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Superluminal group velocity
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Abstract. – Based on boundary condition and dispersion relation, the superluminal group
velocity in an anisotropic metamaterial (AMM) is investigated. The superluminal propagation
is induced by the hyperbolic dispersion relation associated with the AMM. It is shown that a
modulated Gaussian beam exhibits a superluminal group velocity which depends on the choice
of incident angles and optical axis angles. The superluminal propagation does not violate the
theory of special relativity because the group velocity is the velocity of the peak of the localized
wave packet which does not carry information. It is proposed that a triglycine sulfate (TGS)
crystal can be designed and the superluminal group velocity can be measured experimentally.

Introduction. – In 1968, Veselago firstly introduced the concept of left-handed material
(LHM) in which both the permittivity ε and the permeability μ are negative [1]. Veselago
predicted that LHM would have unique and potentially interesting properties, such as the
negative refraction index, the reversed Doppler shift and the backward Čerenkov radiation.
LHM did not receive much attention as it only existed in a conceptual form. After the first
experimental observation using a metamaterial composed of split ring resonators (SRR) [2,3],
the study of such materials has received increasing attention over the last few years [4–15]. As
noted earlier, both ε and μ are necessarily frequency dispersive in LHM. Since the frequency
dispersion is important, the superluminal propagation in the LHM takes place [6–8].

While negative refraction is most easily visualized in an isotropic metamaterial, negative
refraction can also be realized in anisotropic metamaterial (AMM), which does not necessarily
require that all tensor elements of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ have negative values [9–13]. Recently, Smith *et
al.* have shown experimentally that an AMM slab designed to provide a permeability equal
to -1 along the optical axis, will redirect *E*-polarized electromagnetic waves from a nearby
source to a partial focus [14,15].

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In the present letter, we will investigate the superluminal group velocity of waves in an AMM. The superluminal propagation is induced by the hyperbolic dispersion relations associated with the AMM. We describe a modulated Gaussian beam incident on the triglycine sulfate (TGS) crystal, which demonstrates in a straightforward manner that the peak of the localized wave packet displays interesting superluminal behavior.

Hyperbolic dispersion relation. – For anisotropic media, one or both of the permittivity and permeability are second-rank tensors. We assume that the permittivity and permeability tensors are simultaneously diagonalizable. If we take the optical axis as the z-axis, the permittivity and permeability tensors have the following forms:

$$
\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}, \qquad \boldsymbol{\mu} = \begin{pmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{pmatrix}, \tag{1}
$$

where ε_i and μ_i are the relative permittivity and permeability constants in the principal coordinate system ($i = x, y, z$). It should be noted that the real AMM constructed by SRR is highly dispersive, both in spatial sense and frequency sense [14,15]. So these relative values are functions of the angle frequency ω. A general study on the shape of the dispersion relation as a function of the sign of these parameters has already been offered in ref. [12]. In this work, we are interested in the case of AMM with hyperbolic dispersion relation.

Without loss of generality, we assume that the wave vector locates at the x-z plane ($k_y = q_y = 0$). Maxwell's equations yield a scalar wave equation for H-polarized field . In free space, the accompanying dispersion relation has the familiar form

$$
k_x^2 + k_z^2 = \frac{\omega^2}{c^2}, \tag{2}
$$

where k_x and k_z are the x and z components of the incident wave vector, ω is the frequency, and c is the speed of light in vacuum. We assume that there is an angle φ between the optical axis and the z-axis. For the given polarization, the waves equation yield the dispersion relation in AMM as

$$
\alpha q_x^2 + \beta q_z^2 + \gamma q_x q_z = \frac{\omega^2}{c^2}, \tag{3}
$$

where q_x and q_z represent the x and z components of refracted wave vector; α, β and γ are given by

$$
\begin{aligned}
\alpha &= \frac{1}{\varepsilon_x \varepsilon_z \mu_y} (\varepsilon_x \cos^2 \varphi + \varepsilon_z \sin^2 \varphi), \\
\beta &= \frac{1}{\varepsilon_x \varepsilon_z \mu_y} (\varepsilon_x \sin^2 \varphi + \varepsilon_z \cos^2 \varphi), \\
\gamma &= \frac{1}{\varepsilon_x \varepsilon_z \mu_y} (\varepsilon_z \sin 2\varphi - \varepsilon_x \sin 2\varphi).
\end{aligned} \tag{4}
$$

We show the dispersion geometry in fig. 1, where a plane electromagnetic wave is incident from free space into the AMM. We choose the z-axis to be normal to the interface, the x-axis in the plane of the figure, and the y-axis out of the plane of the figure.

We assume here that the electric field is polarized along the y-axis. The z-component of the wave vector can be found by the solution of eq. (3), which yields

$$
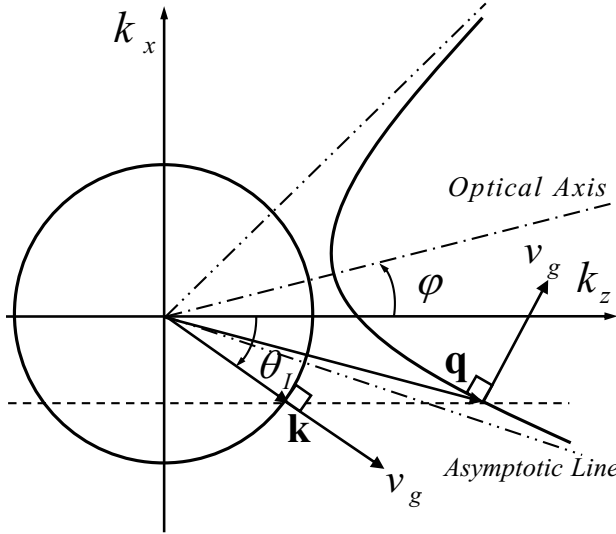q_z = \frac{1}{2\beta} \left[\sqrt{4\beta \frac{\omega^2}{c^2} + (\gamma^2 - 4\alpha\beta) q_x^2} - \gamma q_x \right]; \tag{5}
$$

Fig. 1 – The circle and the hyperbola represent the dispersion relations of free space and AMM, respectively. The incident wave vector $\boldsymbol{k}$ is parallel to the group velocity $\boldsymbol{v}_g$ in free space. Because of the anisotropy, $\boldsymbol{v}_g = \nabla_{\boldsymbol{q}}\omega(\boldsymbol{q})$ in the AMM is not necessarily parallel to the refracted wave vector $\boldsymbol{q}$. $\nabla_{\boldsymbol{q}}\omega(\boldsymbol{q})$ must lie normal to the frequency contour, $\omega(\boldsymbol{q}) = \text{const}$.

the choice of sign of q_z ensures that light power propagates away from the surface to the $+z$ direction. Due to the anisotropy, the transmitted wave components may refract at slightly different angles. The values of refracted wave vector can be found by using the boundary condition and hyperbolic dispersion relation.

We now determine the angle of phase refraction. The incident angle of light in free space is $\theta_I = \tan^{-1}(k_x/k_z)$ and the refraction angle of the transmitted wave vector in AMM can be found by $\theta_T = \tan^{-1}(q_x/q_z)$. From the boundary condition at the interface $z = 0$, the tangential components of the wave vectors must be continuous, *i.e.*, $q_x = k_x$. Thus the refracted angle of wave vector in the AMM can be easily obtained.

It is well known that the group velocity in anisotropic media differs from the direction of its wave vector. The group velocity is a very important physical quantity because it identifies the speed of the maximum intensity of wave packet. The group velocity in anisotropic media can be defined as [16, 17]

$$\boldsymbol{v}_g = \nabla_{\boldsymbol{q}}\omega(\boldsymbol{q}) = \frac{\partial\omega}{\partial q_x}\boldsymbol{e}_x + \frac{\partial\omega}{\partial q_z}\boldsymbol{e}_z, \tag{6}$$

where $\nabla_{\boldsymbol{q}}$ denotes the gradient of $\omega(\boldsymbol{q})$ in the wave vector space, $\boldsymbol{e}_x$ and $\boldsymbol{e}_z$ are unit Cartesian vectors. Because of the anisotropy, $\boldsymbol{v}_g = \nabla_{\boldsymbol{q}}\omega(\boldsymbol{q})$ is not necessarily parallel to the wave vector $\boldsymbol{q}$.

The group velocity specifies the velocity of the peak of the wave group, and is not necessarily parallel to the wave vector. When the interface is aligned with an angle with the optical axes of the AMM, the hyperbolic dispersion relations will exhibit some interesting effects. As shown in fig. 1, the magnitude of $\boldsymbol{q}$ varies as a function of its direction. When the refractive wave vector $\boldsymbol{q}$ is approximately parallel to the asymptotic line of hyperbola (dash-double-dotted line), $\boldsymbol{q}$ can be very large. The wavefront travels in the AMM with the velocity of $v_p = \omega/q$, so the ultra-slow phase velocity can be expected in the medium with
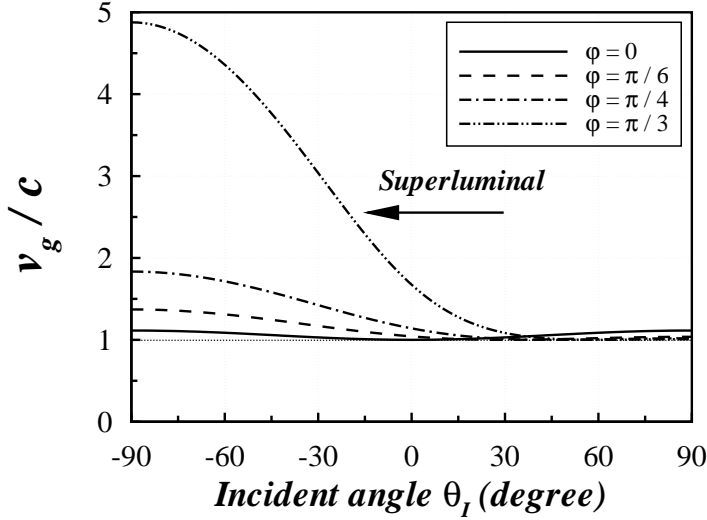
Fig. 2 – The superluminal group velocity in an AMM with different optical axial angles of $\varphi = 0$ (solid), $\pi/6$ (dashed), $\pi/4$ (dash-dotted) and $\pi/3$ (dash-double-dotted). The parameters of AMM are chosen as $\varepsilon_x = 1$, $\varepsilon_z = -5$ and $\mu_y = 1$ (TGS crystal).

hyperbolic dispersion relation. The waves propagate in different directions or in the AMM, $\Delta \boldsymbol{q}$ may not be parallel to $\boldsymbol{q}$. Therefore, the group velocity is generally not parallel to the phase velocity. If $\boldsymbol{q}$ propagates in some special direction, $\Delta q \to 0$ and the superluminal group velocity can be deduced. It should be mentioned that the interesting properties never exist in the medium with circle or ellipse dispersion relation.

Superluminal group propagation. – The negative refraction index imposes that the SRR presents frequency dispersion [2, 3]. This, in turn, conveys that actual anisotropic SRR is highly lossy. So the superluminal group velocity is not easily experimentally verified in SRR metamaterial. However, an extremely promising material, such as the TGS crystal, satisfies the conditions $\varepsilon_x > 0$, $\varepsilon_z < 0$. In principle, at far infrared frequencies due to polarization dependence of certain phonon resonances, the material with suitably low absorption exist [18, 19]. Thus the TGS slab can be prepared at low temperature and the superluminal group velocity can be measured experimentally.

Based on the dispersion relation of eq. (3), the group velocity can be derived as

$$v_g = \frac{c(2\alpha q_x + \gamma q_z)\boldsymbol{e}_x + c(\gamma q_x + 2\beta q_z)\boldsymbol{e}_z}{2\sqrt{\alpha q_x^2 + \gamma q_x q_z + \beta q_z^2}}\Bigg|_{\bar{\boldsymbol{q}}}. \tag{7}$$

If the width of the distribution $\boldsymbol{q}(\boldsymbol{k})$ is small compared to the range of $\boldsymbol{k}$ over which $\boldsymbol{k}_0$ varies significantly, we can simply evaluate $\bar{\boldsymbol{q}}(\boldsymbol{k}) = \boldsymbol{q}(\boldsymbol{k}_0)$ to obtain a good approximation. In fig. 2, the group velocity in the AMM is plotted. It should be noted that the superluminal group velocity in the AMM is induced by the hyperbolic dispersion relation. The superluminal group velocity does not mean the propagation of energy will be superluminal. Using the new definition of electric and magnetic energy [20,21], the energy velocity cannot be superluminal and the conservation of energy is not violated.
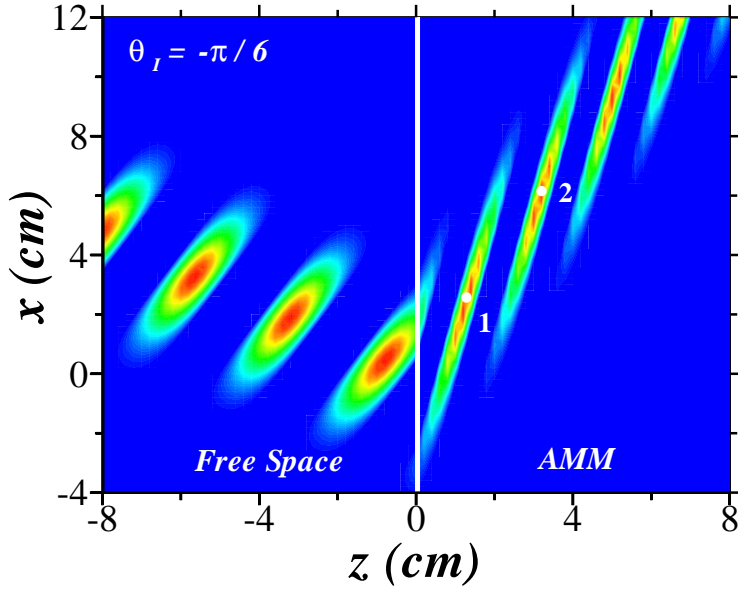
Fig. 3 – The modulated Gaussian beam incident from free space to AMM. The center wave number is $k_0 = \sqrt{5}$ with different incident angles $\theta_I = -\pi/6$. The optical axis angle $\varphi = \pi/4$ and the spatial extent of the incident wave packet is $a = 1$. The peak of the localized wave packet travels from point 1 to point 2 during a cycle Δt with superluminal group velocity. The anisotropic parameters are the same as those used in fig. 2.

Note that a wave group can be formed from planes with different frequencies ω or from plane waves with different $\boldsymbol{k}$ vectors [17]. To obtain a better physical picture of superluminal group velocity in AMM, a modulated Gaussian beam of finite width can be constructed. The field intensity distribution in free space is obtained by the Fourier integral and angular spectrum representation [22]. Following the method outlined by Lu *et al.* [23], let us consider a modulated beam incident from free space:

$$H_I(x, z) = \int_{-\infty}^{+\infty} \mathrm{d}k_\perp f(k_\perp) \exp[i(\boldsymbol{k}_0 + \boldsymbol{k}_\perp) \cdot \boldsymbol{r} - i\omega_0 t], \tag{8}$$

where $\boldsymbol{k}_\perp$ is perpendicular to $\boldsymbol{k}_0$ and $\omega_0 = ck_0$. A general incident wave vector is written as $\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{k}_\perp$. We assume that its Gaussian weight is

$$f(k_\perp) = \frac{a}{\sqrt{\pi}} \exp[-a^2 k_\perp], \tag{9}$$

where a is the spatial extent of the incident beam. We want the modulated Gaussian beam to be aligned with the incident direction defined by the vector $\boldsymbol{k}_0 = k_0 \cos\theta_I \boldsymbol{e}_x + k_0 \sin\theta_I \boldsymbol{e}_z$, which makes an angle θ_I with the surface normal.

Due to the anisotropy, the transmitted wave components may refract at slightly different angles. When the beam enters the AMM, it will no longer maintain Gaussian, but becomes a localized wave packet. Matching the boundary conditions for each k component at $z = 0$

gives the complex field in the form

$$H_T(x, z) = \int_{-\infty}^{+\infty} dk_\perp f(k_\perp) T(\boldsymbol{k}) \exp[i(\boldsymbol{q} \cdot \boldsymbol{r} - \omega_0 t)]. \tag{10}$$

where $T(\boldsymbol{k})$ is the transmission coefficient of the wave packet. The transmission coefficient can obtain a good approximation to simply evaluate this quantity at $\boldsymbol{k}_0$. Then the transmission coefficient of the wave packet is simply given by $T(\boldsymbol{k}_0)$. The normal component of refracted wave vector q_z can be expanded in a Taylor series to first order in $\boldsymbol{k}_0$ to obtain a better approximation,

$$q_z(\boldsymbol{k}) = q_z(\boldsymbol{k}_0) + (\boldsymbol{k} - \boldsymbol{k}_0) \cdot \left.\frac{\partial q_z(\boldsymbol{k})}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_0}. \tag{11}$$

The distribution of the transmission field in the AMM can be derived from eq. (10) under the above approximation.

A close look at the localized wave packet shows that the superluminal propagation of the peak is induced by the hyperbolic dispersion relation in the AMM. Figure 3 shows a close view of the field intensity distribution of the wave packet propagating from free space into the AMM. In a circle $\Delta t = 2\pi/\omega_0$ of the modulated Gaussian beam, the peak of the localized wave packet travels from point 1 to point 2. In fig. 3 we set the center wave vector with incident angles $\theta_I = -\pi/6$. We mark the peak of the wave packet at the two times. In the $\Delta t = 0.094\,\text{ns}$, the peak of the wave packet moves from (1.284, 2.571) to (3.144, 6.147). This propagating velocity corresponds to $0.427\,\text{m/ns}$, which is almost exactly the analytical group velocity of $1.424c$ in eq. (7), and the superluminal group velocity is demonstrated theoretically.

Discussion and conclusion. – It should be noted that the superluminal group velocity has completely different origin from those described in isotropic LHM [6–8] or ultracold gas of atoms [24,25], where the superluminality is caused by the frequency dispersion of the medium. In the case discussed here, the superluminal group velocity in the medium is induced by the hyperbolic dispersion relation in the AMM. As far as we know, this kind of superluminal group velocity has not been recognized before. It should be mentioned, however, that the shape of localized wave packet is distorted once the modulated Gaussian beam is incident into the AMM. Our results do not violate causality or special relativity, because the group velocity is the velocity of the peak of the wave packet, which does not carry information. It is shown that the TGS crystal slab can be designed and the superluminal group velocity can be measured experimentally.

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