A LETTERS JOURNAL EXPLORING
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EUROPHYSICS LETTERS 15 June 2006

Europhys. Lett., **74** (6), pp. 1013–1019 (2006)
DOI: 10.1209/epl/i2005-10592-1

Indeterminacy of amplitude and phase variables in classical dynamical systems: The harmonic oscillator

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received 25 August 2005; accepted in final form 13 April 2006
published online 12 May 2006

PACS. 45.30.+s – General linear dynamical systems.
PACS. 45.50.Dd – General motion.
PACS. 02.30.Hq – Ordinary differential equations.

Abstract. – The amplitude and phase are shown to be variables that are not uniquely determined for the one-dimensional time-dependent harmonic-oscillator equation. Even when the parameter is time independent, it is shown that the amplitude and frequency may be time dependent and the phase nonlinear with respect to time. The consequences of this indeterminacy in the energy and the orthogonal-functions invariant are evaluated. The measurement of these quantities is discussed.

Introduction. – The position of a body in a discrete mechanical system is often expressed in terms of amplitude and phase variables in periodic or quasi-periodic motion. The relationships governing the position and amplitude variables in one dimension produce an Ermakov pair of equations [1]. These equations lead to exact invariants even in the case where the energy of the system is not conserved [2]. Recall that exact invariants are constants of motion for arbitrary time-dependent parameters. In contrast, adiabatic invariants are approximately constant for slow variations of time-dependent parameters. The amplitude and phase representation is concomitant to the derivation of exact invariants. For this reason, the determination of these quantities is crucial in order to establish the eigenvalue of the invariants. The relevance of Ermakov systems in a wide variety of models and disciplines of physics has been recently reviewed [3].

Propagation of disturbances in continuum mechanics is also frequently written in terms of these variables. Such a disturbance may correspond to a position, velocity or force field. In wave phenomena, be it mechanical or electromagnetic nature, the amplitude and phase representation is in fact the most common representation. The propagation of a scalar monochromatic wave train in an inhomogeneous medium in one dimension is formally equivalent to the time-dependent harmonic-oscillator (TDHO) problem [4]. Therefore, the conclusions drawn from the study of the one-dimensional TDHO system may be readily extended to an important class of propagating systems. On the other hand, the Ermakov systems formalism has been

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generalized to an arbitrary number of dimensions [5,6]. In particular, the orthogonal-functions invariant procedure has been recently extended to the $3+1$ scalar wave propagation equation where the concept translates into a conservation equation with the appropriate complementary fields density [7].

Ermakov invariants have been widely used in quantum mechanics in order to find exact or approximate solutions to the Schrödinger equation. The amplitude representation [8] and the phase integral approximation method have been successfully implemented in different quantum problems [9]. A smooth quantum amplitude formulation has been recently developed that yields the corresponding classical variables in the $\hbar \to 0$ limit [10, 11], provided that the classical amplitude is smooth. It has been pointed out that there are numerous Hamiltonians either in the classical or quantum realm that correspond to the differential equation general solution in the former case or the quantum average in the latter case [12]. On the other hand, the quantum uncertainty between conjugate variables necessarily holds for the amplitude and phase observables. However, the nature of these two indeterminacies are quite independent of the *classical* amplitude-phase indeterminacy discussed here. In order to prevent confusions, the present letter avoids any further reference to the implications of the present results in the quantum domain.

It is hardly necessary to stress the importance of the amplitude and phase representation in different fields of physics. In recent years, from a theoretical point of view, this approach has been used to study scattering problems in electromagnetic random media [13] and acoustical inhomogeneous materials [14]; differential equations solutions for the two-center Coulomb [15], d'Alambert [16] and TDHO [17] problems; quantum-mechanical systems such as correlation entanglement by EPR pair eigenstates [18], quantum wells [19], quantum oscillators [20], barrier transmission [21] and minimum uncertainty states [22]. There is also a widespread use of these variables in detection and analysis of experimental data. Some recent examples in observational astronomy are cosmic microwave background maps [23] and correlations in the solar cycle [24]; in nonlinear optics: ultrashort pulse two-photon absorption [25], optical amplifiers [26] and time-dependent optical signals [27]; in physical optics: interferometric noise measurements [28] and standards [29]; in scientific instruments: optical design of confocal microscopes [30], apodization of telescope pupils [31] and analysis of reflectance spectra [32].

In this letter, the amplitude and phase variables are first shown not to be uniquely determined for a given trajectory that is a solution of the classical harmonic-oscillator equation in one dimension. This indeterminacy does not alter the energy eigenvalue in the time-independent case but it does modify the eigenvalue of the orthogonal functions invariant. The usual amplitude and phase measurement procedures are then revised in the light of the previous derivations. Rather counter-intuitively, from an experimentalist point of view, it is shown that oscillatory motion with constant maximum displacement does not necessarily imply a constant amplitude.

Consider the one-dimensional harmonic-oscillator equation with time-dependent parameter $\Omega(t)$

$$\ddot{q} + \Omega^2(t)\, q = 0, \tag{1}$$

where the overdot represents differentiation with respect to time. The real valued solution in terms of amplitude "A" and phase "ϕ" variables is written as

$$q = A(t) \cos(\phi(t)). \tag{2}$$

The amplitude is then solution to the nonlinear differential equation

$$\ddot{A} + \Omega^2(t)\, A - \frac{W^2}{A^3} = 0, \tag{3}$$

where W represents the Wronskian also referred as orthogonal-functions exact invariant [33]. The explicit time-dependence notation of $A(t), \phi(t)$ is dropped hereafter. This exact invariant is given in amplitude and phase variables by

$$W = A^2 \frac{\mathrm{d}\phi}{\mathrm{d}t}. \tag{4}$$

A closely related constant of motion is the Ermakov-Lewis invariant [34] given by $I = 1/2 \left[W^2 q^2/A^2 + \left(q\dot{A} - A\dot{q} \right)^2 \right]$, where the invariant is being expressed in terms of coordinate and amplitude variables.

The general solution when Ω is time independent is usually written as $q = A_0 \cos(\omega_0 t + \theta)$, where A_0 stands for a constant amplitude and the argument of the trigonometric function represents a linear phase. In this case, ω_0 represents the constant frequency and θ is an initial phase. The two arbitrary constants A_0 and θ correspond to the two parameters of the general solution to a second-order differential equation.

Amplitude and phase indeterminacy. – Consider the function

$$\cos\left[\arctan\left(\frac{B}{A}\tan\phi\right)\right],$$

where B is an arbitrary function of time. The trigonometric identity $\cos[\arctan(\alpha/\beta)] = \beta/\sqrt{\alpha^2+\beta^2}$, taking $\alpha = \tan\phi$ and $\beta = A/B$, leads to

$$\cos\left[\arctan\left(\frac{B}{A}\tan\phi\right)\right] = \frac{A\cos\phi}{\sqrt{A^2\cos^2\phi + B^2\sin^2\phi}}.$$

Therefore

$$\underbrace{A}_{\text{amplitude}} \cos \underbrace{\phi}_{\text{phase}} = \underbrace{\sqrt{A^2\cos^2\phi + B^2\sin^2\phi}}_{\text{amplitude}} \cos \underbrace{\left[\arctan\left(\frac{B}{A}\tan\phi\right)\right]}_{\text{phase}} \tag{5}$$

for any real function B. The amplitude function is then

$$g(t) = \sqrt{A^2\cos^2\phi + B^2\sin^2\phi}, \tag{6}$$

whereas the phase is

$$\gamma(t) = \arctan\left(\frac{B}{A}\tan\phi\right). \tag{7}$$

If A satisfies the amplitude differential equation, the amplitude $g(t)$ is also a solution if B satisfies the amplitude equation (3). This assertion is in fact the statement of the nonlinear superposition principle applicable to the nonlinear amplitude differential equation [35]. This principle is analogous to the linear superposition principle that governs the coordinate linear differential equation (1). Therefore, if we cast the coordinate solution in terms of amplitude and phase variables, these variables are indeterminate up to a function B according to equations (6) and (7) provided that this function satisfies the amplitude nonlinear equation (3).

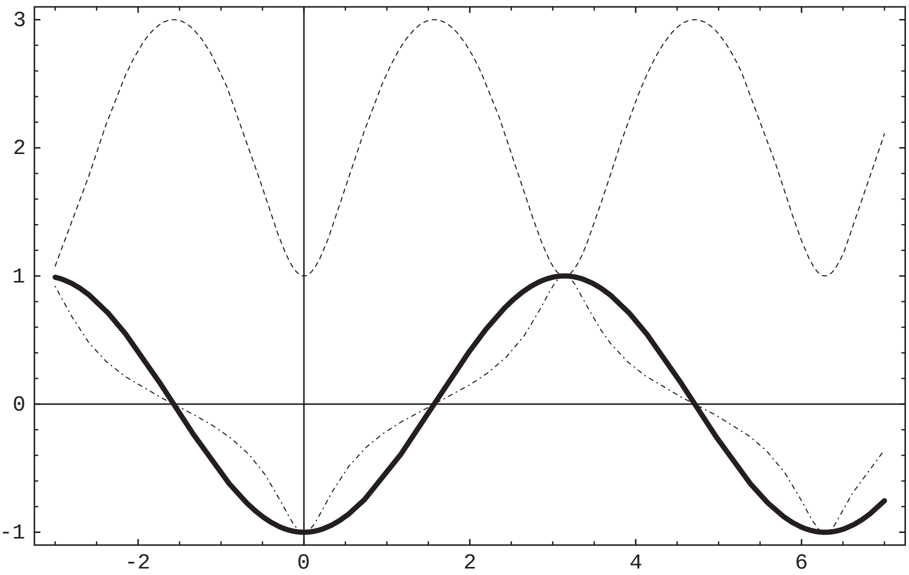

Fig. 1 – Amplitude $g_c(t)$ (dashed line), $\cos[\gamma_c(t)]$ (dot-dashed line) and their product $q = g_c(t)\cos[\gamma_c(t)]$ (solid line) functions. In this example, $A_0 = 1,\ B_0 = 2$.

Constant parameter. – In the particular case of a constant parameter Ω, a solution with constant amplitude and a linear phase is admissible. The function B can then be set to a constant B_0 that fulfills the amplitude differential equation. The identity (5) is then

$$A_0 \cos(\omega_0 t + \theta) = g_c(t, A_0, B_0) \cos\left(\int \omega_c(t, A_0, B_0)\, \mathrm{d}t\right), \tag{8}$$

where the amplitude

$$g_c(t, A_0, B_0) = \sqrt{1/2\left\{(A_0^2 + B_0^2) + (A_0^2 - B_0^2)\cos[2(\omega_0 t + \theta)]\right\}} \tag{9}$$

is now time dependent. The phase $\gamma_c(t, A_0, B_0) = \arctan\left(\frac{B_0}{A_0}\tan(\omega_0 t + \theta)\right)$ is nonlinear and the frequency is also a function of time,

$$\frac{\mathrm{d}\gamma_c(t, A_0, B_0)}{\mathrm{d}t} \equiv \omega_c(t, A_0, B_0) = \frac{A_0 B_0 \omega_0}{1/2\left\{(A_0^2 + B_0^2) + (A_0^2 - B_0^2)\cos[2(\omega_0 t + \theta)]\right\}}. \tag{10}$$

If we choose $B_0 = 0$, then the frequency $\omega_c(t, A_0, B_0)$ is zero and the amplitude $g_c(t, A_0, B_0)$ carries all the function information. In the opposite case when $B_0 = A_0$, a trivial identity is recovered where the amplitude is constant and the phase is linear. Therefore, the amplitude of a harmonic oscillator with time-independent parameter is not necessarily constant but may be time dependent and its frequency is not *a fortiori* constant. Nonetheless, the product of the time-dependent amplitude times the cosine of the nonlinear phase $q = g_c(t)\cos[\gamma_c(t)]$ yields the same trajectory as that given by the constant amplitude and linear phase dependence with constant frequency $q = A_0 \cos[\omega_0 t + \theta]$. The product of these functions is depicted in fig. 1.

Conserved quantities. – The orthogonal-functions invariant, from (4) and the nonlinear relationships for the amplitude (6) and phase (7) is given by

$$W = g^2\dot{\gamma} = \frac{1}{2}\left(A\dot{B} - B\dot{A}\right)\sin(2\phi) + AB\dot{\phi}. \tag{11}$$

This quantity remains invariant even for an arbitrary time-dependent parameter. This property has been exploited in order to find solutions in both classical and quantum time-dependent one-dimensional problems [17, 20]. Furthermore, it has been stressed that this invariant in the time-dependent case, plays the role that the energy operator does when the parameters are time independent [3, 22]. The value of the invariant depends on the choice of B and clearly vanishes if B is set to zero. If the quantities A, B are time independent, the orthogonal-functions invariant simplifies to

$$W = A_0 B_0 \omega_0. \tag{12}$$

The eigenvalue of the invariant clearly differs for different values of B_0.

In contrast, if there is a time varying parameter, the Hamiltonian is then time dependent and it no longer represents the energy of the system. In the particular case of constant parameter $k = \Omega^2 m$, the energy of an oscillator is given by the sum of the kinetic and potential energies $\mathcal{E} = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2$. In terms of amplitude and phase variables

$$\mathcal{E} = \frac{1}{2}m\left(\dot{g}_c\cos\gamma_c - g_c\dot{\gamma}_c\sin\gamma_c\right)^2 + \frac{1}{2}kg_c^2\cos^2\gamma_c. \tag{13}$$

Substitution of the $g_c(t)$ and $\gamma_c(t)$ for a constant parameter (9) and (10) yield

$$\mathcal{E} = \frac{1}{2}mA_0^2\left(\dot{\phi}\sin\phi\right)^2 + \frac{1}{2}kA_0^2\cos^2\phi. \tag{14}$$

Notice that although (13) involves B_0 through the amplitude $g_c(t)$ and phase $\gamma_c(t)$ functions, this last expression is no longer dependent on the constant B_0. Substitution of $\sqrt{k/m} = \dot{\phi} = \omega_0$, yields the well-known result $\mathcal{E} = \frac{1}{2}mA_0^2\omega_0^2$. Therefore, the energy function is identical for either $A_0\cos(\omega_0 t + \theta)$ or $g_c(t)\cos\gamma_c(t)$, provided that we calculate it from the sum of kinetic and potential energies. This assertion is true even if the amplitude is time dependent and the phase nonlinear. It is worth noting that the expression $\frac{1}{2}mg_c^2(t)\dot{\gamma}_c^2(t)$ does not represent the energy of the system if the amplitude and frequency are time dependent.

Measurement. – From an experimental point of view, the amplitude and phase of harmonic-oscillatory motion are measured as follows. The amplitude A_0 is set equal to the maximum displacement from equilibrium and is considered to be constant if the system attains the same maximum amplitude after each cycle. The frequency is in turn evaluated from the number of oscillations in a time interval or from the time duration of a period. In either case, the motion is tagged at a given position and velocity and one oscillation is completed when the body returns to the same position and velocity coordinates. The frequency is considered constant if the period of the oscillations is time independent. The phase, linear in time, is then considered to be equal to the frequency multiplied by time with an arbitrary initial phase constant.

However, the present results show that it is too swift to consider that if the maximum displacement is constant it follows that the amplitude is time independent. It is equally valid to consider that the amplitude is time dependent according to (9). On the other hand, even if the period of oscillation is constant, the phase may be nonlinear and the frequency time-dependent according to (10). It may be argued, with good reasons, that since the constant amplitude and linear phase are equally valid they should be preferred since they are a simpler choice.

Nonetheless, it is not always possible to make such a choice without incurring into inconsistencies. For example, consider a system with an initial constant amplitude A_{0i} and frequency $\omega_i = \Omega_i$ governed by the oscillator equation with constant parameter $\Omega^2(t) = \Omega_i^2$. Let the parameter Ω change abruptly when the displacement is maximum from Ω_i to a different constant Ω_f. According with the conventional measurement procedure, since the maximum displacement remains constant and unaltered, the amplitude also remains constant $A_i = A_f = A_0$. The invariant relationship (4) then leads to $A_0^2\Omega_i = A_0^2\Omega_f$. Then either the parameter $\Omega_i = \Omega_f$ remains unchanged, which contradicts the initial supposition, or the invariant relationship is not fulfilled $W_i \neq W_f$. However, the exact invariant must be constant throughout the motion even if the time-dependent parameter varies in a time span much shorter than the characteristic period of the system. This apparent contradiction is solved if the amplitude and frequency are allowed to be time dependent in either the initial or the final state. Let the initial amplitude be constant $A_i = g_c(t, A_0, A_0) = A_0$, where $B_{0i} = A_{0i}$ in this initial state. Since the position is located at maximum displacement when the abrupt change takes place $q_i(\phi = 0) = q_f(\phi = 0)$, then $A_{0i} = A_{0f} = A_0$. In the final state B_{0f} is obtained from the invariant relationship under steady-state conditions (12), namely, $W = A_0^2\Omega_i = A_0 B_{0f}\Omega_f$. The amplitude in the final state is then $A_f = g_c(t, A_0, B_0)$ with $B_0 = A_0\Omega_i/\Omega_f$, which is clearly a time-dependent amplitude. Therefore, the constant amplitude choice is not always possible. This particular problem of a TDHO with a subperiod time-dependent parameter has been recently treated in detail using an approximate analytical solution [17].

Adiabatic processes require that the time-dependent parameter Ω varies slowly with respect to time. This condition, in turn, implies that the amplitude and frequency must be quasi-static functions. Therefore, time-dependent amplitude and frequency functions should be associated with the nonadiabatic regime. Thus, harmonic-oscillator systems with fast time-dependent parameters with respect to the period of the system, *i.e.* nonadiabatic, must exhibit rapid variations in the amplitude and frequency variables. Whether this assertion may be extended to other dynamical systems is still an open issue.

Conclusions. – The trajectory of a particle may be described in amplitude and phase variables in a nonunique way. This assertion has been demonstrated for a one-dimensional harmonic-oscillator with time-dependent parameters. Nonetheless, it seems plausible that the present derivation may be extended to three-dimensional motion, more general potentials nonlinear in the spatial variables and problems involving propagation. The amplitude and phase representation is then also expected not to be unique in these more general contexts. The energy of the one-dimensional oscillator with time-independent parameters has been shown to remain the same even if the amplitude and frequency solutions are time dependent. For a harmonic-oscillator with time-dependent parameters, the orthogonal-functions invariant is the conserved quantity. The value of this invariant depends on the initial choice of the amplitude and phase functions. From the viewpoint of measurement theory, a constant maximum displacement has been shown not to necessarily imply a time-independent amplitude. Similarly, a period that does not vary from one oscillation to another also admits a time-dependent frequency. It has been exhibited that a TDHO with a sudden frequency change requires *a fortiori* such a time-varying amplitude and frequency behaviour. Furthermore, the amplitude and frequency becomes time dependent when the adiabatic condition is not fulfilled.

REFERENCES

[1] Haas F. and Goedert J., *Phys. Lett. A*, **279** (2001) 181.
[2] Moyo S. and Leach P. G. L., *J. Phys. A*, **35** (2002) 5333.

[3] Haas. F., *J. Phys. A*, **35** (2002) 2925.
[4] Fernández Guasti M., Gil Villegas A. and Diamant R., *Rev. Mex. Fis.*, **46** (2000) 530.
[5] Bassom A. P., Schief W. K. and Rogers C., *J. Phys. A*, **29** (1996) 903.
[6] Simic S., *J. Phys. A*, **33** (2000) 5435.
[7] Fernández-Guasti M., *J. Phys. A*, **37** (2004) 4107.
[8] Milne W. E., *Phys. Rev.*, **35** (1930) 863.
[9] Froman N., *Physical Problems Solved by the Phase-Integral Method* (Cambridge University Press) 2002.
[10] Matzkin A., *Phys. Rev. A*, **63** (2000) 012103.
[11] Matzkin A., *J. Phys. A*, **34** (2001) 7833.
[12] Im C. I., George T. F., Pandey L. N., Yeon K. H. and Kim D. H., *Phys. Rev. A*, **55** (1997) 4023.
[13] Jian Z., Pearce J. and Mittleman D. M., *Phys. Rev. Lett.*, **91** (2003) 033903.
[14] Burov V. A. and Morozov S. A., *Acoust. Phys.*, **47** (2001) 659.
[15] Froman P. O. and Larsson K., *J. Math. Phys.*, **43** (2002) 2169.
[16] Basarab-Horwarth P., Euler N., Eulrer M. and Fushchych W. I., *J. Phys. A*, **28** (1995) 6193.
[17] Fernández Guasti M., *Physica D*, **189** (2004) 188.
[18] Fan H., *J. Phys. A*, **36** (2003) 293.
[19] Monteiro T. S. and Saraga D. S., *Found. Phys.*, **31** (2001) 355.
[20] Fernández-Guasti M. and Moya Cessa H., *J. Phys. A*, **36** (2003) 2069.
[21] Thylwe K. E., *J. Phys. A*, **38** (2005) 235.
[22] Moya Cessa H. and Fernández Guasti M., *Phys. Lett. A*, **311** (2003) 1.
[23] Naselsky P., Novikov D. and Silk J., *Astrophys. J.*, **565** (2002) 655.
[24] Minnini P. D., Gómez D. O. and Mindin G. B., *Solar Phys.*, **208** (2002) 167.
[25] Cormack I. G., Sibbett W. and Reid D. T., *J. Opt. Soc. Am. B*, **18** (2001) 1377.
[26] Romstad F., Borri P., Langbein W., Mork J. and Hvam J. M., *IEEE Photonics Technol. Lett.*, **12** (2000) 1674.
[27] Zubov V. A., *Quantum Electron.*, **32** (2002) 479.
[28] Rubiola E. and Giordano V., *Rev. Sci. Instrum.*, **73** (2002) 2445.
[29] Fick S. E. and Palmer C. H., *J. Res. Natl. Inst. Stand. Technol.*, **106** (2001) 833.
[30] Sawyer N. B. E. *et al.*, *Rev. Sci. Instrum.*, **72** (2001) 3793.
[31] Guyon O., *Astron. Astrophys.*, **404** (2003) 379.
[32] Golubev V. G., Dukin A. A., Medvedev A. V., Pevtsov A. B., Selkin A. V. and Feoktistov N. A., *Phys. Solid State*, **46** (2004) 1815.
[33] Fernández Guasti M. and Gil-Villegas A., *Phys. Lett. A*, **292** (2002) 243.
[34] Lewis H. R., *Phys. Rev. Lett.*, **18** (1967) 510.
[35] Reid J. L. and Ray J. R., *J. Math. Phys.*, **21** (1980) 1583.