logarithmic derivative $\phi_{n\ell}(r)$. When $\ell = 0$, we define $r_A = 0$ with $U(r_A) = -e^2/r_A \sim -\infty$, and r_B is solved from $U(r_B) = E_{n\ell}$. Thus, eq. (32) still holds for $\ell = 0$. The momentum between two turning points is $k_{n\ell}(r) = (\hbar r)^{-1}[-2ME_{n\ell}(r-r_A)(r_B-r)]^{1/2}$. For $\ell = 0$, near the origin we have $k_{n\ell}(r) \sim r^{-1/2}$, and $\phi_{n\ell}(r) \sim r^{-1}$, so that the limit terms in eq. (4) still vanish. The solution with only one node has the form $\phi_{n\ell}(r) = C_1 r^{-1} + C_2$, where $C_1 > 0$. Combining it with the Riccati equation (3) for the hydrogen atom, we find $\phi_{n\ell}(r) = (\ell+1)/r - Me^2/[(\ell+1)\hbar^2]$ with $E_{n\ell} = -Me^4/[2(\ell+1)^2\hbar^2]$ and $n = \ell+1$.

Two integrals in the quantization rule (8) are calculated to be

$$\int_{r_A}^{r_B} \phi_{n\ell}(r)\left[\frac{\mathrm{d}k_{n\ell}(r)}{\mathrm{d}r}\right]\left[\frac{\mathrm{d}\phi_{n\ell}(r)}{\mathrm{d}r}\right]^{-1}\mathrm{d}r = \left[\ell - \sqrt{\ell(\ell+1)}\right]\pi, \quad n = \ell+1, \tag{33}$$

$$\int_{r_A}^{r_B} k_{n\ell}(r)\mathrm{d}r = \left[\frac{e^2}{\hbar}\sqrt{\frac{M}{-2E_{n\ell}}} - \sqrt{\ell(\ell+1)}\right]\pi. \tag{34}$$

The quantization rule (8) becomes $\int_{r_A}^{r_B} k_{n\ell}(r)\mathrm{d}r = [n - \sqrt{\ell(\ell+1)}]\pi$. Thus, the energy levels for the hydrogen atom are [1]

$$E_{n\ell} = -\frac{Me^4}{2n^2\hbar^2}\,. \tag{35}$$

In this letter we present an exact quantization rule (6) for the one-dimensional Schrödinger equation and (8) for the three-dimensional Schrödinger equation with a spherically symmetric potential. We find that the quantum correction term is independent of the number of nodes in the wave function for the exactly solvable quantum systems. In such cases, the energy levels of the quantum system can be easily solved from the exact quantization rule and the solution of the ground state calculated directly from the Riccati equation. For the non-exactly solvable systems, one can use the series form of the quantization rule for numerical calculation [8].

As is well known, the wave functions and the energy levels for the exactly solvable systems can also be solved by the supersymmetry in the recursive way [9]. The logarithmic derivatives $\phi(x)$ are nothing but the superpotentials in the supersymmetric quantum mechanics. The shape invariance of the superpotentials seems to be related to the condition of the quantum correction being an invariant. As far as the wave function $\psi_N(x)$ with $(N-1)$ nodes is concerned, we prefer to calculate its logarithmic derivatives $\phi_N(x)$ directly from the Riccati equation (3). Although there are more or less differences for different examples, the solution for $\phi_N(x)$ can be taken as a fraction, where the numerator is a polynomial of order N and the denominator is a polynomial of order $(N-1)$, because $\phi_N(x)$ contains N nodes and $(N-1)$ poles. Without loss of generality, the coefficient of x^{N-1} in the denominator can be chosen to be one. Substituting it into the Riccati equation (3), one obtains a coupled algebraic equation of order two for the coefficients. Therefore, $\phi_N(x)$ can be solved. We have solved some examples for the lower excited states.

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