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Promotion of cooperation induced by nonlinear attractive effect in spatial Prisoner's Dilemma game

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Abstract. – We introduce nonlinear attractive effects into a spatial Prisoner's Dilemma game where the players located on a square lattice can either cooperate with their nearest neighbors or defect. In every generation, each player updates its strategy by firstly choosing one of the neighbors with a probability proportional to $\mathcal{A}^{\alpha}$ denoting the attractiveness of the neighbor, where $\mathcal{A}$ is the payoff collected by it and α (≥ 0) is a free parameter characterizing the extent of the nonlinear effect; and then adopting its strategy with a probability dependent on their payoff difference. Using Monte Carlo simulations, we investigate the density ρ_C of cooperators in the stationary state for different values of α. It is shown that the introduction of such attractive effect remarkably promotes the emergence and persistence of cooperation over a wide range of the temptation to defect. In particular, for large values of α, *i.e.*, strong nonlinear attractive effects, the system exhibits two absorbing states (all cooperators or all defectors) separated by an active state (coexistence of cooperators and defectors) when varying the temptation to defect. In the critical region where ρ_C goes to zero, the extinction behavior is power-law–like $\rho_C \sim (b_c - b)^{\beta}$, where the exponent β accords approximatively with the critical exponent ($\beta \approx 0.584$) of the two-dimensional directed percolation and depends weakly on the value of α.

Introduction. – Cooperation plays an important role in real world, ranging from biological systems to economic and social systems [1]. Scientists from many different fields of natural and social sciences often resort to Evolutionary Game Theory [2,3] as a common mathematical framework and the Prisoner's Dilemma game (PDG) as a metaphor for studying cooperation between unrelated individuals [3]. The original PDG describes the pairwise interactions of individuals with two behavioral options: the two players must simultaneously decide whether to cooperate or to defect. For mutual cooperation both players receive the rewards R, but only the punishment P for mutual defection. A defector exploiting a cooperator gets an amount T (temptation to defect) and the exploited cooperator receives S (sucker's payoff). These elements satisfy the following two conditions: $T > R > P > S$ and $2R > T + S$. It is

easy to see that defection is the better choice irrespective of the opponent's decision. Thus, the undesired outcome of mutual defection emerges in well-mixed populations [4], which has inspired numerous investigations of suitable extensions that enable cooperative behavior to emerge and persist.

Some previous works have suggested several mechanisms (*e.g.*, kin selection [5], the introduction of "tit-for-tat" [6,7] strategy, and voluntary participation [8–10]) to facilitate the emergence and persistence of cooperation in the populations. The spatial versions [11,12] of the evolutionary PDGs can explain the maintenance of cooperation for the iterated games with a limited range of interaction if the players follow one of the two simplest strategies (defection (D) and cooperation (C)). Recently, the effect of heterogeneous influence of different individuals on the maintenance of cooperative behavior has been studied on regular small-world networks [13]. Ren *et al.* have studied the evolutionary PDG and the snowdrift game with preferential learning mechanism on the Barabási-Albert networks [14]. In the evolutionary games the players wish to maximize their total payoffs, coming from PDGs with the neighbors, by adopting either the deterministic rule introduced by Nowak and May [11,15] or the stochastic evolutionary rule by Szabó and Tőke [16].

In the present work, we make further studies of the evolutionary PDG on square lattice mainly according to the stochastic update rule. It is natural to consider that different individuals may have different attractiveness in social systems, so when updating their strategies, the individuals may not completely randomly choose a neighbor to refer to. Here, we introduce the nonlinear attractive effect into the game (see the model below). Interestingly, we find that the introduction of this effect can remarkably promote cooperative behavior in the PDG in comparison with the random choice case on square lattice.

The model. – We consider the evolutionary PDG on square lattice with periodic boundary conditions. Each player interacts only with its four nearest neighbors (self-interaction is excluded) [17], and collects payoffs dependent on the payoff-matrix parameters. The total payoff of a certain player is the sum over all its interactions. We have inspected that if every player interacts with its first and second nearest neighbors or the self-interactions are included, the qualitative results are unchanged. Following common practices [15,16], we start by rescaling the game to make $T = b$, $R = 1$, and $P = S = 0$, where b represents the advantage of defectors over cooperators [15], being typically constrained to the interval $1.0 < b < 2.0$, such that it depends on a single parameter b. We have checked that the qualitative results do not change if we make $S = -\epsilon < 0 (\epsilon \ll 1)$ in order to strictly enforce a PD setting.

During the evolutionary process, each player is allowed to select one of its neighbors as a reference with a probability proportional to the neighbors' attractiveness, and then decides whether to change its strategy or not depending on their payoff difference. We define the selection probability $P_{x \to y}$ of x selecting a neighbor y as

$$P_{x \to y} = \frac{\mathcal{A}_y^\alpha}{\sum_{z \in \Omega_x} \mathcal{A}_z^\alpha}, \tag{1}$$

where the numerator denotes the attractiveness of the neighbor y, and α is a tunable parameter describing the extent of the nonlinear effect, and $\mathcal{A}_y$ is the total payoff of that neighbor. The denominator is the sum of attractiveness that runs over all neighbors of x. The basic ingredient which determines the choice of one neighbor is the *selection kernel* $\mathcal{A}^\alpha$. On general grounds, this *selection kernel* should be a nondecreasing function of $\mathcal{A}$, namely individuals with better performance may have much stronger attractiveness than the average individual. We note that the selection probability depends only on the extent of a nonlinear effect α since the total payoff collected by any player satisfies $\mathcal{A} \geq 0$ in the present model. Thus we will
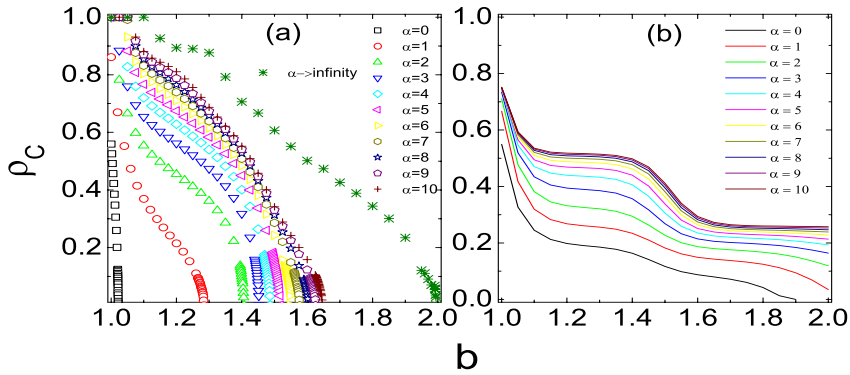
Fig. 1 – (Color online). Monte Carlo results (a) and theoretical analysis (b) for cooperator density ρ_C in the steady state as a function of the temptation to defect b for several different values of α (see the plot). The pair-approximation correctly predicts the trends that ρ_C changes with b and α, but significantly overestimates the extinction thresholds in contrast to the simulation results (see the text for details).

consider the model with $\alpha \in [0, \infty)$. For $\alpha = 0$, the neighbor is randomly selected so that the game is reduced to the original one in refs. [16,17]. The case $\alpha = 1$ leads to the proportional selection rule (exclude the player itself) [18]. While in the limit of $\alpha \rightarrow \infty$, the neighbor whose payoff is the highest among the neighbors is selected, which resembles to the deterministic selection rule [11,15]. For other values of α, the attractiveness of the neighbors is a nonlinear function of their total payoffs. In this way, we consider the general situations of the nonlinear attractive effect on the dynamical behavior of the game.

The player x adopts the selected y neighbor's strategy in the next round with a probability depending on their total payoff difference presented in refs. [16,17,19,20] as

$$W(x \leftarrow y) = \frac{1}{1 + \exp[(\mathcal{A}_x - \mathcal{A}_y)/\kappa]}, \qquad (2)$$

where $\mathcal{A}_x$, $\mathcal{A}_y$ denote the total payoffs of individuals x and y, respectively, and κ characterizes the noise effects, including fluctuations in payoffs, errors in decision, individual trials, etc. The effect of noise has been reported by Szabó *et al.* [20]. In this paper, we make $\kappa = 0.1$. Qualitatively, the results remain unaffected when changing the parameter κ.

Simulations and analysis. – Simulations were carried out for a population of $N = 400 \times 400$ individuals. We study the key quantity of cooperator density ρ_C in the steady state. Initially, the two strategies of C and D are randomly distributed among the individuals with equal probability $1/2$. The above model was simulated with synchronous updating [21]. No qualitative changes occur if we adopt an asynchronous updating [16]. Eventually, the system reaches a dynamic equilibrium state. The simulation results were obtained by averaging over the last 5000 Monte Carlo time steps of the total 50000.

Figure 1 shows the results of both simulations and theoretical analysis of ρ_C when increasing b for several different values of α (see the plot). We can find that, compared with the well-mixed situation, if b is sufficiently small, cooperators can persist in spatial settings in the case of $\alpha = 0$, which indicates that spatial structure can promote cooperation [15]. For $b > b_{c0}$ (≈ 1.0217), where b_{c0} is the extinction threshold of cooperators when $\alpha = 0$, the benefits of
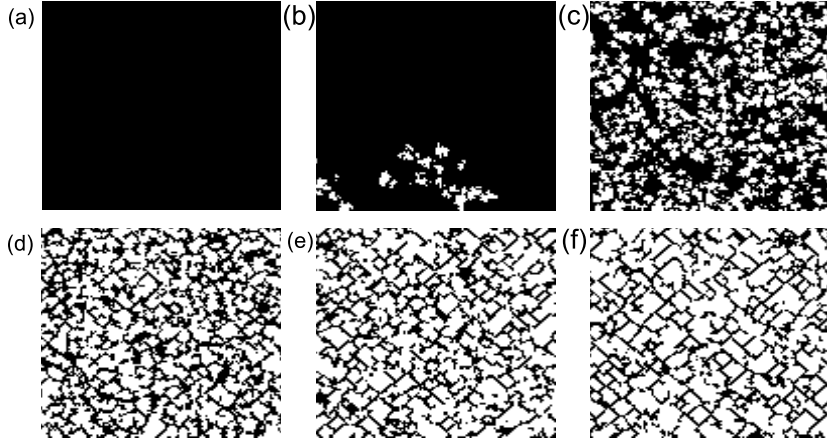
Fig. 2 – A series of snapshots of typical distributions of cooperators (white) and defectors (black) on square lattice for $b = 1.283$ (the value just below the extinction threshold $b_{c1} \approx 1.284$ when $\alpha = 1$) for several different values of α: (a) $\alpha = 0$, (b) $\alpha = 1$, (c) $\alpha = 2$, (d) $\alpha = 5$, (e) $\alpha = 8$, (f) $\alpha = 10$.

spatial clustering are no longer sufficient to offset the losses along the boundary, hence the cooperators vanish [17]. For each positive value of α, the system evolves to the absorbing state of all defectors at certain values of b. The extinction threshold of cooperators $b_{c\alpha}$ clearly increases with α, which indicates that the emergence of cooperation is enhanced.

We know that cooperators survive by forming compact clusters and thus cooperators along the boundary can outweigh their losses against defectors by gains from interactions within the cluster [17]. The payoffs collected by the inner cooperators are, in most cases, larger than the boundary defectors. When considering the attractiveness of the individuals, near the extinction threshold, for cooperator-clusters which are surrounded by defectors, the cooperators along the boundary can keep their cooperative states more easily under stronger preferential selection effect according to the dynamic updating eq. (1) and eq. (2). Thus the strong attractiveness of individuals can favor the spreading of cooperators, hence promote the persistence of cooperation. We can also see that, for very large α, the homogeneous cooperation state ($\rho_C = 1$) emerges when b is very small. Since both the small temptation to defect and the strong nonlinear attractive effect are advantageous to the persistence of cooperation, it is not surprising that ρ_C approaches the maximal fraction 1. Moreover, for the same value of b, ρ_C obviously increases with α. The existence of strong nonlinear effect can facilitate the formation of cooperator clusters, hence enhance the persistence of cooperators (see fig. 2). In addition, the cooperator density ρ_C and the extinction thresholds $b_{c\alpha}$ change more slowly with increasing α. We will consider this point in the following. The pair approximation method, which models the frequency of strategy pairs rather than that of strategies, is usually regarded as an analytical approximation of the spatial dynamics [17]. It is worth noting that the results obtained by pair approximation are directly associated with the local topological structure of the players and the strategy updating dynamics. Whether the strategy updating is implemented synchronously or asynchronously is inessential [21]. Therefore we can apply the pair approximation method to predict qualitatively the evolving behavior of ρ_C. Here we modify the original method by introducing the turnable parameter α, *i.e.*, we substitute the new transition probability $f'(P_B - P_A)$ for the original transition probability $f(P_B - P_A)$ in eq. (A1) in ref. [18],
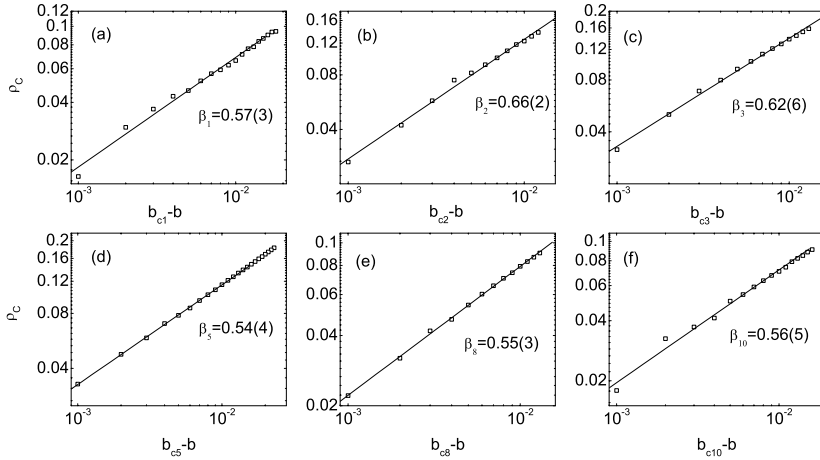
Fig. 3 – Log-log plots of the average cooperator density ρ_C as a function of the distance to the extinction threshold $b_{c\alpha} - b$ for several values of α: (a) $\alpha = 1$, $b_{c1} \approx 1.284$, (b) $\alpha = 2$, $b_{c2} \approx 1.407$, (c) $\alpha = 3$, $b_{c3} \approx 1.453$, (d) $\alpha = 5$, $b_{c5} \approx 1.518$, (e) $\alpha = 8$, $b_{c8} \approx 1.603$, (f) $\alpha = 10$, $b_{c10} \approx 1.646$. The solid lines are the power laws $\sim (b_{c\alpha} - b)^{\beta_\alpha}$ which fit the data correspondence with the exponents of β_α (the detailed values are given in the plots, where the figures between parentheses indicate the statistical uncertainties of the last digit).

where $f'(P_B - P_A) = f(P_B - P_A)\dfrac{P_B^\alpha}{\sum P_B^\alpha}$ and the denominator of the fraction denotes the sum of all possible values of P_B^α when α is certain. Then we adjust eq. (A2a) and eq. (A2b) in ref. [17] by substituting f' for f. The equilibrium values are obtained by numerical integration. We can see from fig. 1 that the pair approximation correctly predicts the trends, that is, the changes of cooperation for b and α. However, it is unable to estimate exactly the extinction thresholds of cooperator density, namely, it overestimates the extinction thresholds (see the plot).

A series of snapshots for several different values of α are shown in fig. 2 for the same value of b. These snapshots are a 120×120 portion of the full 400×400 lattice. We can find that, for random selection case $\alpha = 0$, cooperators are doomed and defectors reign because the value $b = 1.283$ is larger than $b_{c0} \approx 1.0217$ (fig. 2(a)). But for $\alpha = 1$, when b is just below b_{c1}, the cooperators can survive by forming compact clusters which minimize the exploitation by defectors (see fig. 2(b)). For larger α, *i.e.*, stronger nonlinear attractive effect, more cooperator clusters emerge, which illustrates that cooperators can survive against the invading of defectors more easily. On the contrary, the defector clusters decrease with the value of α. Interestingly, the spatial patterns adopted by defectors are completely different from that of cooperators when they are the minority in the populations. As shown in fig. 2 (f), defectors exist in the fashion of zigzag pattern (or step-like). Because if defector clusters are surrounded by cooperators, in the subsequent generations, the defectors along the boundary would transform probably to cooperators according to the selection rule with nonlinear attractive effects eq. (1) and updating rule eq. (2). Eventually, defectors exist in zigzag pattern from which they can benefit maximumly when interacting with their cooperator neighbors (even can do better than a cooperator surrounded by four cooperators, so that when updating, they "always"ask their defector neighbors for strategy transformation), which in return makes the zigzag pattern stably in the evolution of the game.

We also investigate the divergence behaviors of cooperators near the extinction thresholds $b_{c\alpha}$ for several values of α (see fig. 3). We constrained the system size as 800×800 individuals
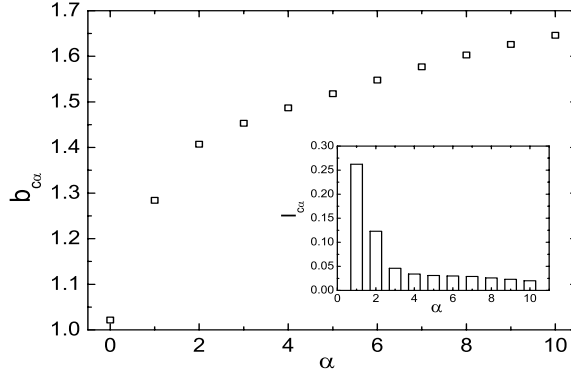
Fig. 4 – The extinction threshold $b_{c\alpha}$ changes with α (from zero to ten). The inset shows that the increment $I_{c\alpha}$ of the extinction threshold $b_{c\alpha}$ changes with α (see the text).

and the results were obtained by averaging over the last 10000 time steps of the total 150000. According to our simulations, near $b_{c\alpha}$ the average fraction of cooperators vanishes as power-law like behavior $\rho_c \sim (b_{c\alpha}-b)^{\beta_\alpha}$, where β_α are a set of exponents corresponding with the value of α (see the plots for detailed values). In physics, such thresholds are usually associated with phase transitions, and indeed, the transitions from persistent levels of cooperation ($b < b_{c\alpha}$) to absorbing states of defection ($b > b_{c\alpha}$) bear the hallmarks of critical phase transitions. These values of β_α are nearly consistent with the critical exponent ($\beta \approx 0.584$) of the two-dimensional directed percolation [22] and depend weakly on α. The estimated errors of β_1, β_2, β_3, β_5, β_8, β_{10} are $0.019, 0.133, 0.072, 0.068, 0.053, 0.033$, respectively. We consider that the large errors are due to the limits of computational conditions.

The variation of extinction threshold $b_{c\alpha}$ with the value of α (from zero to ten) is shown in fig. 4. We can see that the extinction threshold clearly increases with the value of α. Now we define a quantity characterizing the increment of the extinction threshold as $I_{c\alpha} = b_{c\alpha}-b_{c(\alpha-1)}$. We note that this quantity decreases with the value of α (inset in fig. 4). This indicates that, as $\alpha \to \infty$, the increment $I_{c\alpha}$ will approach to the minimal value 0. In other words, the extinction threshold $b_{c\alpha}$ will tend towards the maximal value $b_{c\infty}$, where $b_{c\infty} \approx 1.995$ is the extinction threshold when $\alpha \to \infty$, *i.e.*, when the neighbor whose payoff is the highest of the neighboring is selected to refer to.

Conclusions. – In summary, we have investigated the promotion of cooperation in the context of evolutionary PDG resulting from the nonlinear attractive effect of the neighbors on square lattice. A nonlinear function $\mathcal{A}^\alpha$, in terms of the performance of the players, is used as an estimator of their attractiveness. We have considered the general situations and shown that, compared with the random selection case, the introduction of the nonlinear attractive effect can remarkably promote the cooperative behavior over a wide range of b. Particularly, the stronger the extent of the nonlinear effect is, the more prominent the cooperative behavior will be, and for some large α values, a homogeneous state of all cooperators can emerge. Interestingly, the spatial patterns adopted by cooperators and defectors are completely different when they are the minority in the populations: Cooperators can survive by forming compact clusters, and along the boundary, cooperators can outweigh their losses against defectors by gains from interactions within the cluster; Whereas defectors exist in the way of zigzag pattern (or step-like), from which defectors can benefit maximumly when interacting with their cooperator neighbors. The extinction of cooperators under harsh conditions when $b \to b_{c\alpha}$ displays

a power law-like behavior $\rho_C \sim (b_{c\alpha} - b)^{\beta}$. The introduction of the nonlinear attractive effect can partially resolve the dilemma of cooperation and may shed new lights on the evolution of cooperation in the society.

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