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The Casimir effect for the Bose-gas in slabs

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Abstract. – We study the Casimir effect for the perfect Bose-gas in the slab geometry for various boundary conditions. We show that the grand-canonical potential per unit area at the bulk critical chemical potential $\mu = 0$ has the standard asymptotic form with universal Casimir terms.

In contrast to the well-known *Casimir effect* for the photon gas, see *e.g.* [1], this effect for the massive quantum particles (*quantum gases*) is much less explored. In the present letter we consider the case of the perfect Bose-gas. The perfect Bose-gas is the simplest quantum-mechanical system showing the spontaneous breaking of a continuous symmetry (Bose-Einstein condensation) for which the Casimir effect should appear. To our knowledge, this has surprisingly not been studied before.

We first recall the definition of the *Casimir amplitude* at critical points or for phases with broken symmetry [2, 3]. Let

$$\varphi_d(T) = \lim_{L \rightarrow \infty} \frac{1}{L^2} \Phi_L(T, d) \quad (1)$$

be a limiting thermodynamic potential per unit area in the *3-dimensional slab geometry* $\infty \times \infty \times d$, where $\Phi_L(T, d)$ is the finite-volume thermodynamic potential in the box $\Lambda = L \times L \times d$ and T denotes the temperature. The finite-size scaling analysis in the *critical regime* shows that large d -asymptotics of (1) usually has the form:

$$\varphi_d(T) = d \varphi_{bulk}(T) + \varphi_{surf}(T) + \frac{\Delta(T)}{d^2} + \dots \quad (2)$$

Here $\varphi_{bulk}(T)$ is the potential density in thermodynamic limit and $\varphi_{surf}(T)$ is the surface potential correction. The coefficient $\Delta(T)$ in the third term in (2) is called the *Casimir*

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amplitude. The *Casimir force* between the slab faces is defined as

$$F(d) = -\partial_d[\varphi_d(T) - d\varphi_{bulk}(T)] = 2 \frac{\Delta(T)}{d^3} + \dots \quad (3)$$

In this regime, *i.e.* at a critical point or in phases with long-range correlations generated by the broken symmetry, the value of $\Delta(T)$ is expected to be non-zero and *universal*, depending only on the system and the boundary condition universality classes. Out of the *critical regime* the finite-size corrections in slab are expected to be exponentially small, thus the Casimir amplitude $\Delta(T) = 0$.

The perfect Bose-gas *grand-canonical thermodynamic potential* in the box $L \times L \times d$ has the form [4]

$$\Phi_{L,d}(T, \mu) = \beta^{-1} \sum_{\mathbf{k}} \ln \left\{ 1 - e^{-\beta(\varepsilon(\mathbf{k}) - \mu)} \right\}, \quad \varepsilon(\mathbf{k}) = \frac{\hbar^2}{2m} (k_x^2 + k_y^2 + k_z^2), \quad (4)$$

where $\beta = (k_B T)^{-1}$ and μ is the chemical potential. The sum in (4) runs over the set defined by boundary conditions (b.c.) for the Laplacian operator in the box $L \times L \times d$. Below we shall consider (for simplicity) the periodic b.c. in the x - y directions:

$$k_x = \frac{2\pi}{L} n_x, \quad k_y = \frac{2\pi}{L} n_y \quad n_x, n_y = 0, \pm 1, \pm 2, \dots, \quad (5)$$

and three cases of b.c., Dirichlet (D), Neumann (N) and periodic (P), in the z -direction:

$$(D) \quad k_z = \frac{\pi}{d} n_z, \quad n_z = 1, 2, \dots; \quad (N) \quad k_z = \frac{\pi}{d} n_z, \quad n_z = 0, 1, 2, \dots; \quad (6)$$

$$(P) \quad k_z = \frac{2\pi}{d} n_z, \quad n_z = 0, \pm 1, \pm 2, \dots \quad (7)$$

Therefore, in the *slab limit*, $L \rightarrow \infty$, the grand-canonical thermodynamic potential per unit area is

$$\begin{aligned} \varphi_d(T, \mu) &= \lim_{L \rightarrow \infty} \frac{1}{L^2} \Phi_{L,d}(T, \mu) = \frac{1}{\beta(2\pi)^2} \int_{\mathbb{R}^2} d^2 \mathbf{q} \sum_{k_z} \ln \left\{ 1 - e^{-\beta[\varepsilon(\mathbf{q}) + \varepsilon(k_z) - \mu]} \right\} = \\ &= -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d^2 \mathbf{q} \sum_{k_z} \frac{\varepsilon(\mathbf{q})}{e^{\beta[\varepsilon(\mathbf{q}) + \varepsilon(k_z) - \mu]} - 1}, \end{aligned} \quad (8)$$

where $\mathbf{q} = (q_x, q_y)$ is a two-dimensional wave vector in the (x, y) -plane, $\varepsilon(\mathbf{q}) = \hbar^2(q_x^2 + q_y^2)/2m$ and $\varepsilon(k_z) = \hbar^2 k_z^2/2m$. The last equality results from an integration by part with respect to $q = |\mathbf{q}|$.

It is well known that in the *bulk limit*, $L \rightarrow \infty$, $d \rightarrow \infty$, the Bose-Einstein condensation for the perfect Bose-gas may occur only for $\mu = 0$. Therefore in the sequel we distinguish the two regimes: $\mu < 0$ (*normal phase*) and $\mu = 0$ (*condensed phase*). Our results for the asymptotic expressions of $\varphi_d(T, \mu)$ as $d \rightarrow \infty$ are the following:

a) If $\mu < 0$, then

$$\varphi_d(T, \mu) = -d p_{bulk}(T, \mu) + \varphi_{surf}(T, \mu) + O\left(e^{-\text{Const} \cdot \sqrt{-\beta\mu} d/\lambda}\right), \quad (9)$$

where

$$p_{bulk}(T, \mu) = -\frac{1}{\beta(2\pi)^3} \int d^3 \mathbf{k} \ln \left(1 - e^{-\beta[\varepsilon(\mathbf{k}) - \mu]} \right) \quad (10)$$

is the standard Bose-gas pressure and $\lambda = \hbar\sqrt{\beta/m}$ denotes the *thermal wavelength*. We get that $\varphi_{surf}^{(P)}(T, \mu) = 0$ for periodic b.c., but

$$\varphi_{surf}^{(D,N)}(T, \mu) = \pm \frac{1}{2(2\pi)^2} \int d^2\mathbf{q} \frac{\varepsilon(\mathbf{q})}{e^{\beta[\varepsilon(\mathbf{q}) - \mu]} - 1} \quad (11)$$

with + for Dirichlet and – for Neumann b.c.

b) If $\mu = 0$, then we obtain for the periodic b.c. the asymptotics

$$\begin{aligned} \varphi_d(T, 0) &= -d p_{bulk}(T, 0) + \frac{k_B T}{d^2} \left[\Delta^{(P)}(T) + O((\lambda/d)^M) \right], \quad \text{for any } M \geq 1, \\ \Delta^{(P)}(T) &= -\frac{k_B T \zeta(3)}{\pi}, \end{aligned} \quad (12)$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the *Riemann zeta-function*. For the Dirichlet and Neumann b.c. we find

$$\begin{aligned} \varphi_d(T, 0) &= -d p_{bulk}(T, 0) + \varphi_{surf}^{(D,N)}(T, 0) + \frac{k_B T}{d^2} \left[\Delta^{(D,N)}(T) + O((\lambda/d)^M) \right], \quad \text{for any } M \geq 1, \\ \Delta^{(D,N)}(T) &= -\frac{k_B T \zeta(3)}{8\pi}, \end{aligned} \quad (13)$$

and $\varphi_{surf}^{(D,N)}(T, 0)$ is given by (11).

We see that for Dirichlet and Neumann b.c. the Casimir amplitudes (13) are the same, implying identical *attractive* forces in slab in the presence of the Bose-Einstein condensation. It is also interesting to note that these amplitudes are twice larger than that of a massless relativistic scalar field, see formula (2.68) in [5].

To obtain these results we consider separately the cases $\mu < 0$ and $\mu = 0$.

For $\mu < 0$ and Dirichlet b.c. (see (6)) the result a) follows from the representation of (8) as the sum:

$$\varphi_d(T, \mu) = \sum_{n=1}^{\infty} \phi(d^{-1}n), \quad \phi(u) = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d^2\mathbf{q} \frac{\varepsilon(\mathbf{q})}{e^{[\beta\varepsilon(\mathbf{q}) + (\pi\lambda u)^2/2 - \beta\mu]} - 1}. \quad (14)$$

Since for $\mu < 0$ the function $\phi(u)$ is infinitely differentiable at $u = 0$ and all its derivatives vanish at infinity, we can use the *Euler-MacLaurin theorem* [6, 7] in the form

$$\sum_{n=1}^{\infty} F(n) = \int_0^{\infty} ds F(s) - \frac{1}{2}F(0) - \frac{B_2}{2!}F^{(1)}(0) - \frac{B_4}{4!}F^{(3)}(0) - \dots, \quad (15)$$

where $\{B_{2n}\}_{n=1}^{\infty}$ are the Bernoulli numbers and $\{F^{(2n-1)}(s)\}_{n=1}^{\infty}$ denote the corresponding derivatives. When applying (15) to (14), the two first terms of (15) yield, respectively, the bulk and surface contributions in (9). Moreover, since $\phi(u) = \phi(-u)$ all odd derivatives $\{\phi^{(2n-1)}(u)\}_{n=1}^{\infty}$ vanish at $u = 0$. So for large d the corrections to the first two terms are smaller than any power of d^{-1} . In fact, a more elaborated estimate gives an exponentially small correction $O(e^{-\text{Const} \cdot \sqrt{-\beta\mu} d/\lambda})$, see (21) below.

If $\mu = 0$, all derivatives $\phi^{(2n-1)}(u)$ of order $2n - 1 > 2$ diverge at $u = 0$. So instead of the Euler-MacLaurin formula one has to use a more refined representation for the sum (14). To this end we use the *Jacobi identity* (see, e.g. [8], Chapt. 11):

$$\sum_{n=1}^{\infty} e^{-\pi n^2 a} = \left(\frac{1}{2\sqrt{a}} - \frac{1}{2} \right) + \frac{1}{\sqrt{a}} \sum_{n=1}^{\infty} e^{-\pi n^2/a} , \quad a > 0 , \quad (16)$$

which can be obtained as a simple consequence of the Poisson summation formula. First, we represent (14) by its low-activity series for $\mu < 0$ (setting $v = \beta\epsilon(\mathbf{q}) = \lambda^2 q^2/2$):

$$\begin{aligned} \varphi_d(T, \mu) &= -\frac{1}{2\pi\beta\lambda^2} \int_0^{\infty} dv v \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} e^{\beta r \mu} e^{-r[v+(\pi\lambda n/d)^2/2]} = \\ &= -\frac{1}{2\pi\beta\lambda^2} \sum_{r=1}^{\infty} \frac{e^{\beta r \mu}}{r^2} \sum_{n=1}^{\infty} e^{-\pi n^2 (r\pi(\lambda/d)^2/2)} = -\frac{d}{\beta(\sqrt{2\pi}\lambda)^3} \sum_{r=1}^{\infty} \frac{e^{\beta r \mu}}{r^{5/2}} + \frac{1}{4\pi\beta\lambda^2} \sum_{r=1}^{\infty} \frac{e^{\beta r \mu}}{r^2} - \\ &\quad - \frac{2d}{\beta(\sqrt{2\pi}\lambda)^3} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{e^{\beta r \mu}}{r^{5/2}} e^{-2(nd/\lambda)^2/r} , \end{aligned} \quad (17)$$

where the last equation follows from the the Jacobi identity (16) with $a = r\pi(\lambda/d)^2/2$. These series are absolutely convergent when $\mu = 0$ and one recognises that the first term in the right-hand side of (17) is nothing but $-d p_{bulk}(T, 0)$ and the second one coincides with $\varphi_{surf}^{(D)}(T, 0)$. To obtain for $\mu = 0$ the large d asymptotics of the last term we write the sum over r in the form $(\lambda/nd)^3 \delta_n(d) S_n(d)$, where

$$S_n(d) = \sum_{r=1}^{\infty} \psi(\delta_n(d)r) , \quad \psi(x) = \frac{e^{-2/x}}{x^{5/2}} , \quad \delta_n(d) = \left(\frac{\lambda}{nd} \right)^2 . \quad (18)$$

Since the function $\psi(x)$ is infinitely differentiable for $x \geq 0$ and all its derivatives vanish at $x = 0$ and at infinity, the Euler-MacLaurin formula (15) yields for large d and n

$$\delta_n(d) S_n(d) = \int_0^{\infty} dx \psi(x) + O((\delta_n(d))^M) = \frac{\sqrt{\pi}}{2^{5/2}} + O((\delta_n(d))^M) , \quad \text{for any } M \geq 1 , \quad (19)$$

i.e. the corrections to the integral are smaller than any power of $\delta_n(d)$. Inserting (19) in the last term of the right-hand side of (17) we get

$$\frac{2d}{\beta(\sqrt{2\pi}\lambda)^3} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{r^{5/2}} e^{-2(nd/\lambda)^2/r} = \frac{1}{8\pi\beta d^2} \left[\sum_{n=1}^{\infty} \frac{1}{n^3} + O((\lambda/d)^M) \right] , \quad \text{for any } M \geq 1 , \quad (20)$$

that proves (13) for the Dirichlet b.c. By (6) for Neumann b.c. sums in (16) and (17) start with $n = 0$, which implies changing the sign of the surface term (11) and gives (13) for Neumann b.c.

For periodic b.c. (7) we have to use instead of (14) the representation

$$\varphi_d(T, \mu) = \sum_{n=-\infty}^{\infty} \phi((d/2)^{-1}n) = 2 \sum_{n=1}^{\infty} \phi((d/2)^{-1}n) + \phi(0) .$$

By consequence, the contribution of the $n = 0$ term, which corresponds to the surface corrections, will disappear and we obtain (12).

The Jacobi identity allows also to prove the asymptotic (9). To this end notice that the double sum in the last term of (17) has for $\mu < 0$ the estimate

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{e^{\beta r \mu}}{r^{5/2}} \sum_{n=1}^{\infty} e^{-2(nd/\lambda)^2/r} &\leq \sum_{r=1}^{\infty} \frac{1}{r^{5/2}} \sum_{n=1}^{\infty} e^{\max_{r \geq 1} (\beta r \mu - 2(nd/\lambda)^2/r)} = \\ &= \frac{\zeta(5/2)}{e^{2\sqrt{-2\beta\mu}d/\lambda} - 1} = O(e^{-\sqrt{-8\beta\mu}d/\lambda}), \end{aligned} \quad (21)$$

that proves (9) for large d .

Now few remarks and comments are in order.

The grand potential of a free Fermion gas does not have a Casimir term for any value of the chemical potential. Indeed replacing the Bose by the Fermi distribution in (8) gives a corresponding $\phi(u)$ -function (14) that is infinitely differentiable at $u = 0$ for all μ . Since the function and all its derivatives vanish at $u = 0$ the Euler-MacLaurin formula yields corrections smaller than any inverse power of d .

According to common wisdom, the Casimir force is due to Goldstone modes that will occur in the bulk limit when a continuous symmetry is spontaneously broken. Usually, they are taken *ad hoc* as a part of phenomenological models, see *e.g.* [9] for the case of superfluid films. In the free Bose-gas it is explicit: the excitations $\varepsilon(\mathbf{k}) - \mu$ become gapless when $\mu = 0$. Then by the London-Placzek formula [10, 11], this generates long-range particle-particle correlations

$$\rho^{(2)}(\mathbf{r}_1, \mathbf{r}_2) - \rho^2 \sim \rho_0(T) |\mathbf{r}_1 - \mathbf{r}_2|^{-1}, \quad (22)$$

as $|\mathbf{r}_1 - \mathbf{r}_2| \rightarrow \infty$ in the condensed phase. Here ρ_0 denotes the Bose-Einstein condensation density of the perfect Bose-gas. Casimir forces are usually attributed to such correlations in the critical regime. One can check that it is true for the perfect Bose-gas because of long-range correlations that appear in the slab limit.

It is known that the perfect Bose-gas particle number fluctuations in the condensed phase are abnormal, *i.e.* proportional to volume V in contrast to the normal thermodynamical fluctuations \sqrt{V} , because of lack of superstability, see *e.g.* [12]. So we would like to warn that the results for Casimir amplitudes might be very different from (12), (13) in the presence of a superstable interaction.

In the case of electromagnetic interactions, the Casimir term is always present as a result of the long range of the forces. In the standard calculation of the zero-temperature Casimir force between perfect conductors [1] the Casimir term appears because the third-order derivative occurring in the appropriate Euler-MacLaurin expansion does not vanish, contrary to the case a) above. This is due to the linear form $\hbar\omega_{\mathbf{k}} = \hbar c|\mathbf{k}|$ of the photon spectrum (non-analytic at $\mathbf{k} = 0$). On the other hand, in the high-temperature regime, the Casimir term $-\zeta(3)/16\pi\beta d^2$ becomes purely classical and is entirely due to fluctuating Coulomb potentials in the conductor walls [13–15]. Moreover, a classical Coulomb system enclosed in a slab with perfectly conducting walls manifests a universal finite-size correction of the Casimir type of the opposite sign $\zeta(3)/16\pi\beta d^2$ [16]. Such Coulomb systems are critical at all temperatures because the potential fluctuations are always of a long range, although charge correlations are themselves of a short range as a consequence of screening.

We note that the Casimir terms found in b) are classical and universal, namely they do not depend on the Planck constant and the particle mass. In fact, for the free Bose-gas, it follows from a simple dimensional analysis that such terms are necessarily of the form $C k_B T/d^2$ where C is a numerical constant. They are present at all positive temperature provided that the density ρ of the gas is higher than the critical density $\rho_c(\beta)$. Since M is arbitrary in (12)

and (13), we see that the Casimir amplitude is dominant as soon as $d > \lambda$. For instance in liquid helium at $T = 2\text{ K}$, one has $\lambda \sim \rho^{-1/3} \sim 4\text{ \AA}$, *i.e.* a slab of thickness slightly larger than the interatomic distance already shows the Casimir effect. If we fix d and let $T \rightarrow 0$ we enter in the regime $\lambda \gg d$. By (17) it is sufficient to keep the $n = r = 1$ term which gives

$$\varphi_d(T, \mu) \sim -\frac{e^{\beta r \mu}}{2\pi\beta\lambda^2} e^{-\pi^2(\lambda/d)^2/2}, \quad \mu \leq 0, \quad \lambda^2 = \frac{\hbar^2}{mk_B T}. \quad (23)$$

So in contrast to the photon case the grand potential is exponentially small as $T \rightarrow 0$, which excludes the Casimir effect in this regime.

* * *

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