A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

Synchronization in the presence of memory

To cite this article: R. Morgado *et al* 2007 *EPL* **79** 10002

A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

July 2007

EPL, **79** (2007) 10002
doi: 10.1209/0295-5075/79/10002

www.epljournal.org

Synchronization in the presence of memory

R. MORGADO[1,2], M. CIEŚLA[1], L. LONGA[1,2] and F. A. OLIVEIRA[1,2]

[1] *Marian Smoluchowski Institute of Physics, Jagellonian University, Department of Statistical Physics and Mark Kac Complex Systems Research Center - Reymonta 4, Kraków, Poland*
[2] *Institute of Physics and International Center of Condensed Matter Physics, University of Brasília, Campus Universitário Darcy Ribeiro, CP 04513, CEP 70919-970 Brasília, DF, Brazil*

received 25 January 2007; accepted in final form 16 May 2007
published online 5 June 2007

PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 05.45.Jn – High-dimensional chaos

Abstract – We study the effect of memory on synchronization of identical chaotic systems driven by common external noises. Our examples show that while in general the synchronization transition becomes more difficult to meet when the memory range increases, for intermediate ranges the synchronization tendency of systems can be enhanced. Generally the synchronization transition is found to depend on the memory profile and range and the ratio of noise strength to memory amplitude, which indicates a possibility of optimizing synchronization by memory. We also point out a close link between dynamics with memory and noise, and recently discovered synchronizing properties of networks with delayed interactions.

Since a generic study of Fahy and Hamann [1] synchronization of dynamical systems has become an active field of research. Examples of synchronous behavior are found in physical, biological, chemical and social systems [2], from a road traffic anticipation [3], population growth [4] and secure communications [5] through biophysics [6], chemistry [7] and laser optics [8] to computer science [9]. At least four types of synchronization scenarios have been identified [10] of which the synchronization between identical systems coupled by a common noise has received much attention due to its relative simplicity and importance [11].

In recent years studies of synchronization were extended further to obey an important case of interactions that are delayed in time [12–14]. One of the most striking observations here was the enhancement of synchrony for networks of many nonlinear interacting units by time-delayed transmission of the signals [14–17]. More specifically, in a neural-network model [15] an enhanced synchronization of neurons by delays has been observed. A similar effect was found for a network of coupled logistic maps [16]. Even a network of logistic maps with random delay times was able to sustain synchronization [17].

A purpose of this work is to show that the constructive influence of delays on synchronization can be extended further to obey a broad class of nonlinear systems with noise and memory, where the latter is understood as the auto-feedback with delay having distribution in time. A close link between the synchronizing networks under delay and synchronization by a common noise of (effectively decoupled) nonlinear systems with memory is also demonstrated. Detailed numerical calculations are carried out for generalized logistic maps and chaotic Fahy-Hamann systems described by non-Markovian Langevin equations. In both cases the choice of the models has been motivated by their well-documented synchronizing properties in the limit of vanishing memory. Importantly, our analysis demonstrates that the presence of memory and noise not only can sustain synchronization existing in equivalent, memoryless systems, but also can enhance it. The models chosen, though governed by different dynamics, collectively display this possibility, which suggests that the effect can be quite common for nonlinear interacting systems.

Memory and randomness naturally link with the time evolution of interacting dynamical systems [18–20]. Indeed, the evolution of a system coupled with "external" degrees of freedom (*e.g.* open systems, nonlinear networks) can, at least in principle, be reduced to the dynamics of an effective single system, but with memory and noise. The effective system is usually more amenable to numerical, analytical and formal considerations. Perhaps the simplest and exact example of such reduction is that of a nonlinear system coupled bi-linearly to harmonic oscillators. The elimination of the oscillator degrees of freedom results in a generalized Langevin equation for the dynamics of the system, where memory and noise terms are fully specified by the properties of the oscillators and by their coupling with the system [18]. Similarly, the dynamics of a network with delay times can be approximated by, or in some cases reduced to an effective dynamics of single nodes with memory and noise. Consider, for example, a network of coupled logistic maps with discrete time [16,17]. The state $x_i(t+1)$ of the node "i" at time $t+1$ depends on the state $x_i(t)$ of that node at time t and on the states $x_j(t-\tau_{ij})$ of the nodes $\{$"$j\neq i$"$\}$ that couple to "i" at earlier times $\{t-\tau_{ij}\}$, where $\{\tau_{ij}\}$ are the delay times (see, *e.g.*, eq. (1) in [16,17]). By iterating the equations for $x_j(t-\tau_{ij})$ $(j\neq i)$ back in time and re-substituting them to $x_i(t+1)$, we arrive at the effective, single-node equations for $x_i(t+1)$ expressed in terms of $x_i(t)$ and the nonlinear auto-feedback (memory) Γ. For the network of logistic maps Γ is a polynomial in $x_i(t-n), 1\leqslant n\leqslant t$ with coefficients depending on the network's connectivity and the initial values $x_j(0)$ for the nodes. Any randomness in the original network like random delay times, random elements in the network's connectivity, or averaging over $x_j(0)$ goes into additive/multiplicative noise terms in the effective equations of motion.

One of the simplest, but important, class of Γ's is a linear auto-feedback with $\Gamma\sim\sum_{k=1}^{N}\Gamma_k x_i(t-k)$, which for example, can represent a "mean-dynamics" of the above-mentioned networks of logistic maps. For Γ given by a polynomial in $x_i(t-n)$ a recipe for getting $\{\Gamma_k\}$ of the "mean-dynamics" would be, *e.g.*, a replacement of $x_i(t-n)$ by $x_i(t-n)=\langle x\rangle+[x_i(t-n)-\langle x\rangle] \;=\langle x\rangle+\delta x_i(t-n)$ and neglect of terms that are nonlinear in δx_i; $\langle\cdot\rangle$ denotes the average over the trajectory and over initial conditions.

The discussion as given clearly shows that memory and noise are intrinsic to the dynamical evolution of a system. It is then important to know in which way they affect synchronization. We report on the noise-induced synchronization, which is generic for this case. The synchronizing system is characterized by master trajectories that divide the whole phase space into basins of attraction such that all trajectories initiated in the same basin and subjected to the same sequence of the noise evolve to the same master trajectory [1,9].

As our first model we consider an ensemble of chaotic logistic maps coupled by common, additive noises. Since the coupling is realized only through the noise terms, it is sufficient to explore just two such systems, which we define as

$$x_{n+1}^i=4x_n^i(1-x_n^i)+I\sum_{k=1}^{N}\Gamma_k x_{n-k}^i+\xi_n+\epsilon_n^i \mod 1, \tag{1}$$

where $i=1,2$ and ξ_n is the non-symmetric, δ-correlated noise taken uniformly from the interval $[a,a+b]$. To avoid roundoff-induced synchronization, we also add an extremely small independent uniform noise ϵ_n^i (in the interval $[-10^{-12},10^{-12}]$) to each particle at every iterated step [21]. We extend our analysis to a symmetric version of (1) by defining a new variable

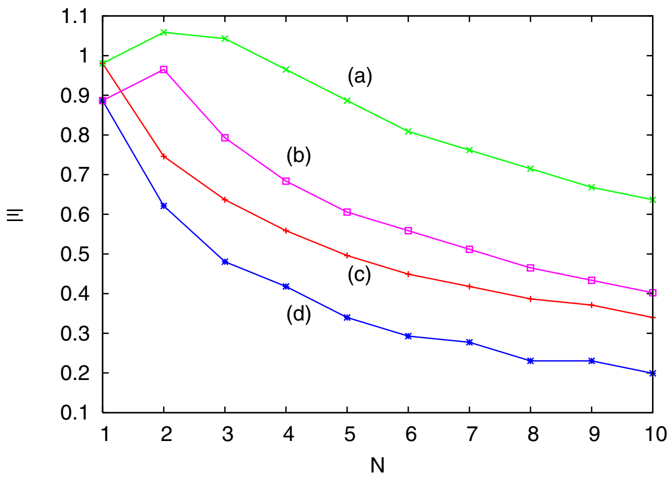

Fig. 1: Threshold lines for symmetric map (2): (a) Γ_n^e, $I<0$, (b) Γ_n^e, $I>0$, (c) Γ_n^c, $I<0$, and (d) Γ_n^c, $I>0$. For Γ_n^e cases we take $\lambda=\frac{2}{N}$.

$$z_n^i=x_n^i-\langle x\rangle\,, \tag{2}$$

where $\langle x\rangle\equiv\langle x_n^i\rangle$. For $I=0$ the noise-induced synchronization in the $[a,b]$-plane has recently been studied in detail by Rim *et al.* [22]. We restrict ourselves to linear auto-feedback with $\Gamma=I\sum_{k=1}^{N}\Gamma_k x_{n-k}$. The set of coefficients $\{I\Gamma_k\}$ is the "memory profile" and N is proportional to the memory range. Two models for the memory profile are considered in detail: the constant memory profile with

$$\Gamma_n\equiv\Gamma_n^c=\begin{cases}1, & \text{for} \quad n\leqslant N,\\ 0, & \text{for} \quad n>N,\end{cases} \tag{3}$$

and the exponentially decaying memory profile

$$\Gamma_n\equiv\Gamma_n^e=\begin{cases}\exp(-\lambda n), & \text{for} \quad n\leqslant N,\\ 0, & \text{for} \quad n>N.\end{cases} \tag{4}$$

For $N\gg 1/\lambda$ the inversion of λ is the memory range.

The stability of the synchronized states is determined by the sign of the (maximal) transversal Lyapunov exponent Λ for the dynamics of difference $\delta x_n=x_n^1-x_n^2=z_n^1-z_n^2$. In the numerical calculations we choose at random the initial states $\{x_N^1,x_{N-1}^1,\ldots,x_0^1\}$ from the allowed interval and nearby states $\{x_N^2,x_{N-1}^2,\ldots,x_0^2\}$. Then we iterate the equations of motion to construct statistics of the expansion and contraction rates: $\Lambda_i=\ln\left(\frac{|\delta x_{N+i+1}|}{|\delta x_{N+i}|}\right)$ of δx_n. The procedure, repeated for many randomly chosen initial states, allows us to calculate the average of Λ_i, which approximates Λ [13,15,22].

In case of non-zero memory we generally find that for large enough absolute memory intensity (or strength of the coupling to the past), $|I|$, the synchronization is destroyed for all $N\geqslant 1$. Results are shown in fig. 1 where $|I|$, above

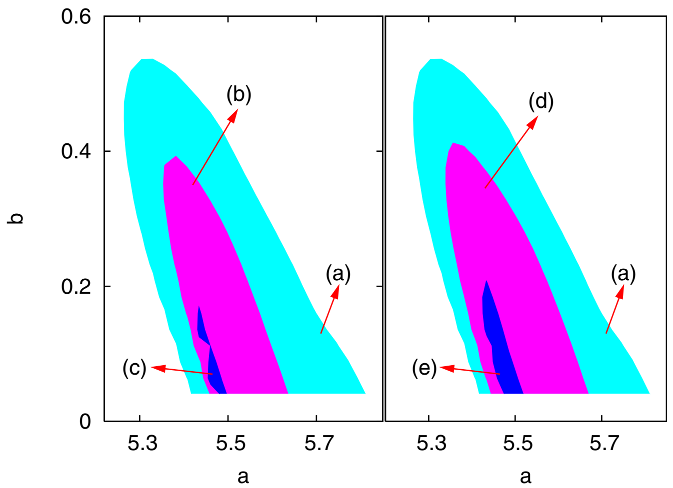

Fig. 2: Synchronization areas for the symmetric map. Region (a) corresponds to a system without memory [22]. The left plot is done for Γ_n^c with (b) $N=5$, $I=0.1$ and (c) $N=5$, $I=0.3$. The right plot corresponds to Γ_n^e with $\lambda=\frac{2}{N}$ and with (d) $N=5$, $I=0.1$ and (e) $N=5$, $I=0.5$.

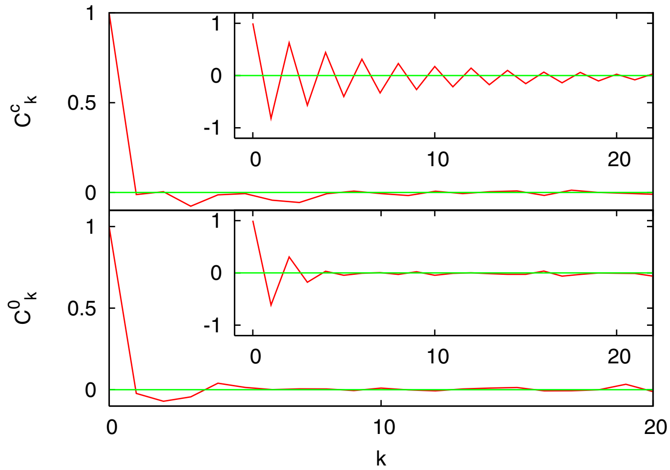

Fig. 3: Correlation functions for the symmetric map with constant memory ($C_k \equiv C_k^c$) and without memory ($C_k \equiv C_k^0$). Here $N=5$, $I=0.1$, $b=0.1$, $a=5.3$ ($a=5.55$ for insets).

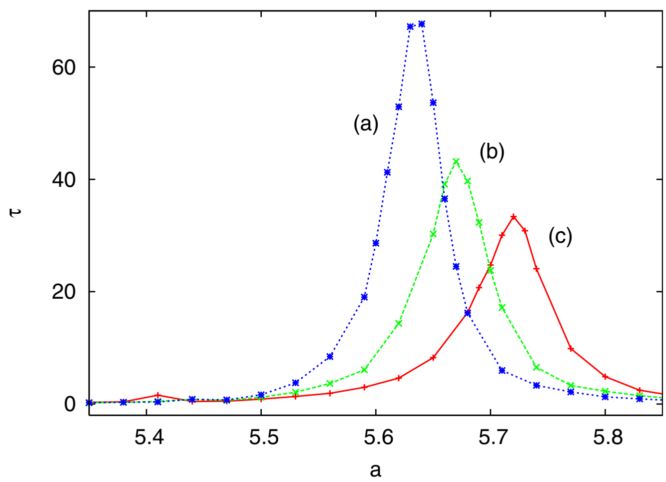

Fig. 4: Decay time of C_k as a function of noise parameter a. Cases studied are: (a) constant memory and (b) exponential memory for $I=0.1$ and $\lambda=2/N$. Case (c) corresponds to $I=0$. The remaining parameters are $N=5$ and $b=0.1$.

which the synchronization region disappears, is sketched. For $|I|$ exceeding the threshold value the systems desynchronize for all a and b. Choosing $\lambda=\frac{2}{N}$ we find that the results compare well for both memory profiles. In this case Γ_N^e at $n=N$ is about an order of magnitude smaller than at $n=0$ ($\Gamma_N^e=e^{-2}$). The effect is illustrated further in fig. 2, where the evolution of the synchronizing boundaries, $\Lambda(a,b)=0$, are shown with increasing (positive) intensity for the symmetric map (2) and for different memory profiles. The case without memory [22] is also shown for comparison. Please note that the synchronization area shrinks with increasing intensity and range of the memory. This behavior is observed for $N \geqslant 1$, for positive and negative intensities, and for all maps studied. Interestingly, the maxima in fig. 1 prove that *memory can also act on synchronization in a constructive way by enhancing it.* We observe the enhancement of synchrony by memory for the exponential memory profile, but not for the constant memory. Nevertheless, the phenomenon appears quite general. As demonstrated below, systems with a more complex dynamics, governed by the integro-differential, generalized Langevin equation, show a similar behavior. We discuss possible implications of these results toward the end.

The memory profile is closely linked to a time-time autocorrelation function

$$C_k = \frac{\langle (x_n - \langle x \rangle)(x_{n-k} - \langle x \rangle) \rangle}{\left\langle (x_n - \langle x \rangle)^2 \right\rangle}. \qquad (5)$$

We monitored the behavior of this function inside and outside of the synchronizing area. Exemplary results are shown in figs. 3 and 4. The decay time τ in fig. 4 was calculated by assuming that C_k is a linear combination of trigonometric functions multiplied by an exponential decay of the form $|C_k| = \exp\left(-\frac{k}{\tau}\right)$. We observe in figs. 2 and 4, that close to the border of the synchronizing area τ is enhanced by the presence of memory, with the maximum positioned outside this area. That is, the chaotic systems with larger τ "remember for longer" about their initial conditions, which makes synchronization more difficult and explains intuitively the shrinkage of the areas in fig. 2.

The second model with which we explore the influence of memory on noise-induced synchronization is the generalization of the Fahy-Hamann system [1]. We analyze trajectories of two identical particles in a two-dimensional potential well given by

$$V(x_1,x_2) = \frac{\sin 2\pi x_1}{2\pi x_1} + \frac{\sin 2\pi x_2}{2\pi x_2} + \frac{(x_1^2+x_2^2)^2}{16\pi^2}, \qquad (6)$$

with different initial conditions. The motion of the i-th particle ($i = 1, 2$) is governed by the generalized Langevin equation:

$$m\ddot{x}^i_\alpha(t) = -\frac{\partial V(x^i_1, x^i_2)}{\partial x^i_\alpha} - mI\int^t \Gamma(t-t')\,\dot{x}^i_\alpha(t')\,\mathrm{d}t' + \xi(t), \tag{7}$$

where m is the mass. I is again the memory intensity or, in this case, the friction constant. The noise ξ, common for both particles, is a Γ-correlated stochastic force with zero mean, where correlations are obeying the fluctuation-dissipation theorem

$$\langle\langle\,\xi(t)\xi(t')\,\rangle\rangle = 2mIk_BT\,\Gamma(t-t'), \tag{8}$$

with T being the absolute temperature and k_B the Boltzmann constant. In what follows we restrict ourselves to the exponentially correlated noise by choosing, as in eq. (4), $\Gamma(t-t') = e^{-\lambda(t-t')}$. Double angular brackets denote averaging over a noise realization. The equations of motion (7) are integrated numerically using stochastic version of the Euler algorithm. Discretization of the eqs. (7) entails re-scaling the noise strength by a factor $1/\sqrt{\Delta t}$, where Δt is the time step. Finally, the exponentially correlated noise is generated from uniformly distributed random numbers through the Ornstein-Uhlenbeck process. As previously, simulations are carried out to determine the maximal Lyapunov exponent, Λ, as a function of memory range ($1/\lambda$) and I. We used a natural system of units: energy $\epsilon_u = V(0,0) - V_{min} \approx 2.41$, time $t_u = \sqrt{3}$, measuring the curvature of the potential at the origin and length $l_u = 1$, giving the period of the oscillating part of the potential. In the absence of memory the system was originally studied by Fahy and Hamann [1] using a regular Andersen thermostat. The results unambiguously showed that trajectories were exponentially convergent to a common trajectory after a transient period. The same phenomenon has been reported for a Langevin dynamics without memory of a one-dimensional Lennard-Jones chain [23] and of other systems [9]. We should add that the purely deterministic evolution in the potential (6) exhibits chaotic behavior.

Calculations of Λ in the presence of memory as a function of $1/\lambda$ are shown in fig. 5. Please note that the dependence of Λ on $1/\lambda$ is much more complex now than that previously observed for the maps. At least three regimes can be identified. In the first regime, corresponding to a short memory range, we observe the de-synchronization of the system by memory. However, after reaching the maximum, Λ drops down and for the intermediate memory range synchronization is considerably enhanced. The strongest synchronization conditions are met for $1/\lambda \approx 0.14$, where Λ approaches the minimum. Interestingly, the positions of the extremes are practically independent of temperature T.

We also studied the border between synchronization areas and chaotic regions in the simulated system, in analogy to fig. 2 for the symmetric map. Result is shown in fig. 6. For a memoryless system ($1/\lambda = 0$) we observe chaotic behavior both for small temperatures and small frictions. Starting from initially small values of I the solution for I of the $\Lambda(T, I) = 0$ condition approaches the maximum of 1.6 for $T \approx 0.5$ and then drops down to $I \approx 1$ with increasing temperature. Adding a small memory ($1/\lambda = 0.01$) makes the chaotic region grow with synchronization appearing only for large damping coefficient I, not shown in fig. 6. But when the memory range exceeds $1/\lambda \approx 0.05$ the situation radically changes and for $1/\lambda = 0.125$ all the plotted area of fig. 6 corresponds to

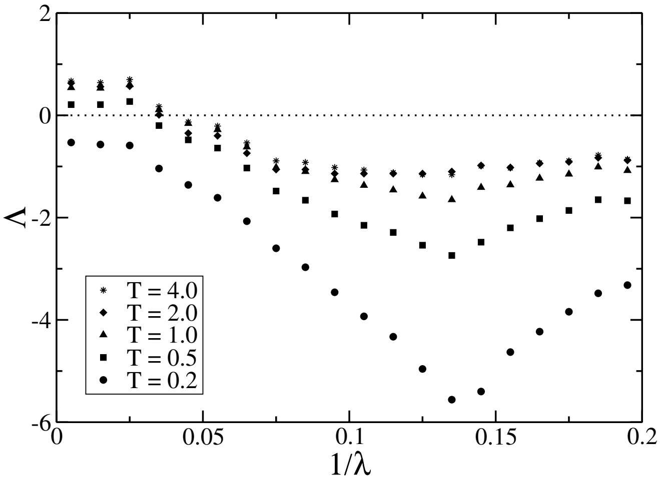

Fig. 5: Maximal Lyapunov exponent Λ against memory range $1/\lambda$ for different temperatures. The friction constant is $I = 1.0$.

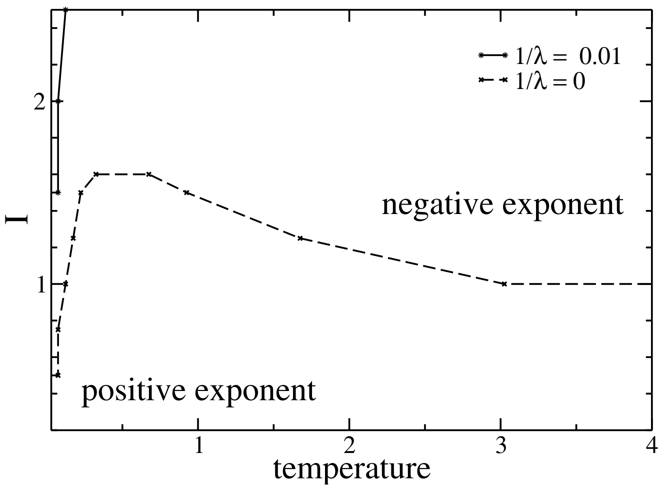

Fig. 6: Regions of positive and negative Lyapunov exponent Λ against friction constant and temperature for two different memory ranges $1/\lambda = 0.01, 0.0$. Lines correspond to border between synchronization and chaos ($\Lambda = 0$). When parameters are taken from above the border line the system synchronizes; for frictions and temperatures taken from below of this line, nearby trajectories diverge exponentially. For $1/\lambda = 0.01$ almost the whole area shown, corresponds to chaos; for $1/\lambda = 0.125$ we observed synchronization only.

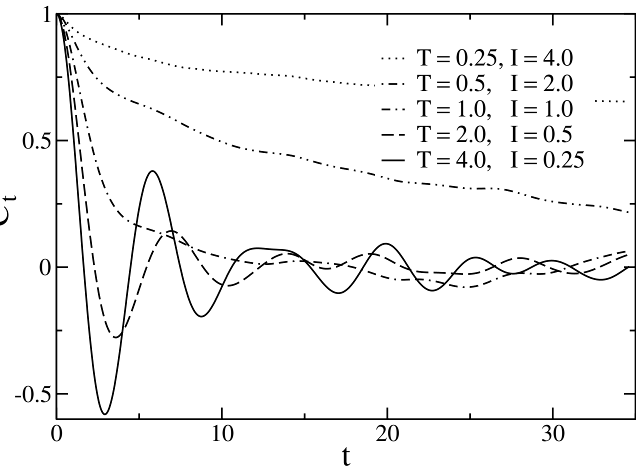

Fig. 7: Correlation functions for different temperatures and friction constants in case of ordinary (memoryless) Langevin dynamics with $\Gamma(t-t') = \delta(t-t')$. For $T < 1.0$ and $I > 1.0$ the maximal Lyapunov exponent is negative, while for the remaining (T, I) parameters the system's dynamics is chaotic.

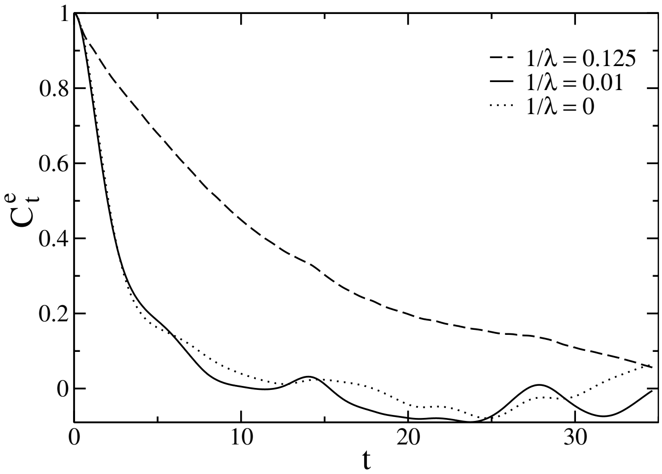

Fig. 8: Correlation functions for $I = 1$ and $T = 1$ and for different memory ranges $1/\lambda = 0.125$, 0.01, 0.0. The largest range of correlations is observed for synchronizing dynamics when $1/\lambda = 0.125$.

synchronization. Chaotic behavior is observed only for very small temperatures and $I \to 0$, where the dynamics becomes deterministic.

Finally, we turn to the analysis of the autocorrelation function $C_k \equiv C_t$, eq. (5), for the generalized Fahy-Hamman system, where we identify k with $t = k\Delta t$. In the deterministic limit the Fahy-Hamann dynamics is characterized by the chaotic (quasi-periodic) behavior. Since the noise and friction terms are not present in this limit we generally expect to have weaker decay of autocorrelations in the absence of noise than for the Langevin dynamics. Indeed a direct simulation supports this expectation. Adding the noise and friction terms to the chaotic dynamics induces areas of synchronization in the parameter space. For the Γ-correlated stochastic force obeying the fluctuation-dissipation theorem (8) we compare autocorrelations recorded for trajectories characterized by the same intensity of the mean-square noise $\langle\langle\xi(t)^2\rangle\rangle$, equal to $2mIk_BT$. The exemplary results for the Langevin system without memory are shown in fig. 7. The case with exponential memory is illustrated in fig. 8. In both cases a qualitative behavior of autocorrelations resembles that observed for logistic maps. As previously the autocorrelation function decays faster in the chaotic regime.

Summarizing, the results obtained for the generalized logistic maps and for the dynamical system evolving according to the generalized Langevin equation uncover a possibility of having the *constructive* influence of memory on the noise-induced synchronization. The results are quite counterintuitive for, in the first place, we would expect that memory by introducing extra dimensions [24] should act in just the opposite way, *i.e.* making synchronization more difficult [16,17]. Though the proposed models are relatively simple they represent quite different dynamics, which suggests that the uncovered enhancement of synchrony by memory can be a general phenomenon, occurring for a wide class of nonlinear dynamical systems with memory and noise. They rise a possibility of seeking for a right memory profile to create optimal conditions for synchronization to occur *i.e. optimizing synchronization by memory.* Additionally, our findings apart from giving yet another example of the nontrivial interplay between memory and noise, show a close relation to the synchronizing properties of coupled networks with delays that have recently been discovered.

* * *

This work was supported by Brazilian CNPq Grant No 201388/03-0, Grant N202 169 31/3455 of Polish Ministry of Science and by the EC Marie Curie Host Grant for Transfer of Knowledge "*COCOS*" (contract MTKD-CT-2004-517186). We also gratefully acknowledge the computing grant G27-8 towards the use of the ICM computers (University of Warsaw) and would like to thank dr P. Góra for useful discussions.

REFERENCES

[1] Fahy S. and Hamann D. R., *Phys. Rev. Lett.*, **69** (1992) 761.
[2] Acebron J. A., Binilla L. L., Perez Vicente C. J., Ritort F. and Spigler R., *Rev. Mod. Phys.*, **77** (2005) 137.
[3] Treiber M. and Helbing D., *Phys. Rev. E*, **68** (2003) 046119.
[4] Grenfell B. T. *et al.*, *Nature (London)*, **394** (1998) 647.
[5] Zhang Y., Tao C., Du G. and Jiang J. J., *Phys. Rev. E*, **71** (2005) 016217.

[6] Mainen Z. F. and Sejnowski T. J., *Science*, **268** (1995) 1503; Belykh I., de Lange E. and Hasler M., *Phys. Rev. Lett.*, **94** (2005) 188101.

[7] Kiss I. Z. and Hudson J. L., *Phys. Rev. E*, **64** (2001) 046215.

[8] DeShazer D. J., Breban R., Ott E. and Roy R., *Phys. Rev. Lett.*, **87** (2001) 044101.

[9] Uberuaga B. P., Anghel M. and Voter A. F., *J. Chem. Phys.*, **120** (2004) 6363.

[10] Bove I., Boccaletti S., Bragard J., Kurths J. and Mancini H., *Phys. Rev. E*, **69** (2004) 016208; McGraw P. N. and Menzinger M., *Phys. Rev. E*, **72** (2005) 015101(R); Guan S., Lai C.-H. and Wei G. W., *Phys. Rev. E*, **72** (2005) 016205.

[11] Zhou C. and Kurths J., *Phys. Rev. Lett.*, **88** (2002) 230602; Chen H. F. and Liu J. M., *Phys. Rev. E*, **71** (2005) 046216.

[12] Schuster H. G. and Wagner P., *Prog. Theor. Phys.*, **81** (1989) 939.

[13] Buchner T. and Zebrowski J. J., *Phys. Rev. E*, **63** (2000) 016210.

[14] *See e.g.* Voss H. U., *Phys. Rev. E*, **61** (2000) 5115; Earl M. G. and Strogatz S. H., *Phys. Rev. E*, **67** (2003) 036204; Jirsa V. K. and Ding M., *Phys. Rev. Lett.*, **93** (2004) 070602; Dodla R., Sen A. and Johnston G. L., *Phys. Rev. E*, **69** (2004) 056217.

[15] Dhamala M., Jirsa V. K. and Ding M., *Phys. Rev. Lett.*, **92** (2004) 074104.

[16] Atay F. M., Jost J. and Wende A., *Phys. Rev. Lett.*, **92** (2004) 144101.

[17] Masoller C. and Marti A. C., *Phys. Rev. Lett.*, **94** (2005) 134102.

[18] Zwanzig R., *Nonequilibrium Statistical Mechanics*, (Oxford University Press) 2001.

[19] Morgado R., Oliveira F. A., Batrouni G. G. and Hansen A., *Phys. Rev. Lett.*, **89** (2002) 100601; Costa I. V. L., Morgado R., Lima M. V. B. T. and Oliveira F. A., *Europhys. Lett.*, **3** (2003) 173.

[20] Lee M. H., *Phys. Rev. Lett.*, **87** (2001) 250601.

[21] Longa L., Curado E. M. F. and Oliveira F. A., *Phys. Rev. E*, **54** (1996) R2201.

[22] Rim S., Hwang D. U., Kim I. and Kim C. M., *Phys. Rev. Lett.*, **85** (2000) 2304.

[23] Ciesla M., Dias S. P., Longa L. and Oliveira F. A., *Phys. Rev. E*, **63** (2001) 065202(R).

[24] Farmer J. D., *Physica D*, **4** (1982) 366.