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To cite this article: Ö. F. Dayi and L. T. Kelleyane 2007 *EPL* **78** 21004

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

April 2007

EPL, **78** (2007) 21004
www.epljournal.org
doi: 10.1209/0295-5075/78/21004

N = 1/2 supersymmetric gauge theory in noncommutative space

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received 17 January 2007; accepted in final form 7 March 2007
published online 3 April 2007

PACS 11.10.Nx – Noncommutative field theory
PACS 12.60.Cn – Extensions of electroweak gauge sector

Abstract – A formulation of (non-anticommutative) $N=1/2$ supersymmetric $U(N)$ gauge theory in noncommutative space is studied. We show that at one loop UV/IR mixing occurs. A generalization of Seiberg-Witten map to noncommutative and non-anticommutative superspace is employed to obtain an action in terms of commuting fields at first order in the noncommutativity parameter θ. This leads to Abelian and non-Abelian gauge theories whose supersymmetry transformations are local and non-local, respectively.

Introduction. – Deformation of superspace where fermionic coordinates are non-anticommuting appeared in some different contexts [1–6]. At the start one can simultaneously deform bosonic coordinates allowing them to be noncommuting, in terms of a star product embracing both of the deformations [7]. However, as far as gauge theories are concerned, usually non-anticommutativity is considered alone. Instead of introducing noncommutativity of bosonic coordinates and non-anticommutativity of fermionic ones simultaneously from the beginning, we may do it in two steps: $N=1/2$ supersymmetric gauge theory action in components includes ordinary fields and non-anticommutativity parameter [2]. Thus its noncommutative generalization can be obtained as usual. However, the same action would result using the superfield formulation given in [7]. Hence, two approaches are equivalent. We study this non-anticommutative as well as noncommutative theory.

One of the most important features arising in field theories in noncommutative space is the UV/IR mixing [8]. In supersymmetric gauge theory in noncommuting space, linear and quadratic poles in the noncommutativity parameter θ are absent at one loop, due to the fact that contributions from fermionic and bosonic degrees of freedom cancel each other. First loop Feynman graph calculations for noncommutative supersymmetric gauge theory with Abelian gauge group was studied in [9–11] and $U(N)$ case was considered in [12–14].

Renormalization of $N=1/2$ supersymmetric Yang-Mills theory was discussed in [7,15–23]. For the gauge group $U(N)$, renormalizability at one loop requires to alter the original action. In [16] it was commented that in supersymmetric gauge theory where both noncommutativity and non-anticommutativity are present, there would be UV/IR mixing. Although we do not study renormalizability properties of the non-anticommutative and noncommutative theory, we will show that UV/IR mixing is present by an explicit calculation for $U(1)$ case.

Seiberg and Witten [24] introduced an equivalence relation between the gauge fields $\hat{A}$ taking values in noncommutative gauge group and the ordinary gauge fields A as

$$\hat{A}(A)+\hat{\delta}_{\hat{\phi}}\hat{A}(A)=\hat{A}(A+\delta_{\phi}A). \qquad (1)$$

Here $\hat{\phi}$ and ϕ denote gauge parameters of the noncommutative and ordinary cases. Seiberg-Witten (SW) map allows one to deal with noncommutative gauge theory in terms of an action expanded in the noncommutativity parameter θ with ordinary gauge fields. We would like to study its generalization to superspace. Gauge transformations of $N=1/2$ supersymmetric theory in component fields does not depend on the non-anticommutativity parameter C, owing to the parametrization of vector superfield given in [2]. As we will explicitly show, this is a generalization of SW map to non-anticommutative superspace. Generalizations of SW map to C deformed superspace are studied in [25] and [26]. It is also studied in harmonic superspace [27]. We will discuss in detail how generalizations of SW map to superspace can be obtained in terms of component fields. When only non-anticommutativity is present one can chose to work either with ordinary supersymmetry transformations with a deformed gauge transformation or without deforming gauge transformations but changing supersymmetry transformations as in [2]. On the other hand, for only θ deformed superspace SW map includes deformation of supersymmetry as well as gauge transformations [28–31]. As is clear from its definition (1), SW map refers only to gauge transformations. Hence, it is not *a priori* guaranteed that a noncommutative gauge theory will be gauge invariant after performing SW map. However, as we will show in the generalization of SW map there is a freedom of choosing $C\theta$-dependent terms once the C and θ deformed parts are fixed separately. Thus, $C\theta$-dependent terms can be chosen such that the resultant theory becomes gauge invariant but supersymmetry transformations should be deformed.

In the second section we present $N=1/2$ supersymmetric gauge theory action in noncommutative space exhibiting its gauge and supersymmetry invariance. Moreover, we show that UV/IR mixing occurs. In the third section first we discuss how to generalize SW map to noncommuting and/or non-anticommuting superspace gauge transformations. Then, we apply SW map to $U(1)$ and non-Abelian noncommutative and noncommutative gauge theory to obtain θ-expanded actions. Supersymmetry transformations of the latter become non-local to preserve gauge invariance of the resultant action.

Noncommutative $N=1/2$ supersymmetric gauge theory. – In terms of constant, respectively, symmetric and antisymmetric parameters $C_{\alpha\beta}$ and $\theta^{\rho\sigma}$, let the Grassmann coordinates θ_α, $\alpha=1,2$, $\bar\theta_{\dot\alpha}$, $\dot\alpha=1,2$, and bosonic coordinates $y^\mu=\tilde x^\mu+i\theta\sigma^\mu\bar\theta$, $\mu=0,\ldots,3$, satisfy the deformed brackets [2]

$$\{\hat\theta_\alpha,\hat\theta_\beta\}=C_{\alpha\beta},\qquad [\hat y^\rho,\hat y^\sigma]=i\theta^{\rho\sigma}, \tag{2}$$

$$\{\hat\theta_\alpha,\bar\theta_{\dot\alpha}\}=0,\qquad \{\bar\theta_{\dot\alpha},\bar\theta_{\dot\beta}\}=0, \tag{3}$$

$$[\hat y^\rho,\bar\theta_{\dot\alpha}]=0,\qquad [\hat y^\rho,\hat\theta_\alpha]=0. \tag{4}$$

This is possible only in Euclidean space. Although we deal with Euclidean $\mathbb{R}^4$, we use Minkowski space notation and follow the conventions of [32]. We will also use the antisymmetric parameter $C^{\mu\nu}=C^{\alpha\beta}\epsilon_{\beta\gamma}\sigma_\alpha^{\mu\nu\gamma}$ which satisfies the self-duality property

$$C^{\mu\nu}=\frac{i}{2}\epsilon^{\mu\nu\rho\lambda}C_{\rho\lambda}. \tag{5}$$

An associative star product embracing both of the deformations was introduced in [7]

$$f(y,\theta)\tilde\star g(y,\theta)=f(y,\theta)\exp\Big(\tfrac{i}{2}\theta^{\mu\nu}\frac{\overleftarrow{\partial}}{\partial y^\mu}\frac{\overrightarrow{\partial}}{\partial y^\nu} -\tfrac12 C^{\alpha\beta}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\Big)g(y,\theta)\equiv f(y,\theta)\overset{C}{\star}\overset{\theta}{\star}g(y,\theta), \tag{6}$$

where the derivatives $\partial/\partial\theta^\alpha$ are defined to be at fixed y_μ and $\bar\theta$. In fact, one can separate C- and $\theta^{\rho\sigma}$-dependent parts which we denote $\overset{C}{\star}$ and $\overset{\theta}{\star}$.

Instead of dealing with the $\tilde\star$ product we will proceed in a different way. Seiberg considered the case $\theta^{\rho\sigma}=0$ in terms of a vector superfield written in a Wess-Zumino gauge which was employed to write the action in commuting coordinates x^μ as (note that we do not deal with $\hat{\tilde x}$ coordinates appearing in $\hat y$ which would be noncommuting)

$$S=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big\{-\frac14F^{\mu\nu}F_{\mu\nu}-i\lambda \not{D}\bar\lambda+\frac12D^2 -\frac{i}{2}C^{\mu\nu}F_{\mu\nu}\bar\lambda\bar\lambda+\frac{|C|^2}{8}(\bar\lambda\bar\lambda)^2\Big\}. \tag{7}$$

$F_{\mu\nu}$ is the non-Abelian field strength related to the gauge field A_μ. λ, $\bar\lambda$ are independent fermionic fields and D is auxiliary bosonic field. Covariant derivative is defined as $D_\mu=\partial_\mu+i[A_\mu,\cdot]$. The action (7) is invariant under the usual gauge transformations and it possesses $N=1/2$ supersymmetry.

Obviously, (7) is a theory in commuting coordinates though the constant parameter C appears. Hence, considering it in noncommuting space letting the coordinates satisfy

$$[\hat x^\mu,\hat x^\nu]=i\theta^{\mu\nu} \tag{8}$$

is legitimate. We introduce the star product

$$f(x)\star g(x)=f(x)e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}g(x) \tag{9}$$

and work with the commuting coordinates x_μ satisfying the Moyal bracket

$$[x^\mu,x^\nu]_\star\equiv x^\mu\star x^\nu-x^\nu\star x^\mu=i\theta^{\mu\nu}. \tag{10}$$

By replacing ordinary products with the star product (9) in (7), one obtains the action

$$I=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\Big\{-\frac14\hat F^{\mu\nu}\hat F_{\mu\nu}-i\hat\lambda\not{\hat D}\star\hat{\bar\lambda}+\frac12\hat D^2 -\frac{i}{2}C^{\mu\nu}\hat F_{\mu\nu}\hat{\bar\lambda}\star\hat{\bar\lambda}+\frac{|C|^2}{8}(\hat{\bar\lambda}\star\hat{\bar\lambda})^2\Big\}. \tag{11}$$

Here we adopted the definitions

$$\begin{aligned}\hat F_{\mu\nu}&=\partial_\mu\hat A_\nu-\partial_\nu\hat A_\mu+i[\hat A_\mu,\hat A_\nu]_\star,\\ \not{\hat D}\star\hat{\bar\lambda}&=\not{\partial}\hat{\bar\lambda}+i[\not{\hat A},\hat{\bar\lambda}]_\star.\end{aligned}$$

This noncommutative gauge theory would also have resulted from the superfield formulation of $N=1/2$ supersymmetric theory discussed in [7] by making use of the parametrization given in [2] for vector superfields.

We assume that surface terms are vanishing, so that the following property is satisfied:

$$\int d^4x f(x)\star g(x)=\int d^4x f(x)g(x).$$

Gauge transformations of the fields are

$$\begin{aligned}\delta\hat A_\mu&=\partial_\mu\hat\phi-i\big[\hat\phi,\hat A_\mu\big]_\star,\\ \delta\hat\lambda_\alpha&=-i\big[\hat\phi,\hat\lambda_\alpha\big]_\star,\\ \delta\hat{\bar\lambda}_{\dot\alpha}&=-i\big[\hat\phi,\hat{\bar\lambda}_{\dot\alpha}\big]_\star,\\ \delta\hat D&=-i\big[\hat\phi,\hat D\big]_\star,\end{aligned} \tag{12}$$

where $\hat{\phi}$ denotes gauge parameter. Making use of (12) one can observe the following transformations:

$$\begin{aligned} \delta\hat{F}_{\mu\nu} &= -i\big[\hat{\phi},\hat{F}_{\mu\nu}\big]_\star, \\ \delta(\not{D}\star\hat{\bar{\lambda}}) &= -i[\hat{\phi},\not{D}\star\hat{\bar{\lambda}}]_\star, \\ \delta(\hat{\bar{\lambda}}\star\hat{\bar{\lambda}}) &= -i[\hat{\phi},\hat{\bar{\lambda}}\star\hat{\bar{\lambda}}]_\star. \end{aligned}$$

Therefore, we can conclude that the action (11) is gauge invariant under noncommutative $U(N)$ gauge transformations.

On the other hand, supersymmetry transformations of the component fields can be defined as

$$\delta_S\hat{\lambda} = i\xi\hat{D} + \sigma^{\mu\nu}\xi(\hat{F}_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\hat{\bar{\lambda}}\star\hat{\bar{\lambda}}), \tag{13}$$

$$\delta_S\hat{A}_\mu = -i\hat{\bar{\lambda}}\bar{\sigma}_\mu\xi, \tag{14}$$

$$\delta_S\hat{D} = -\xi\sigma^\mu D_\mu\star\hat{\bar{\lambda}}, \tag{15}$$

$$\delta_S\hat{\bar{\lambda}} = 0, \tag{16}$$

where ξ^α is a constant Grassmann parameter. To discuss supersymmetry properties of the action (11) one needs to make use of the relation

$$\sigma^{\rho\lambda}\sigma^\mu = \frac{1}{2}(-\eta^{\mu\lambda}\sigma^\rho + \eta^{\mu\rho}\sigma^\lambda + i\epsilon^{\mu\rho\lambda\kappa}\sigma_\kappa).$$

The $C=0$ part can be shown to be supersymmetric using the Bianchi identity $\epsilon^{\mu\nu\lambda\rho}D_\mu\star\hat{F}_{\nu\lambda}=0$, which is due to the associativity of star product. On the other hand, the $C_{\mu\nu}$-dependent terms yield

$$\begin{aligned} \int \mathrm{d}^4x\xi\Big\{&2(\sigma_\nu C^{\mu\nu}D_\mu\star\hat{\bar{\lambda}})(\hat{\bar{\lambda}}\star\hat{\bar{\lambda}}) + \epsilon^{\mu\nu\rho\lambda}\sigma_\nu C_{\rho\lambda}(\hat{\bar{\lambda}}\star\hat{\bar{\lambda}}) \\ &\times(D_\mu\star\hat{\bar{\lambda}}) - 4(\sigma_\nu C^{\mu\nu}D_\mu\star\hat{\bar{\lambda}})(\hat{\bar{\lambda}}\star\hat{\bar{\lambda}})\Big\} = 0, \end{aligned}$$

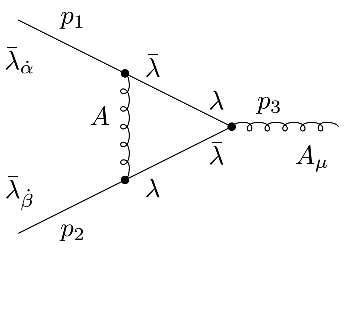
where the self-duality condition (5) is utilized. Hence, (11) is a noncommutative $N=1/2$ supersymmetric $U(N)$ gauge theory action.

To perform perturbative calculations one should introduce ghost fields to fix the gauge. Moreover, matter fields may also be added. Let us consider noncommutative $U(1)$ gauge group. In this case, Feynman rules can be read from the $N=1/2$ supersymmetric $U(N)$ gauge theory [15] by the replacement of the structure constants:

$$f_{a_1a_2a_3} \longrightarrow 2\sin\left(\tilde{k}_2k_3\right), \tag{17}$$

$$d_{a_1a_2a_3} \longrightarrow 2\cos\left(\tilde{k}_2k_3\right), \tag{18}$$

where we denoted $\tilde{k}^\mu \equiv \theta^{\mu\nu}k_\nu$. Here, k_2 and k_3 are the momenta of the lines corresponding to the indices a_2 and a_3, respectively. Instead of giving a full discussion of one-loop calculations, we would like to consider only the following non-planar one-loop diagram:



which is typical of the $N=1/2$ supersymmetric gauge theory. The amplitude is proportional to

$$\begin{aligned} \propto g^3C^{\kappa\nu}\sigma_{\kappa\beta\dot{\beta}}\sigma^\mu_{\gamma\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\nu(\not{k}-\not{p}_1)^{\dot{\gamma}\gamma}(\not{k}+\not{p}_2)^{\dot{\delta}\beta}}{k^2(k-p_1)^2(k+p_2)^2} \\ \times\cos(\tilde{k}p_1)\sin(\tilde{k}p_2)\sin(\tilde{k}p_3). \end{aligned}$$

Using the calculation methods of [9], one can observe that this amplitude produces low momenta poles as

$$g^3C^{\kappa\nu}\sigma_{\kappa\beta\dot{\beta}}\sigma^\mu_{\gamma\dot{\delta}}\epsilon_{\dot{\alpha}\dot{\gamma}}\frac{\tilde{l}_\nu(\tilde{\not{l}})^{\dot{\gamma}\gamma}(\tilde{\not{l}})^{\dot{\delta}\beta}}{\tilde{l}^4}, \tag{19}$$

where l_μ are some definite functions of p:

$$l = l(p_1,p_2,p_3).$$

To get the correct factors we should take into account contributions coming from all of the diagrams including ghosts and also matter if they are coupled. Nevertheless, calculation of the above diagram shows that UV/IR mixing occurs.

Generalized SW **map and $\theta^{\rho\sigma}$-expanded action.** – To attain $\theta^{\rho\sigma}$-expanded action in terms of ordinary component fields we first should discuss in detail how SW map (1) can be generalized to noncommutative and/or non-anticommutative superspace. SW map (1) clearly alludes only to gauge transformations, it does not refer to any gauge theory action. Hence, although one applies the map to a gauge invariant noncommuting and/or non-anticommuting theory it is not guaranteed that the resultant action will possess the ordinary gauge invariance. However, we will show that in the superspace with noncommuting and non-anticommuting coordinates it can be chosen appropriately such that the resultant action is gauge invariant.

In the ordinary (non-deformed) superspace infinitesimal gauge transformations of component fields derived from

$$\delta_\Lambda e^V = -i\bar{\Lambda}e^V + ie^V\Lambda. \tag{20}$$

Let us deal with $U(1)$ gauge group to illustrate how generalizations of SW map can be obtained. Using the parametrization of [2] we define the non-deformed vector superfield as

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D - i\partial_\mu A^\mu\right), \tag{21}$$

which satisfies $V^2 = -(1/2)\bar{\theta}\bar{\theta}\theta\theta A_\mu A^\mu$ and $V^3=0$. The appropriate gauge parameters are

$$\Lambda = \phi + \frac{i}{2}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi, \tag{22}$$

$$\bar{\Lambda} = \phi - \frac{i}{2}\theta\sigma^\mu\bar{\theta}\partial_\mu\phi + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\partial^2\phi. \tag{23}$$

To obtain infinitesimal gauge transformations we need to deal not only with V but with $\Sigma = V + \frac{1}{2}V^2$. Indeed,

$$\delta_\Lambda\Sigma = -i\left(\bar{\Lambda} - \Lambda + \bar{\Lambda}\Sigma - \Sigma\Lambda\right) \tag{24}$$

yields the ordinary infinitesimal gauge transformations of the component fields. Noncommutative gauge transformations are defined as

$$\delta_{\bar{\Lambda}}\hat{\Sigma}_{\Lambda}=-i\left(\hat{\bar{\Lambda}}-\hat{\Lambda}+\hat{\bar{\Lambda}}\tilde{\star}\hat{\Sigma}-\hat{\Sigma}\tilde{\star}\hat{\Lambda}\right), \qquad (25)$$

by replacing multiplication of bilinear components with the star product $\tilde{\star}$, in (24), so that we also need

$$\hat{V}\tilde{\star}\hat{V}=-\frac{1}{2}\bar{\theta}\bar{\theta}\left(\theta\theta\hat{A}_{\mu}\overset{\theta}{\star}\hat{A}^{\mu}+\frac{1}{4}|C|^{2}\hat{\bar{\lambda}}\overset{\theta}{\star}\hat{\bar{\lambda}}\right),\ \hat{V}\tilde{\star}\hat{V}\tilde{\star}\hat{V}=0.$$

Note that the star product $\overset{\theta}{\star}$ is in terms of y coordinates.

Generalization of SW map to noncommutative and non-anticommutative superspace gauge transformations can be defined by the equivalence relation

$$\hat{\Sigma}(\Sigma)+\hat{\delta}_{\hat{\Lambda}}\hat{\Sigma}(\Sigma)=\hat{\Sigma}(\Sigma+\delta_{\Lambda}\Sigma). \qquad (26)$$

This is obtained by replacing the gauge field A with the vector superfield Σ and the gauge parameter ϕ with the supergauge parameter Λ in (1).

We deal with $U(1)$ gauge group to solve the equivalence relation (26). Keeping terms first order in $\theta_{\mu\nu}, C_{\alpha\beta}$ and $C\theta$ which are denoted as

$$\hat{\Sigma}=\Sigma+\Sigma_{(C)}+\Sigma_{(\theta)}+\Sigma_{(C\theta)}\equiv\Sigma+\Sigma_{(1)}, \qquad (27)$$

$$\hat{\Lambda}=\Lambda+\Lambda_{(C)}+\Lambda_{(\theta)}+\Lambda_{(C\theta)}\equiv\Lambda+\Lambda_{(1)}, \qquad (28)$$

$$\hat{\bar{\Lambda}}=\bar{\Lambda}+\bar{\Lambda}_{(C)}+\bar{\Lambda}_{(\theta)}+\bar{\Lambda}_{(C\theta)}\equiv\bar{\Lambda}+\bar{\Lambda}_{(1)}, \qquad (29)$$

eq. (26) leads to

$$\begin{aligned}&\Sigma_{(1)}\left(\Sigma+\partial_{\Lambda}\Sigma\right)-\Sigma_{(1)}\left(\Sigma\right)+i\bar{\Lambda}_{(1)}-i\Lambda_{(1)}=\\&\quad i\left(\Sigma+\Sigma_{(1)}\right)\left(\tilde{\star}-1\right)\left(\Lambda+\Lambda_{(1)}\right)\\&\quad -i\left(\bar{\Lambda}+\bar{\Lambda}_{(1)}\right)\left(\tilde{\star}-1\right)\left(\Sigma+\Sigma_{(1)}\right).\end{aligned} \qquad (30)$$

To acquire a better understanding, let us discuss it first for only non-anticommutative superspace by setting $\theta^{\rho\sigma}=0$. When only $\overset{C}{\star}$ survives (30) leads to

$$\begin{aligned}&\Sigma_{(C)}\left(\Sigma+\partial_{\Lambda}\Sigma\right)-\Sigma_{(C)}\left(\Sigma\right)+i\bar{\Lambda}_{(C)}-i\Lambda_{(C)}=\\&\quad -C^{\alpha\beta}\left(\partial_{\alpha}\Sigma\partial_{\beta}\Lambda-\partial_{\alpha}\bar{\Lambda}\partial_{\beta}\Sigma\right),\end{aligned} \qquad (31)$$

where we denoted $\partial/\partial\theta^{\alpha}\equiv\partial_{\alpha}$. There are two different ways of solving this equation: the first one is to choose

$$\Sigma_{(C)}=0,\ \Lambda_{(C)}=0,\ \bar{\Lambda}_{(C)}=\frac{1}{2}\bar{\theta}\bar{\theta}\theta_{\alpha}C^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}\phi\bar{\lambda}^{\dot{\alpha}}, \qquad (32)$$

so that the gauge transformations are changed, though supersymmetry transformations are given by the ordinary ones. The second solution is not to retain ordinary gauge transformation but to change supersymmetry transformations by deforming the vector superfield as

$$\Sigma_{(C)}=-\frac{i}{2}\bar{\theta}\bar{\theta}\theta_{\alpha}C^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\alpha}}A_{\mu}\bar{\lambda}^{\dot{\alpha}},\ \Lambda_{(C)}=0,\ \bar{\Lambda}_{(C)}=0. \qquad (33)$$

Indeed, this is Seiberg's solution which resulted in $N=1/2$ supersymmetric gauge theory. In the following we will deal only with the latter solution.

Now, let only $\overset{\theta}{\star}$ survive by setting $C=0$ in (30). Hence, in terms of the component fields $V_i\equiv(A,\lambda,\bar{\lambda},D)$, and the ordinary gauge transformations δ_{ϕ}, (30) yields

$$A_{(\theta)\mu}(V_i+\delta_{\phi}V_i)-A_{(\theta)\mu}(V_i)=-\partial_{\mu}\phi_{(\theta)}+\theta^{\rho\sigma}\partial_{\rho}\phi\partial_{\sigma}A_{\mu}, \qquad (34)$$

$$\lambda^{\alpha}_{(\theta)}(V_i+\delta_{\phi}V_i)-\lambda^{\alpha}_{(\theta)}(V_i)=\theta^{\rho\sigma}\partial_{\rho}\phi\partial_{\sigma}\lambda^{\alpha}, \qquad (35)$$

$$\bar{\lambda}^{\dot{\alpha}}_{(\theta)}(V_i+\delta_{\phi}V_i)-\bar{\lambda}^{\dot{\alpha}}_{(\theta)}(V_i)=\theta^{\rho\sigma}\partial_{\rho}\phi\partial_{\sigma}\bar{\lambda}^{\dot{\alpha}}, \qquad (36)$$

$$D_{(\theta)}(V_i+\delta_{\phi}V_i)-D_{(\theta)}(V_i)=\theta^{\rho\sigma}\partial_{\rho}\phi\partial_{\sigma}D. \qquad (37)$$

They had already been obtained in [30] and can be solved as

$$A_{(\theta)\mu}=\theta^{\rho\sigma}A_{\rho}(\partial_{\sigma}A_{\mu}-\partial_{\mu}A_{\sigma}/2), \qquad (38)$$

$$D_{(\theta)}=\theta^{\rho\sigma}A_{\rho}\partial_{\sigma}D, \qquad (39)$$

$$\lambda_{(\theta)\alpha}=\theta^{\rho\sigma}A_{\rho}\partial_{\sigma}\lambda_{\alpha}, \qquad (40)$$

$$\bar{\lambda}^{\dot{\alpha}}_{(\theta)}=\theta^{\rho\sigma}A_{\rho}\partial_{\sigma}\bar{\lambda}^{\dot{\alpha}}. \qquad (41)$$

We are concerned with $N=1/2$ supersymmetric gauge theory in noncommuting space. Hence, when we deal with the full-fledged star product $\tilde{\star}$ we would like to keep Seiberg's solution (33) for $\Sigma_{(C)}$. Therefore, we plug (33) into (30) which results in

$$\begin{aligned}&\Sigma_{(\theta)}\left(\Sigma+\delta_{\Lambda}\Sigma\right)+\Sigma_{(C\theta)}\left(\Sigma+\delta_{\Lambda}\Sigma\right)-\Sigma_{(\theta)}\left(\Sigma\right)\\&-\Sigma_{(C\theta)}\left(\Sigma\right)+i\bar{\Lambda}_{(\theta)}+i\bar{\Lambda}_{(C\theta)}-i\Lambda_{(\theta)}-i\Lambda_{(C\theta)}\\&-\frac{1}{2}\bar{\theta}\bar{\theta}\Big[\theta\theta(\partial_{\mu}\phi_{(\theta)}A^{\mu}+\partial_{\mu}\phi A^{\mu}_{(\theta)}+\partial_{\mu}\phi_{(C\theta)}A^{\mu}\\&+\partial_{\mu}\phi A^{\mu}_{(C\theta)})+i\theta_{\alpha}C^{\alpha\beta}\sigma^{\mu}_{\beta\dot{\alpha}}(\partial_{\mu}\phi_{(\theta)}\bar{\lambda}^{\dot{\alpha}}+\partial_{\mu}\phi\bar{\lambda}^{\dot{\alpha}}_{(\theta)})\Big]\\&=\frac{i}{4}\theta^{\rho\sigma}C^{\alpha\beta}\left(\partial_{\alpha}\partial_{\rho}\bar{\Lambda}\partial_{\beta}\partial_{\sigma}\Sigma-\partial_{\alpha}\partial_{\rho}\Sigma\partial_{\beta}\partial_{\sigma}\Lambda\right).\end{aligned} \qquad (42)$$

For only $\theta^{\rho\sigma}$ part we would like to retain the equations (34)-(37) by adopting the solutions (38)-(41). After some calculations, one can show that (42) simplifies and we are left with only $C\theta$-dependent terms:

$$\begin{aligned}&\Sigma_{(C\theta)}\left(\Sigma+\delta_{\Lambda}\Sigma\right)-\Sigma_{(C\theta)}\left(\Sigma\right)+i\bar{\Lambda}_{(C\theta)}-i\Lambda_{(C\theta)}\\&\quad-\frac{1}{2}\bar{\theta}\bar{\theta}\theta\theta(\partial_{\mu}\phi_{(C\theta)}A^{\mu}+\partial_{\mu}\phi A^{\mu}_{(C\theta)})=0.\end{aligned} \qquad (43)$$

In components it yields

$$A^{\mu}_{(C\theta)}(V_i+\delta_{\phi}V_i)-A^{\mu}_{(C\theta)}(V_i)+\partial^{\mu}\phi_{(C\theta)}=0, \qquad (44)$$

$$\lambda^{\alpha}_{(C\theta)}(V_i+\delta_{\phi}V_i)-\lambda^{\alpha}_{(C\theta)}(V_i)=0, \qquad (45)$$

$$\bar{\lambda}^{\dot{\alpha}}_{(C)}(V_i+\delta_{\phi}V_i)-\bar{\lambda}^{\dot{\alpha}}_{(C\theta)}(V_i)=0, \qquad (46)$$

$$D_{(C\theta)}(V_i+\delta_{\phi}V_i)-D_{(C\theta)}(V_i)=0. \qquad (47)$$

We will not work out their solutions[1] but note that there is the trivial solution

$$A^{\mu}_{(C\theta)} = \lambda^{\alpha}_{(C\theta)} = \bar{\lambda}^{\dot{\alpha}}_{(C\theta)} = D_{(C\theta)} = \phi_{(C\theta)} = 0. \quad (48)$$

Instead of dealing with the trivial solution (48) one can add $C\theta$-dependent terms. However, this will result in changing the supersymmetry transformations of the fields as will be discussed.

When component fields are deformed by adding C- and/or $\theta^{\rho\sigma}$-dependent terms, supersymmetry transformations of commuting fields should also be deformed, [2,28–31]. Let us discuss how deformed supersymmetry transformations can be obtained. Denote the original supersymmetry transformations of components as

$$\delta_S V_i = f_i(V_j,\xi). \quad (49)$$

By replacing the original components with the deformed ones one gets

$$\delta_S \hat{V}_i = f_i(\hat{V}_j,\xi). \quad (50)$$

Now, perform the map

$$\hat{V}_i(V) = V_i + V_{i(C)} + V_{i(\theta)} + V_{i(C\theta)}, \quad (51)$$

and plug it in the left as well as in the right-hand side of (50). Then one can read deformed supersymmetry transformation of ordinary component fields as

$$\delta_S V_i = f_i(\hat{V}_j(V),\xi) - \delta_S V_{i(C)} - \delta_S V_{i(\theta)} - \delta_S V_{i(C\theta)}. \quad (52)$$

As an example, one can show that when one deals with only C deformed case and adopts Seiberg's solution (33), the $N=1/2$ supersymmetry transformations (13)-(16) with $\theta^{\rho\sigma}=0$ follow.

Let us apply SW map to $N=1/2$ supersymmetric gauge theory in noncommutative space. For the $U(1)$ case we adopt Seiberg's solution for the C-dependent part (33), hence the action (11) with $U(1)$ results. Then, using the solutions (34)-(37) and (48) for $\theta^{\rho\sigma}$- and $C\theta$-dependent terms, (11) yields the θ-expanded $N=1/2$ supersymmetric gauge theory action up to the first order in θ for $U(1)$

$$\begin{aligned} I^{(1)} = \int \mathrm{d}^4x \Bigg[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\partial\!\!\!/\bar{\lambda} - \frac{i}{2}C^{\mu\nu}F_{\mu\nu}\bar{\lambda}^2 \\ &+\frac{1}{2}D^2 - \theta^{\rho\sigma}\Bigg(-\frac{1}{2}F^{\mu\nu}F_{\nu\sigma}F_{\mu\rho} + \frac{1}{8}F_{\rho\sigma}F_{\mu\nu}F^{\mu\nu} \\ &-\frac{1}{4}D^2F_{\rho\sigma} + \frac{i}{2}F_{\rho\sigma}\lambda\partial\!\!\!/\bar{\lambda} + i\lambda\sigma^{\mu}\partial_{\sigma}\bar{\lambda}F_{\mu\rho} \\ &+\frac{i}{2}C^{\mu\nu}F_{\mu\rho}F_{\nu\sigma}\bar{\lambda}^2 - \frac{i}{4}C^{\mu\nu}F_{\rho\sigma}F_{\mu\nu}\bar{\lambda}^2\Bigg)\Bigg]. \end{aligned} \quad (53)$$

[1] In [25] after making use of Seiberg's solution for C deformed vector superfield (33), additional C-dependent fields and parameters were introduced and equations for components were derived. They are the same with (44)-(47) by replacing $C\theta$ components of fields and parameters with these additional C-dependent ones. Hence, the discussions of [25] regarding solutions can be applied to our case up to an overall $\theta^{\rho\sigma}$-dependence.

Although (53) possesses the usual $U(1)$ gauge invariance its supersymmetry transformations should be altered. Deformed supersymmetry transformations can be read by making use of the general formula (52) with the transformations (13)-(16) as

$$\begin{aligned} \delta_S A_\mu = i\xi\sigma_\mu\bar{\lambda} - \frac{i}{2}\theta^{\rho\sigma}\xi\sigma_\rho\bar{\lambda}(\partial_\sigma A_\mu + F_{\sigma\mu}) \\ +\frac{i}{2}\theta^{\rho\sigma}\xi\sigma_\sigma A_\rho\partial_\mu\bar{\lambda}, \end{aligned} \quad (54)$$

$$\begin{aligned} \delta_S\lambda = i\xi D - \xi\sigma^{\mu\nu}F_{\mu\nu} + \theta^{\rho\sigma}\xi\sigma^{\mu\nu}F_{\mu\rho}F_{\nu\sigma} \\ +\frac{i}{2}\sigma^{\mu\nu}\xi C_{\mu\nu}\bar{\lambda}^2 - i\xi\theta^{\rho\sigma}\partial_\rho\lambda\sigma_\sigma\bar{\lambda}, \end{aligned} \quad (55)$$

$$\delta_S\bar{\lambda} = -i\theta^{\rho\sigma}\xi\partial_\rho\bar{\lambda}\sigma_\sigma\bar{\lambda}, \quad (56)$$

$$\begin{aligned} \delta_S D = -\xi\sigma^\mu\partial_\mu\bar{\lambda} + \theta^{\rho\sigma}\xi\sigma^\mu\partial_\rho\bar{\lambda}F_{\mu\sigma} \\ +i\theta^{\rho\sigma}\xi\sigma_\sigma\partial_\rho D\bar{\lambda}, \end{aligned} \quad (57)$$

which can be shown to yield

$$\begin{aligned} \delta_S F_{\mu\nu} \;=\; & i\xi(\sigma_\nu\partial_\mu\bar{\lambda} - \sigma_\mu\partial_\nu\bar{\lambda}) + i\xi\theta^{\rho\sigma}\sigma_\rho(\partial_\mu\bar{\lambda}F_{\nu\sigma} \\ & -\partial_\nu\bar{\lambda}F_{\mu\sigma}) - i\xi\theta^{\rho\sigma}\sigma_\rho\bar{\lambda}\partial_\sigma F_{\mu\nu}. \end{aligned}$$

In fact, we explicitly checked that the action (53) is invariant under the θ-expanded supersymmetry transformations (54)-(57).

The θ-expanded $U(1)$ gauge theory action (53) can be utilized to study some different aspects of noncommuting $N=1/2$ supersymmetric gauge theory. Similar to noncommuting electrodynamics one can calculate one-loop renormalization properties of this theory [33] and find solutions of equations of motion [34]. Moreover, using the master action of $N=1/2$ supersymmetric $U(1)$ gauge theory given in [35], one can study duality properties of the action (57).

We would like to apply SW map to non-Abelian gauge theory in the light of the approach used for $U(1)$. Thus we adopt Seiberg's solution for only the C-dependent part of SW map which yields the noncommutative $U(N)$ gauge theory action (11). Then, for the $\theta^{\rho\sigma}$ part of the component fields, we adopt the generalization of SW map given in [29]:

$$\begin{aligned} A_{(\theta)\mu} &= \frac{\theta^{\rho\sigma}}{4}\{A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu}\}, \\ F_{(\theta)\mu\nu} &= \frac{\theta^{\rho\sigma}}{4}\left(2\{F_{\mu\rho},F_{\nu\sigma}\} - \{A_\rho,(D_\sigma+\partial_\sigma)F_{\mu\nu}\}\right), \\ D_{(\theta)} &= \frac{\theta^{\rho\sigma}}{4}\{A_\rho,(D_\sigma+\partial_\sigma)D\}, \\ \lambda_{(\theta)\alpha} &= \frac{\theta^{\rho\sigma}}{4}\{A_\rho,(D_\sigma+\partial_\sigma)\lambda_\alpha\}, \\ \bar{\lambda}^{\dot{\alpha}}_{(\theta)} &= \frac{\theta^{\rho\sigma}}{4}\{A_\rho,(D_\sigma+\partial_\sigma)\bar{\lambda}^{\dot{\alpha}}\}. \end{aligned} \quad (58)$$

As we have already emphasized, SW map does not refer to any action but it is an equivalence relation between gauge transformations. Hence, *a priori* one cannot guarantee that a noncommutative gauge theory will remain gauge invariant under SW map. Indeed, if we choose to work with the trivial solution (48) for $C\theta$-dependent terms, the resultant theory will not be gauge invariant. Therefore, as the $C\theta$ term we choose the non-local one:

$$\lambda^{\alpha}_{(C\theta)} = -\frac{\theta^{\rho\sigma}}{8} C^{\mu\nu} F_{\mu\nu} \{[\bar{\lambda}_{\dot{\alpha}}, A_\rho], (\partial_\sigma + D_\sigma)\bar{\lambda}^{\dot{\alpha}}\} (\sigma^\kappa D_\kappa \bar{\lambda})^{-1}_{\alpha}, \tag{59}$$

with the other components vanishing. Obviously, the gauge parameters $\Lambda_{(C\theta)}$ and $\bar{\Lambda}_{(C\theta)}$ should be appropriately chosen such that (43) be satisfied. When we employ the map (58) and (59) in the action (11), up to some surface terms, we attain

$$\begin{aligned} I = \int \mathrm{d}^4x \,\mathrm{tr}\Big[& -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\lambda\sigma^\mu D_\mu \bar{\lambda} + \tfrac{1}{2} D^2, \\ & + \tfrac{\theta^{\rho\sigma}}{8} \Big(4F^{\mu\nu} F_{\mu\rho} F_{\nu\sigma} - F_{\sigma\rho} F^{\mu\nu} F_{\mu\nu} + 2D^2 F_{\rho\sigma}, \\ & -2\{F_{\rho\sigma}, \lambda\}\sigma^\mu D_\mu \bar{\lambda} - 4\lambda\sigma^\mu \{F_{\mu\rho}, D_\sigma \bar{\lambda}\} \Big), \\ & - \tfrac{i}{2} C^{\mu\nu} \Big(F_{\mu\nu} \bar{\lambda}^2 - \theta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \bar{\lambda}^2 - \tfrac{\theta^{\rho\sigma}}{4} \{F_{\sigma\rho}, F_{\mu\nu}\} \bar{\lambda}^2, \\ & + \tfrac{|C|^2}{8} \Big(\bar{\lambda}^2 \bar{\lambda}^2 - \tfrac{\theta^{\rho\sigma}}{4} \{F_{\sigma\rho}, \bar{\lambda}^4\} \Big) \Big]. \end{aligned} \tag{60}$$

The price which we pay for adding some $C\theta$-dependent terms to obtain gauge invariance is to change supersymmetry transformations in terms of (52). Obviously, new supersymmetry transformations of λ will have a non-local part. For non-abelian case even the local parts of new supersymmetry transformations become so complicated that we do not present them here. Obtaining some other solutions of (43) which respects gauge invariance is an open problem which should be studied.

* * *

We would like to thank K. Ülker for fruitful discussions.

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