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# Finite-temperature functional RG, droplets and decaying Burgers turbulence

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**Abstract.** – The functional RG (FRG) approach to pinning of $d$-dimensional manifolds is re-examined at any temperature $T$. The coupling function $R(u)$ is shown to be a physical observable in any $d$, exactly related to a free energy cumulant in a parabolic well. In $d = 0$ its beta function is displayed to a high order, ambiguities resolved; for random field disorder (Sinai model) we obtain exactly the $T = 0$ fixed point $R(u)$ and its thermal boundary layer (TBL) form (*i.e.* for $u \sim T$) at $T > 0$. Connection between FRG in $d = 0$ and decaying Burgers turbulence is discussed. An exact solution to the functional RG hierarchy in the TBL is obtained for any $d$ and related to droplet probabilities.

Elastic manifolds pinned by quenched disorder [1] are the simplest system to study glass phases where (dimensionless) temperature is formally irrelevant, scaling as $\tilde{T}_L = TL^{-\theta}$ with system size $L$. They are parameterized by a displacement ($N$-component height) field $u(x) \equiv u_x$, where $x$ spans a $d$-dimensional internal space. The competition between elasticity and disorder produces rough ground states with sample averages $\overline{(u_x - u_{x'})^2} \sim |x - x'|^{2\zeta}$ ($\theta = d - 2 + 2\zeta$). These are believed to be statistically scale invariant, hence should be described by a critical (continuum) field theory (FT). The latter seems highly unconventional in several respects. First, an infinite number of operators become marginal simultaneously in $d = 4 - \epsilon$. This is handled via functional RG methods where the relevant coupling constant becomes a function of the field, $R(u)$, interpreted as the (second cumulant) disorder correlator [2]. A more formidable difficulty then arises: at $T = 0$ both $R(u)$ and, more generally, the full effective action functional $\Gamma[u]$, appear to be non-analytic([1]) around $u = 0$. A linear cusp in $R''(u)$ was found in one-loop and large-$N$ calculations [2, 3]. Qualitative (two-mode minimization) and mean-field arguments relate this cusp to multiple metastable states and shock type singularities in the energy landscape [4]. As a consequence, ambiguities arise in loop corrections [5]. Although candidate renormalizable FTs have been identified [5, 6] (working directly at $T = 0$) this problem has, until now, hampered derivation of the field theory from first principles (with the notable exception of the $N = 1$ depinning transition [7]).

Working at non-zero temperature $T > 0$ should help define the theory, and $\Gamma[u]$ has been argued to remain smooth within a "thermal boundary layer" (TBL) of width $u \sim \tilde{T}_L$ around $u = 0$. This width however shrinks as $\tilde{T}_L \to 0$ in the thermodynamic limit, and if a fixed-$u$, large-$L$ limit exists for any fixed small $T$ it should unambiguously define the (non-analytic) "zero temperature theory": this program, called "matching", was proposed and studied in ref. [8]. It does, to some extent, rely on a scaling ansatz proposed there for the

([1])The non-analyticity of $\Gamma$ occurs at *finite scale* (the Larkin length) contrarily to, *e.g.*, the critical $\phi^4$ theory.

TBL. This ansatz was shown [8] to be consistent to 1-loop with the droplet picture [9] and, in the (near) equilibrium (driven) dynamics, to account for the phenomenology of ultraslow activated (creep) motion [8, 10]. Although its physics is reasonable, it is not yet established how a critical renormalizable FT emerges as $\tilde{T}_L \to 0$, with a finite unambiguous beta function.

Another field of physics where an (unconventional) field-theoretic description is needed, but remains elusive, is high Reynolds number turbulence. There too the scale-invariant regime, the inertial range, needs regularization at the small dissipation scale set by the (formally irrelevant) viscosity $\nu$ [11]. In the simpler case of Burgers turbulence [12–14] connections to disordered systems were studied [15]. However, even in that case, the interesting connections between the $\nu \to 0$ limit and the $\tilde{T}_L \to 0$ limit of FRG were not addressed.

Given its central role in the FRG, it is of high interest to obtain the *precise* physical content of the function $R(u)$, beyond previous qualitative arguments. In the FT, a precise, but abstract, definition was given, from the replicated effective action at zero momentum, allowing for a systematic dimensional expansion. From it, it was observed that $R''(0)$ gives the exact sample to sample variance of the center of mass of the manifold (a typical observable with a universal $T = 0$ limit), while $R''''(0)$ yields sample-to-sample susceptibility fluctuations (a finite temperature observable which diverges as $T_L \to 0$).

In this letter we first show that, in any $d$, not only $R(u)$, but also higher cumulants and the full (replicated) effective action $\Gamma[u]$ are physical observables. These can be directly *measured* by adding a quadratic external potential and varying the position of its center. It makes explicit the $T = 0$ physics of the FRG in terms of (functional) shocks of a *decaying* (functional) Burgers equation and at $T > 0$ makes precise the relation between the TBL form of the effective action and droplet probabilities. Next, the instructive $d = 0$ case is studied. It amounts to a particle in a $N$-dimensional random potential (described by the partition sum $Z = \int \mathrm{d}u e^{-V(u)/T}$, *i.e.* a simple $N$-integral over a random function $V(u)$) and related to $N$-dim Burgers equation. For $N = 1$, the matching program started in ref. [8] is pushed to obtain here the (unambiguous) beta function to four-loop. There the relation between the inviscid distributional limit of Burgers equation [12] and the FRG in the $T_L \to 0$ limit is made precise. In the sub-case of the Sinai (*i.e.* random field) model, the exact $R(u)$ is computed at $T = 0$. The TBL rounding form at $T > 0$ is also obtained. Obtaining the thermal rounding form in any $d$ amounts to solve an infinite hierarchy of (functional) exact RG equations: remarkably, this can be achieved, the solution being parameterized by droplet probability data. All details are given in a forthcoming publication [16].

The model studied here is defined by the total energy in a given sample ($u \in R^N$):

$$H_V[u] = \frac{1}{2} \int_{xy} g_{xy}^{-1} u_x u_y + \int_x V(u_x, x). \tag{1}$$

The distribution of the random potential is translationally invariant, with second cumulant $\overline{V(u,x)V(u',x')} = \delta^d(x - x')R_0(u - u')$ and $\overline{V(u,x)} = 0$. This implies the statistical (tilt) symmetry (STS) under $(x, u_x) \to (x, u_x + \phi_x)$ [1]. Several results here are valid for arbitrary $g_{xy}$, but we often specialize to $g_q^{-1} = q^2 + m^2$ in Fourier space, where the small mass provides a confining parabolic potential and a convenient infrared cutoff at large scale $L_m = 1/m$. In all formulas below one can replace $\int_x \equiv \int \mathrm{d}^d x \to \sum_x$ and $\delta(x - x') \to \delta_{xx'}$ in the bare disorder correlator, *i.e.* a lattice provides a UV cutoff which preserves STS. Numerous physical systems are modelled by (1), *e.g.* i) CDW or vortex lattices [1], for a periodic $R_0(u)$; ii) magnetic interfaces with bond disorder, for a short-range (SR) $R_0(u)$; iii) magnetic interfaces with random field disorder, of variance $\sigma$, for a long-range $R_0(u) \sim -\sigma|u|$ at large $u$.

Let us recall the definition of $R(u)$ in the FT. Using replica fields $u^a_x$, the bare action is

$$\mathcal{S}[u] = \frac{1}{2T}\sum_a \int_{xy} g^{-1}_{xy} u^a_x u^a_y - \frac{1}{2T^2}\sum_{ab}\int_x R_0(u^a_x - u^b_x), \tag{2}$$

with $a = 1, \dots, p$. Connected correlations of (1) are obtained expanding the $W$ functional, $W[j] = \ln \int \prod_{ax} \mathrm{d}u^a_x e^{\int_x \sum_a j^a_x u^a_x - \mathcal{S}[u]}$ at $p = 0$. The effective action of the replica theory is the Legendre transform $\Gamma[u] = \int_x \sum_a u^a_x j^a_x - W[j]$. It generates (in an expansion in $u$) the renormalized vertices and is the important functional for the FRG. To define the renormalized disorder one performs an expansion in number of replica sums:

$$\Gamma[u] = \sum_a \int_{xy} \frac{g^{-1}_{xy} u^a_x u^a_y}{2T} - \sum_{ab}\frac{R[u^{ab}]}{2T^2} - \sum_{abc}\frac{S[u^{abc}]}{3!T^3} + \cdots, \tag{3}$$

where STS implies that the single-replica term is the bare one, and the form of the $n$-replica terms, *e.g.* $R[u^{ab}]$ is a functional depending only on the field $u^{ab}_x \equiv u^a_x - u^b_x$, whose value for a uniform field (*i.e.* local part) defines $R(u)$, i.e $R[\{u^{ab}_x = u\}] = L^d R(u)$(2). It was used in the FT [5,6] to compute the beta function, $-m\partial_m|_{R_0} R(u) = \beta[R](u)$, in powers of $R$.

We have shown that this abstract definition is equivalent to a physical one: for each realization of the random potential $V$, one defines the renormalized potential functional $\hat{V}[v] = \hat{V}[\{v_x\}]$ as the free energy in the presence of an external quadratic potential centered at $u_x = v_x$:

$$e^{-\frac{1}{T}\hat{V}[\{v_x\}]} = \int \prod_x \mathrm{d}u_x e^{-\frac{1}{T}H_{V,v}[u]}, \quad H_{V,v}[u] = \int_{xy} \frac{g^{-1}_{xy}}{2}(u_x - v_x)(u_y - v_y) + \int_x V(u_x, x). \tag{4}$$

Using STS one sees that the renormalized energy landscape has second cumulant correlations $\overline{\hat{V}[\{v_x\}]\hat{V}[\{v'_x\}]} = \hat{R}[\{v_x - v'_x\}]$ (averages with respect to $V$). The result shown in [16] is that $\hat{R} = R$. Hence one can *measure* the 2-replica part of the effective action by computing the free energy in a well whose position is varied. Choosing a uniform $v_x = v$ yields

$$\overline{\hat{V}(v)\hat{V}(v')} = L^d R(v - v'), \tag{5}$$

where $\hat{V}(v) = \hat{V}[\{v_x = v\}]$ is the local part, using a parabolic potential centered at $u_x = v$. Performing the Legendre transform [16] (more involved) relations are found for higher cumulants, *e.g.* $S = \hat{S} - 3sym_{abc} g_{xy} R'_x[v^{ab}] R'_y[v^{ac}]$. The STS property was used: for a non-STS model, *e.g.* with discrete $u$, either it flows to the STS fixed point as $m \to 0$, and the above holds asymptotically, or it does not and a (more involved) extension holds [16].

From (4) the *renormalized (pinning) force* functional $\mathcal{F}_x = \hat{V}'_x[v] \equiv \delta\hat{V}[v]/\delta v_x$ is related to the thermally averaged position in the presence of the shifted well, via $\mathcal{F}_x = \int_y g^{-1}_{xy}(v_y - \langle u_y \rangle_{H_{V,v}})$. Hence the force correlator functional $R''_{xy}[v]$ has a nice expression. For uniform $v_x = v$ and at $T = 0$ it is simple: denote $u_x(v)$ the minimum energy configuration of $H_{V,v}[u]$ for a fixed $v_x = v$ and $\bar{u}(v) = L^{-d}\int_x u_x(v)$ its center-of-mass position. Then, denoting $\Delta(v) = -R''(v)$:

$$\overline{(v - \bar{u}(v))(v' - \bar{u}(v'))} = \Delta(v - v') L^{-d} m^{-4} \tag{6}$$

which generalizes to non-zero $T$ (replacing $\bar{u}(v)$ by its thermal average) and to the full multi-local functional(3). Note that (6) is *exact for any* $m$, though only the rescaled limit of

(2)Such zero momentum renormalization conditions are standard in a massive theory. It is not presently known how to close FRG using other conditions, *e.g.* symmetric external momenta, as in massless theories.

(3)Equations (5), (6) generalize to any $N$, and to two copies as used in chaos studies [17] $\overline{\hat{V}_i(v)\hat{V}_j(v')} = L^d R_{ij}(v - v')$.

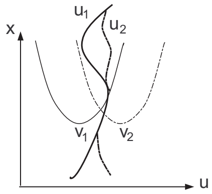


Fig. 1 – Ground state of a directed polymer $d = 1$, $N = 1$: at least one shock occurs (at some $v = v_s$) as the center of the quadratic well changes from $v_1$ to $v_2$ and the GS from $u_1(x)$ to $u_2(x)$.

$m^{-\epsilon+2\zeta}\Delta(um^{-\zeta})$ exhibits universality for $m \to 0$. For fixed $L/a$, $a$ being the UV cutoff scale, the minimum is expected unique for continuous distributions of $V$, except for discrete values $v_s$, the positions of shocks where $\bar{u}(v)$ switches between different values (*e.g.* $u_1$ to $u_2$ see fig. 1) and the force is discontinuous (at $T = 0$): below, the shock strength is noted $u_{21}^{(s)} = u_2 - u_1$.

The renormalized pinning force satisfies an exact RG (ERG) equation (with $\partial g = -m\partial_m g$):

$$-2m\partial_m \mathcal{F}_x[v] = \int_{yz} \partial g_{yz}(T\mathcal{F}''_{xyz}[v] - \mathcal{F}'_{xy}[v]\mathcal{F}_z[v]) \tag{7}$$

a functional generalization of the *decaying Burgers equation* to which it reduces for $d = 0$:

$$\partial_t F(v) = \frac{T}{2}F''(v) - F'(v)F(v) \tag{8}$$

with $t = m^{-2}$, $F(v) = \hat{V}'(v)$, usually written $\partial_t \mathsf{u} + \mathsf{u}'_x\mathsf{u} = \nu\mathsf{u}''_{xx}$, identifying $\mathsf{u}, x, \nu$ in Burgers to $F, v, T/2$ in the FRG. The stochasticity in (7), (8) comes from their (random) initial conditions $F(v) = V'(v)$ and $\mathcal{F}_x[v] = V'_x[v]$ for $t = 0, m = \infty$. Equation (8) (and its primitive) is equivalent to an infinite ERG hierarchy for the $n$-th moments $\bar{S}^{(n)}(v_{1,2,\ldots,n}) = (-)^n\overline{\hat{V}(v_1),\ldots,\hat{V}(v_n)}$ in $d = 0$:

$$-m\partial_m R(v) = \frac{2T}{m^2}R''(v) + \frac{2}{m^2}\bar{S}_{110}(0,0,v), \tag{9}$$

$$-m\partial_m \bar{S}^{(n)}(v_{1,2,\ldots,n}) = \frac{nT}{m^2}[\bar{S}^{(n)}_{20\ldots0}(v_{1,2,\ldots,n})] + \frac{n}{m^2}[\bar{S}^{(n+1)}_{110\ldots0}(v_{1,1,2,\ldots,n})], \tag{10}$$

where $\bar{S} \equiv \bar{S}^{(3)} = \hat{S}$, subscripts denote partial derivatives and $[\ldots]$ is symmetrization. A similar, more formidable looking functional hierarchy exists for any $d$:

$$-m\partial_m R[v] = T\partial g_{xy}R''_{xy}[v] + \partial g_{zz'}\bar{S}^{110}_{zz'}[0,0,v] \tag{11}$$

together with ERG equations for $\bar{S}$ and higher moments. In both cases a related hierarchy exists for the cumulants $R, S, \ldots$ defining $\Gamma[u]$ in (3), studied in [8]. The usual RG strategy is to truncate them to a given order in $R$ yielding the beta function. Ambiguities in the limit of coinciding arguments in (10), (11) may arise in doing so directly at $T = 0$.

We start with $d = 0$ (and $N = 1$), a particle in a 1D random potential $V(u)$, aiming to obtain an unambiguous beta function as $m \to 0$. We define rescaled $\tilde{T} = 2Tm^\theta$ and $R(u) = \frac{1}{4}m^{\epsilon-4\zeta}\tilde{R}(um^\zeta)$ (this should yield a FP when correlations of $V$ grow as $u^{\theta/\zeta}$). Trying first standard loop expansion at $T > 0$ ($\tilde{R}$ analytic), we obtained from (10) the beta function $-m\partial_m\tilde{R}|_{R_0} = \beta[\tilde{R}, T]$. To $n$-loop, it is a sum of terms of order $\tilde{T}^p\tilde{R}^{n+1-p}$, $0 \le p \le n$. The one-loop equation (*i.e.* adding $\tilde{T}\tilde{R}''$ to the first three terms in (12) below) exhibits the

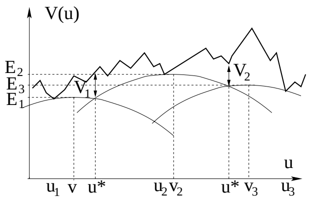


Fig. 2 – Construction of the joint probability $P(\{E_i, v_i\})$ that $\hat{V}(v_i) = E_i$ at points $v_i$: the random walk $V(u)$ must remain above all parabola centered on the $v_i$ of apex $E_i$ intersecting at points $u_i^*$. Each independent interval $[u_i^*, u_{i+1}^*]$ can be treated as in [18].

standard TBL for $u \sim \tilde{T}$ discussed in [8]. To 2-loop a term $\frac{-1}{4}\tilde{T}\tilde{R}''''(0)\tilde{R}''(u)$ appears, and using the TBL identity $\lim_{m\to 0}\tilde{T}\tilde{R}''''(0) = \tilde{R}'''(0^+)^2$, exact at one-loop, produces precisely the 2-loop "anomalous" term in (12) below. Alas, one finds [16] that this procedure fails at 3-loop. One must instead examine the whole ERG hierarchy as in [8]. There, a method to obtain the unambiguous beta function was found by verifying order by order, a continuity property of the Γ-cumulants $S^{(n)}_{11\ldots 1}(u_{1,\ldots,n})$ upon bringing points together. We completed in [16] the derivation of the (local) beta function, obtaining (up to a constant, with $\mathsf{R}'' = \tilde{R}'' - \tilde{R}''(0)$):

$$
\begin{aligned}
-m\partial_m \tilde{R} = {} & (\epsilon - 4\zeta)\tilde{R} + \zeta u \tilde{R}' + [\frac{1}{2}(\tilde{R}'')^2 - \tilde{R}''(0)\tilde{R}''] + \frac{1}{4}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)\mathsf{R}'' + \\
& + \frac{1}{16}(\mathsf{R}'')^2(\tilde{R}'''')^2 + \frac{3}{32}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)^2 + \frac{1}{4}\mathsf{R}''((\tilde{R}''')^2\tilde{R}'''' - \tilde{R}'''(0^+)^2\tilde{R}''''(0^+)) + \\
& + \frac{1}{96}(\mathsf{R}'')^3(\tilde{R}^{(5)})^2 + \frac{3}{16}(\mathsf{R}'')^2\tilde{R}'''\tilde{R}''''\tilde{R}^{(5)} + \frac{1}{8}\mathsf{R}''((\tilde{R}''')^3\tilde{R}^{(5)} - \tilde{R}'''(0^+)^3\tilde{R}^{(5)}(0^+)) + \\
& + \frac{1}{16}(\mathsf{R}'')^2(\tilde{R}'''')^3 + \frac{9}{16}\mathsf{R}''((\tilde{R}''')^2(\tilde{R}'''')^2 - \frac{1}{6}\tilde{R}'''(0^+)^2(\tilde{R}'''')^2 - \frac{5}{6}\tilde{R}'''(0^+)^2\tilde{R}''''(0^+)^2) + \\
& + \frac{5}{16}((\tilde{R}''')^2 - \tilde{R}'''(0^+)^2)((\tilde{R}''')^2\tilde{R}'''' + \frac{1}{10}\tilde{R}''''\tilde{R}'''(0^+)^2 - \frac{11}{10}\tilde{R}''''(0^+)\tilde{R}'''(0^+)^2) + \mathrm{O}(\tilde{R}^6).
\end{aligned}
\tag{12}
$$

The first line are one- and 2-loop terms, the second is 3-loop, the last three are 4-loop. Normal terms (*i.e.* non-vanishing for analytic $R(u)$) are grouped with anomalous "counterparts" to show the absence of $\mathrm{O}(u)$ term, a strong constraint (linear cusp, no supercusp): these combinations can hardly be guessed beyond 3-loop. This shows the difficulty in constructing the FT, already in $d = 0$. We emphasize that (12) results from *a first-principle derivation.*

$R(u)$ being a physical observable, we look for cases where it can be computed. The Brownian landscape $V(u)$, the so-called Sinai model, is interesting as the $d = 0$ limit of random field disorder. Recently, we obtained the full statistics of (deep) extrema in the presence of a harmonic well [18]. This is generalized [16] as described in fig. 2. Graphically the renormalized landscape $\hat{V}(v) = E$ is constructed by raising a parabola $P_v$, $y(u) = -\frac{m^2}{2}(u-v)^2 + E'$ from $E' = -\infty$ until it touches (for $E' = E$) the curve $y = V(u)$ at point $u = u_1(v)$, position of the minimum of $H_{V,v}(u)$, $E$ being the maximum (apex) of the parabola. $P_v$ touching at two points $u_1(v) < u_2(v)$ signals a shock at $u = v$. Computing $P(E_1, v_1; E_2, v_2)$ (see fig. 2) yields [16] (with $m^2 = 1$, $\sigma = 1$, $a = 2^{-1/3}$, $ba^2 = 1$, $\int_\lambda \equiv \int_{-\infty}^{+\infty}\frac{\mathrm{d}\lambda}{2\pi}$ and $\bar{R}(v) = R(v) - R(0)$)

$$
\begin{aligned}
\bar{R}(v) \;=\; & -2^{\frac{1}{3}}\sqrt{\pi v}\, e^{-\frac{v^3}{48}} \int_{\lambda_1}\int_{\lambda_2} \frac{[1 - \frac{2(\lambda_2-\lambda_1)^2}{b^2 v}]e^{i\frac{v}{2b}(\lambda_1+\lambda_2) - \frac{(\lambda_2-\lambda_1)^2}{b^2 v}}}{Ai(i\lambda_1)Ai(i\lambda_2)} \times \\
& \times \left[1 + \frac{v\int_0^\infty \mathrm{d}V e^{\frac{v}{2}V} Ai(aV + i\lambda_1)Ai(aV + i\lambda_2)}{Ai(i\lambda_1)Ai(i\lambda_2)}\right].
\end{aligned}
\tag{13}
$$

One finds $\bar{R}(v) \approx -v + 0.810775$ at large $v$, and recovers [18] $\overline{u^2} = -R''(0) = 1.054238$. Once rescaled, (13) should be a FP of (12) corresponding to $\zeta = \zeta_{RF} = 4/3$.

At non-zero $T$, one re-examines (4) taking into account, within a droplet calculation, the probability density $D(y)\mathrm{d}y$ for two degenerate minima of $V$, spatially separated by $y = u_2 - u_1$ ($D(y) = D(-y)$). It yields the TBL form (for $v \sim Tm^2$):

$$R''(v) = R''(0) + m^4 T \langle y^2 F_2(m^2 y v/T)\rangle_y \tag{14}$$

with $F_2(z) = \frac{z}{4}\coth\frac{z}{2} - \frac{1}{2}$ and $\langle \ldots \rangle_y \equiv \int \mathrm{d}y \ldots D(y)$ is normalized by the STS identity $\langle y^2\rangle_y = 2/m^2$. Since $F_2(z) \sim |z|/4$ at large $z$ (14) yields consistent matching between finite-$T$ (droplet) quantities in the TBL and the cusp of the $T = 0$ FP for $v = \mathrm{O}(1)$, with $R'''(0^+) = \frac{m^4}{2}\frac{\langle |y|^3\rangle_y}{\langle y^2\rangle_y}$. Equation (14) should be more generally valid in $d = 0$ (any $N$), but in the RF case it is known [18] that (setting $m = \sigma = 1$) $D(y) = \frac{1}{2}\int_{\lambda_1,\lambda_2} \frac{Ai'(i\lambda_1)e^{i(\lambda_1-\lambda_2)|y|/b}}{Ai(i\lambda_1)Ai^2(i\lambda_2)}$ is found to be consistent with $R'''(0^+) = 0.901289$ from (13). Remarkably, (14) generalizes to higher moments $\bar{S}^{(n)}$, with functions $F_n$ obtained in [16] yielding an exact "droplet" solution of the hierarchy (10).

Since in $d = 0$, the FRG (8) *identifies* with decaying Burgers, the correspondence

$$-R''(0) \equiv \overline{\mathsf{u}(x)^2}, \qquad \frac{T}{2}R''''(0) \equiv \nu\overline{(\nabla \mathsf{u}(x))^2} = \bar{\epsilon} \tag{15}$$

holds (more generally $\overline{\mathsf{u}(x)\mathsf{u}(0)} \equiv -R''(x)$) with finite limits as $\nu \to 0$. The second is the *dissipative anomaly*, also present in 3D Navier-Stokes. In Burgers it is due to shocks. The (equivalent) finite limit of the l.h.s. implies a thermal boundary layer in the FRG. Dilute shocks in Burgers are equivalent to droplets and a TBL in the FRG where $u_{21} \equiv \mathsf{u}(0^+) - \mathsf{u}(0^-)$, and (14) can be recovered from a single shock solution in Burgers upon averaging over its position [16]. The celebrated Kolmogorov law in the inertial range:

$$\frac{1}{2}\bar{S}_{111}(0,0,u) \sim \bar{\epsilon}u \equiv \frac{1}{12}\overline{(\mathsf{u}(x) - \mathsf{u}(0))^3} \sim -\bar{\epsilon}x \tag{16}$$

corresponds to the non-analytic behaviour of the third cumulant at small argument in the $T = 0$ theory. Identical coefficients in (15) and (16) are a consequence of matching across the TBL (*i.e.* viscous layer), identifying the second derivative of (9) at $v = 0$ (for $\nu > 0$) and $v = 0^+$ (for $\nu \to 0$), *i.e.* $\partial_t R''(0) = TR''''(0) \equiv \partial_t R''(0^+) = \bar{S}_{112}(0,0,0^+)$. Similar relations exist in stirred Burgers (and Navier-Stokes) [11]: there the dissipation rate $\bar{\epsilon}$ is balanced by forcing, instead of scale-invariant time decay of correlations, but small-scale shock properties should be rather similar. Closure of hierarchies similar to (10) was proposed there [13] in terms of an "operator product expansion". Recent studies cast doubt on such simple closures [12]: $N = 1$ decaying Burgers (and stirred [14]) can be constructed in the inviscid limit ($\nu \to 0$) using distributions, *e.g.* $tF'(v) = 1 - \sum_s u_{21}^{(s)}\delta(v - v_s)$. It is shown there that shock "form factors" (*i.e.* size distribution) determines small-distance (non-analytic) behaviour of moments of velocity differences, $\overline{(F(v) - F(0))^p} \sim \mu_p v \mathrm{sign}(v)^{p+1}$, with $\mu_p = \overline{\sum_s (u_{21}^{(s)})^p \delta(v - v_s)}$. In the FRG these are equivalent to droplet distributions as we show [16] that $\mu_p = \langle |y|^{p+1}\rangle_y / \langle y^2\rangle_y$, *e.g.* consistent with $R'''(0^+) \equiv \overline{\mathsf{u}(0^+)\nabla\mathsf{u}(0)} = \mu_2/(2t^2)$ given above. The $T = 0$ distributional limit of (8) derived in [12] is equivalent to $\partial_t \hat{V}(v) = -\frac{1}{2}F(v^+)F(v)$: it validates the first-principle FRG discussed above yielding (12) (the central property being continuity of all $\bar{S}_{1\ldots1}$ since $F(v)$ remains bounded). These considerations should be universal for dilute shocks, i.e. independent of details of shock probability distributions. For RF disorder the full distribution of shock parameters $\{u_{21}^{(s)}, v_s\}$ is known exactly [19]. It is used in [16] to obtain from (6) another expression for $\Delta(v)$ fully consistent with (13).

Similar droplet estimates in higher $d$ yield an exact solution [16] of the *functional* hierarchy (11) for all moments. In the TBL, $m^2 v_x = \mathrm{O}(T)$, the $R[v]$ functional reads:

$$R[v] = \frac{1}{2} v_x R''_{xy}[0] v_y + T^3 \sum_i \left\langle H_2 \left( \int_{xy} v_x g^{-1}_{xy} u^{(i)}_{12,y}/T \right) \right\rangle_D , \qquad (17)$$

where $H_2''(z) = F_2(z)$. To find this solution one considers a small density (of order $T m^\theta$) of well-separated "elementary droplets", *i.e.* local GS degeneracies $u^{(i)}_{12,x} = u^{(i)}_{2,x} - u_{1x}$. $\langle \ldots \rangle_D$ denotes the average over them. Equation (17) relates droplet probabilities to the TBL in the FRG.

To conclude, we related FRG functions, *e.g.* $R(u)$, to observables. This allows to compute them in simple cases, and provides a method to measure them in numerics and experiments. Their relations to shocks in energy landscape was made precise, via a generalized Burgers equation. Shock form factors and droplet distributions were related to the FRG functions. Tantalizing questions such as the extent of universality in the TBL, how do properties of Burgers ($d = 0$) extend to functional shocks (*e.g.* Kolmogorov law) can now be adressed.

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