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# Scattering dynamics of driven closed billiards

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**Abstract** – We investigate the classical scattering dynamics of the driven elliptical billiard. Two fundamental scattering mechanisms are identified and employed to understand the rich behavior of the escape rate. A long-time algebraic decay which can be tuned by varying the driving amplitude is established. Pulsed escape rates and decelerated escaping particles are generic properties of the harmonically breathing billiard. This suggests time-dependent billiards as prototype systems to study the nonequilibrium evolution of classical ensembles encountering a multitude of scattering processes off driven targets.

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**Introduction.** – Many models of statistical mechanics can be reduced to billiard-type dynamical systems. Seminal results like the justification of a probabilistic approach to statistical mechanics [1,2] or a connection between the escape from a billiard and the famous Riemann hypothesis [3] were obtained. Modern studies of billiards are, among others, motivated by corresponding experiments including ultracold atoms confined in a laser potential [4–7], microwave billiards [8–10] or mesoscopic quantum dots [11]. Equally for the design of directional micro-lasers, billiards are of immediate relevance [12]. From the theoretical point of view, investigations on the classical and quantum properties of billiards have pioneered the fields of quantum chaos, modern semiclassicals and transport at the mesoscopic scale (see [8] and refs. therein).

The escape rate, being the fraction of remaining particles as a function of time, is a characteristic of open billiards which is both experimentally accessible and important for transport properties [3]. This key property allows to probe the dynamics from the outside and has been studied thoroughly for billiards with a static boundary [13–15]. Integrable systems exhibit an algebraic, while fully chaotic billiards show an exponential decay of the escape rate [13] (although it is known that in special cases, like the Bunimovich stadium, there is an algebraic

long-time behavior in the decay, due to the slow transport of particles close to the marginally stable bouncing ball orbits, cf. *e.g.* [14]). Here we focus on the case where the billiard shape changes in time according to a certain law. Such a driven billiard not only leads to a higher dimensional phase space of the scattering processes but also to a non-conservation *i.e.* time-evolution of the energy. Driven billiards represent prototype systems for the evolution of ensembles of particles in a closed driven environment where multiple scattering off the driven boundary takes place thereby leading to a dynamical nonequilibrium state. Extremely little is known on the properties of such systems, the few existing investigations dealing with aspects such as Fermi acceleration [16–19] and principal structures of the corresponding phase space [20–22]. In spite of the above-motivated interest escape rates of driven billiards have not been addressed up to date. Using atom-optical techniques such as acousto-optic deflectors or even in the case of microwave cavities moving boundaries are well within reach of experiment.

We investigate the classical scattering dynamics and the time-evolution of ensembles of particles in a harmonically driven elliptical billiard. The decay of the escape rate is traced back to the underlying scattering mechanism by identifying two fundamental scattering processes being key ingredients for the time-evolution. The escape rate behaves asymptotically as  $N_C(t) \sim t^{-w_C}$  and it is shown that the decay constant  $w_C$  can be changed continuously

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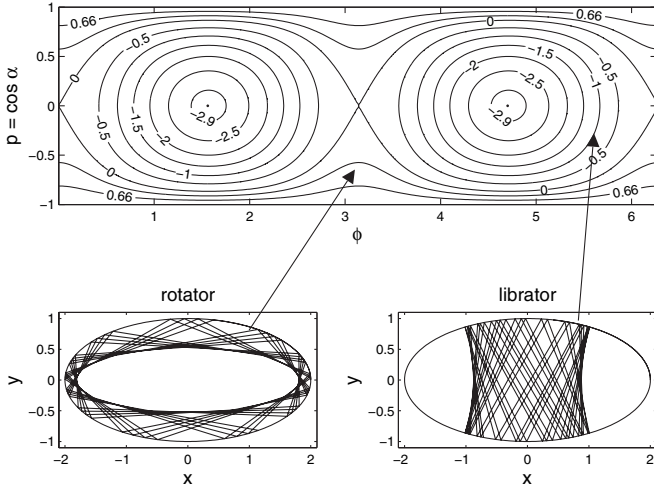


Fig. 1: Phase space of the ellipse, the invariant curves are the contour lines of  $F(\varphi, p)$  (upper part) and typical trajectories in coordinate space (lower part).

by varying the driving amplitude  $C$ . Moreover, phenomena such as a pulsed escape, demonstrate the richness of the properties of driven billiards and suggest specifically the driven ellipse as a prototype system for the nonequilibrium evolution of ensembles that experience multiple scattering processes with moving targets.

**Fundamental properties of the dynamics in the ellipse.** – As a precursor for the driven system let us discuss some relevant features of the static elliptical billiard. The boundary  $\mathcal{B}$  of an ellipse is given by  $\mathcal{B} = \{(x(\varphi) = A \cos \varphi, y(\varphi) = B \sin \varphi) | 0 \leq \varphi < 2\pi\}$  with  $A > B > 0$ , thus  $A$  and  $B$  being the long and the short half-diameter, respectively. The dynamics in the ellipse is completely integrable [20], see the Poincaré surface of section (PSS) shown in fig. 1. In addition to the energy, there is a second constant of motion

$$F(\varphi, p) = \frac{p^2(1 + (1 - \varepsilon^2) \cot^2 \varphi) - \varepsilon^2}{1 + (1 - \varepsilon^2) \cot^2 \varphi - \varepsilon^2}, \quad (1)$$

restricting the orbits to invariant curves in phase space, where  $\varepsilon = \sqrt{1 - B^2/A^2}$  is the eccentricity and  $p = \cos \alpha$  ( $\alpha$  is the angle between the tangent at the boundary and the trajectory of the particle).  $F(\varphi, p)$  can be interpreted as the product of the angular momenta (PAM) about the two foci [23]. There are two different types of orbits in the ellipse, librators and rotators, see fig. 1. Librators cross the  $x$ -axis between the two foci and touch repeatedly a confocal hyperbola. They possess values of  $F$  between  $-\varepsilon^2/(1 - \varepsilon^2)$  and zero and their motion is restricted to a limited range of  $\varphi$ . Rotators travel around the ellipse, eventually exploring every value of  $\varphi$ , repeatedly touching a confocal ellipse. They have values of  $F$  between zero and one.

**Escape rate.** – Let us consider the escape rate of the static billiard by placing a small hole at the very right

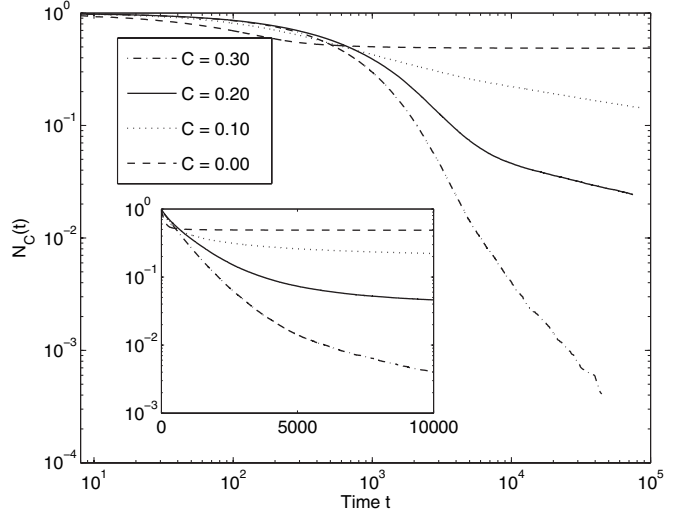


Fig. 2: Fraction of remaining particles  $N_C(t)$  of the IVE as a function of time for different values of the driving amplitude  $C$ . The inset provides a semi-logarithmic plot.

( $\varphi = 0$ ) of the ellipse ( $A = 2$ ,  $B = 1$  in arbitrary units) and iterating the corresponding 2D discrete mapping (for the phase space variables  $\varphi$  and  $p$ ) numerically. We employ an ensemble of  $10^6$  particles with initial conditions being uniform randomly distributed in  $\varphi$ ,  $\alpha$ -space. The result is shown in fig. 2 (curve with  $C = 0.00$ ). The decay approaches a saturation value  $N_s(\varepsilon)$  which is caused by particles traveling on libration orbits that are not connected with the hole [5]. For  $\varepsilon = 0$  the saturation value is zero and it increases monotonically with increasing  $\varepsilon$ . The analytical dependence of  $N_s$  on  $\varepsilon$  can be obtained by determining the number of libration type particles starting from the region bounded by the separatrix [24]

$$N_s(\varepsilon) = \frac{1}{\pi^2} \int_0^{2\pi} d\varphi \arccos \sqrt{\frac{\varepsilon^2}{1 + (1 - \varepsilon^2) \cot^2 \varphi}}. \quad (2)$$

This expression is in excellent agreement with the results of the numerical simulations. One can therefore conclude that varying  $\varepsilon$  allows us to control the total number of particles being emitted.

**Time-dependent ellipse.** – We now apply harmonic oscillations to the boundary of the ellipse

$$\mathbf{b}(\varphi, t) = \begin{pmatrix} (A_0 + C \sin(\omega t + \delta)) \cos \varphi \\ (B_0 + C \sin(\omega t + \delta)) \sin \varphi \end{pmatrix}, \quad (3)$$

$C > 0$  is the driving amplitude and  $\delta$  is a phase shift which will be set to zero in the following. The dynamics of the particles is now governed by a 4D discrete mapping [20,21]: the time and position of a particle-boundary collision is described by the pair  $\varphi$ ,  $t$ , the energy and direction of motion by  $\mathbf{v} = (v_x, v_y)$ . The time  $t_{n+1}$  for the  $(n+1)$ -th

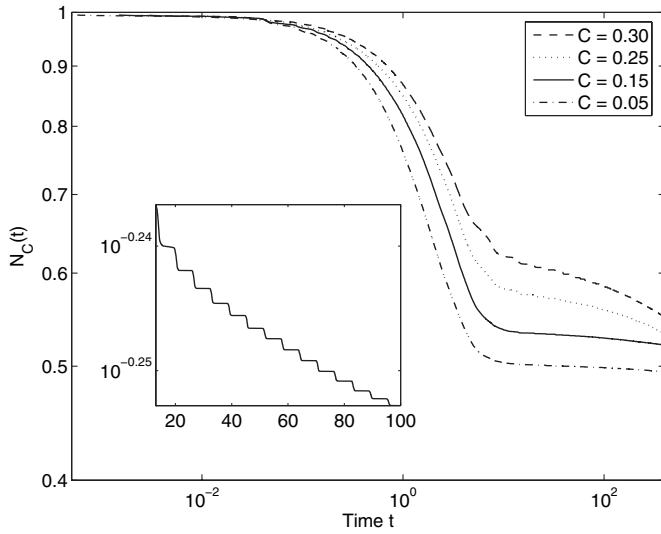


Fig. 3: Fraction of remaining particles  $N_C(t)$  for the HVE,  $2\pi$ -oscillations of the decay are shown in the inset.

collision is determined implicitly by

$$\left( \frac{v_n^x(t_{n+1} - t_n) + x_n}{A_0 + C \sin(\omega t_n + \delta)} \right)^2 + \left( \frac{v_n^y(t_{n+1} - t_n) + y_n}{B_0 + C \sin(\omega t_n + \delta)} \right)^2 - 1 = 0, \quad (4)$$

where the smallest  $t_{n+1} > t_n$  that solves (4) has to be taken and  $\mathbf{x}_n = (x_n, y_n)$  is the  $n$ -th collision point. The  $n+1$  collision point is given by  $\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{v}_n(t_{n+1} - t_n)$  and  $\varphi_{n+1}$  can be obtained by inverting (3). The velocity immediately after the collision  $\mathbf{v}_{n+1}$  is described by  $\mathbf{v}_{n+1} = \mathbf{v}_n - 2[\hat{\mathbf{n}}_{n+1} \cdot (\mathbf{v}_n - \mathbf{u}_{n+1})] \cdot \hat{\mathbf{n}}_{n+1}$ , where  $\mathbf{u}_{n+1}$  is the boundary velocity and  $\hat{\mathbf{n}}_{n+1}$  the inward pointing normal vector of the collisional event occurring at time  $t_{n+1}$  and position  $\mathbf{x}_{n+1}$ . When iterating the underlying implicit mapping numerically the determination of  $t_{n+1}$  from eq. (4) requires sophisticated numerical techniques due to many neighboring roots. As a result the corresponding simulations are computationally very demanding.

**Escape rates.** – We focus on the escape rates of monoenergetic ensembles consisting of  $10^5$  particles<sup>1</sup> with  $\omega = 1$ ,  $A_0 = 2$  and  $B_0 = 1$  for different values of the driving amplitude  $C$ . Two relevant cases can be distinguished:  $|\mathbf{v}_0| \approx \omega C$  being the intermediate-velocity ensemble (IVE) for which the initial particle velocity is of the order of the maximum velocity of the billiard boundary and  $|\mathbf{v}_0| \gg \omega C$  being the high-velocity ensemble (HVE). Naturally, it would be also interesting to examine the case  $|\mathbf{v}_0| \ll \omega C$ . However, the first few collisions then accelerate the particles to velocities  $|\mathbf{v}| \approx \omega C$  and, after a short time, we encounter the situation of the IVE.  $N_C(t)$  of the IVE and HVE are shown in figs. 2 and 3, respectively. In both cases, there exists no saturation value as observed in the

<sup>1</sup>Particles start from the innermost ellipse,  $\alpha$  is distributed uniform randomly and  $\mathbf{v}_0 = v_0 \cdot (\cos \alpha, \sin \alpha)$ , where  $v_0 = 1$  (IVE) and  $v_0 = 100$  (HVE).

static case. A short-time exponential decay is followed by a transient and the long-time behavior ( $t > 10^4$ ) is to a good approximation algebraic,  $N_C(t) \sim t^{-w_C}$ , at least for the case of the IVE (we remark that this algebraic decay has been numerically shown to exist for much longer times than illustrated in fig. 2). The decay constant  $w_C$  increases monotonically with increasing  $C$  (this fact is based not only on the four values of the driving amplitude  $C$  shown here, but on simulations carried out for 20 values of  $C$  between 0.01 and 0.30). It can be therefore concluded that tuning the driving amplitude allows us to control the decay constant  $w_C$  in the long-time behavior of the decay. The division of the behavior of the escape rate into the above-mentioned regimes is even more pronounced in the case of the HVE, see fig. 3, although the long time tail shows substantial deviations from an algebraic behavior.

**Mechanisms for the destruction of the librator orbits.** – The saturation value  $N_s(\varepsilon)$  observed in the static case is caused by librating orbits which are gradually destroyed by the driving (3) thereby resulting in a non-vanishing rate  $\dot{N}_C(t)$  even for large times. Rather than trying to examine the 4D phase space (since 2D intersections or projections of this 4D space are nonrepresentative and difficult to interpret), let us analyze the destruction of the librator orbits as follows. Orbits  $(\varphi'_i, p'_i)$  in the driven ellipse can be compared to the corresponding ones of the static ellipse  $(\varphi_i, p_i)$  by inspecting the angular momentum  $F(\varphi, p)$ . In contrast to the case of the static ellipse where  $F(\varphi_i, p_i) = \text{const}$ ,  $\forall i$ ,  $F(\varphi'_i, p'_i) \neq F(\varphi'_j, p'_j)$  ( $i \neq j$ ) in the driven case, *i.e.*  $F$  is no longer a constant of motion. The difference  $\Delta F = F_{i+1} - F_i$  (see fig. 1) upon a collision is a measure of whether a librator orbit approaches ( $\Delta F > 0$ ) the separatrix or whether it moves in phase space towards ( $\Delta F < 0$ ) the corresponding elliptic fixed points of the static system. Two fundamental scattering processes can be identified that destroy librators by changing them into rotator orbits:

1. *Vertical process:* The angle of incidence of a particle upon collision does not coincide with the reflection angle due to momentum transfer by the moving billiard boundary. This momentum change  $\Delta p$  causes a vertical displacement of the particle in the PSS.
2. *Horizontal process:* A particle that would hit the boundary at a certain point  $\varphi$  in case of a static billiard, hits the boundary in the driven case at  $\varphi'$ , simply because it has moved. Here no change of the momentum takes place. This corresponds to a horizontal displacement  $\Delta \varphi$  of the particle in the PSS.

These processes are fundamental in the sense that every change  $\Delta F$  can be decomposed into  $\Delta F = \Delta F_h + \Delta F_v$ , where  $\Delta F_{h,v}$  denote the individual changes caused by the horizontal, the vertical process, respectively.

In general, these effects do not appear isolated but a combination  $(\Delta \varphi, \Delta p)$  of both will occur at a single

collision. The value  $\Delta F_{h,v}$  at a single collision can be calculated exactly [24]. The changes  $\Delta F_{h,v}$  increase monotonically with the driving amplitude  $C$ . Additionally,  $\Delta F_v$  depends monotonically on the driving frequency  $\omega$ . It is important to note that collisions occurring during the expansion/contraction period of the ellipse always lead to  $\Delta F_v > 0/\Delta F_v < 0$ , whereas such a clear distinction is not possible for  $\Delta F_h$ .

For a single particle undergoing many scattering processes, the effective change  $\Delta F_e$  after a certain time depends on the sequence of these processes, hence  $\Delta F_e = \Delta F_1 + \Delta F_2 + \dots + \Delta F_n$  after  $n$  collisions. When regarding an ensemble of particles, the effective changes  $(\Delta F_e)_j$  (where the index  $j$  indicates the  $j$ -th particle) after  $n$  collisions can vary significantly from particle to particle, since the sequence of the individual  $\Delta F_i$  will vary substantially for different initial conditions which is due to the nonlinear dynamics of the underlying discrete mapping.

**Qualitative model of the decay.** – With the just presented considerations, we can explain qualitatively the observed escape rates and especially the disappearance of the saturation value. Let us focus first on the HVE. The initial fast decay ( $t < 5$ ) is due to the rotator orbits that are connected with the hole (in the PSS) and escape very rapidly. The long-time decay ( $t > 10$ ) is caused by particles starting on librator orbits that have been scattered through horizontal and vertical processes across the separatrix. The closer a particles initial orbit lies near the elliptic fixed points (of the static billiard), the longer it takes until the effective change  $\Delta F$  is large enough to reach the separatrix ( $F = 0$ ), see fig. 4. The monotonic dependence of the  $\Delta F_i$  on  $C$  explains the increasing emission rate  $\dot{N}_C(t)$  with increasing  $C$ , since at a given time  $t$ , the number of particles that can participate in the decay is larger for larger values of  $C$ . The decay in the transient region ( $5 < t < 10$ ) is caused by a superposition of the tail of the initial fast exponential decay and the onset of the slow algebraic decay. With the same arguments, the decay of the IVE can be explained qualitatively. Since  $\mathcal{O}(\mathbf{v}_0) \approx \mathcal{O}(\max(\mathbf{u}))$ , the resulting changes  $\Delta F$  are much larger for the case of the IVE compared to the ones of the HVE. This leads to a very early onset of the slow algebraic decay and consequently the transient region is broadened.

The above presented considerations are confirmed by the results shown in the right inset of fig. 4, where exemplarily the average escape time and the standard deviation  $\sigma$  of the IVE ( $C = 0.10$ ) as a function of the initial PAM  $F(\varphi_0, p_0)$  are plotted. Particles with values of  $F(\varphi_0, p_0)$  between zero and one (rotators) have approximately the same average escape times  $\langle t_{esc} \rangle$ , but the large (compared to  $\langle t_{esc} \rangle$ ) standard deviation  $\sigma$  indicates that the individual escape times can vary significantly from particle to particle. The average escape time of particles starting on librator orbits ( $F(\varphi_0, p_0) < 0$ ) increases with decreasing  $F(\varphi_0, p_0)$ , since the required  $\Delta F_e$  to change a librator into

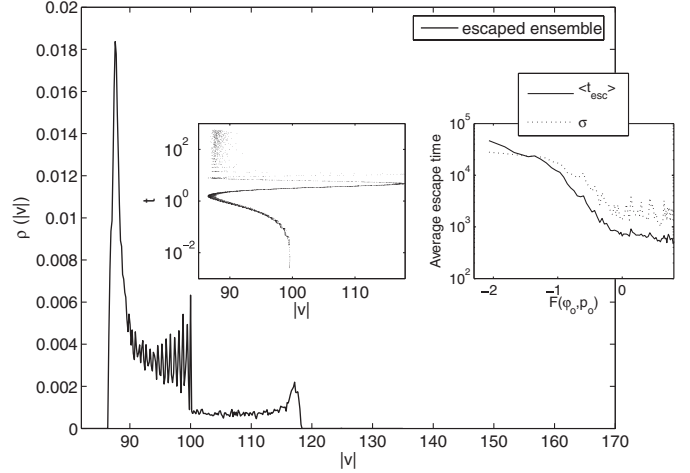


Fig. 4: Distribution  $\rho(|\mathbf{v}|)$  of the velocity of the escaped ensemble (HVE,  $C = 0.25$ ). Left inset: the escape time  $vs.$  the escape velocity is plotted. Right inset: average escape time of the IVE ( $C = 0.10$ ) as a function of the initial  $F(\varphi_0, p_0)$ .

a rotator becomes larger and more and more collisions are necessary to reach these values of  $\Delta F_e$ .

**Modulation of the decay.** – In the inset of fig. 3, a modulation of the escape rate with period  $T = 2\pi$  can be seen, being exactly the period of the applied driving law (3). Specifically, for  $t \geq 10$ , where all particles starting on rotator orbits have already escaped,  $\dot{N}_C(t) \approx \text{const}$  during approximately 11/12 (empirically observed) of one period and subsequently  $\dot{N}_C(t) \neq 0$  during a time interval  $T/12$  only. From this behavior it is evident that the ellipse operates from a certain time on as a pulsed source of particles. These repeated intervals are centered around points  $t_m$  of maximal extension of the ellipse,  $t_m = (4m + 1)\pi/2$ ,  $m = 2, 3, 4, \dots$ . During the expansion period, dominantly vertical but also horizontal processes turn librators into rotators. The moving ellipse remains for a comparatively long time period in the vicinity of the extremal configuration at  $t_m$  and consequently the newly created rotators escape. Therefore, the dynamics is effectively probed during these short time intervals centered around  $t_m$ . During the contraction period, the librators are stabilized via vertical processes, consequently  $\dot{N}_C(t) \approx 0$  during 11/12 of a period  $T$ .

**Escape velocities.** – An astonishing feature of the escape of the HVE is displayed in fig. 4, where the distribution  $\rho(|\mathbf{v}|)$  of the escape velocities and the correlation of the escape time  $vs.$  the escape velocity (left inset) after a certain time is shown for  $C = 0.25$ . There is a strong asymmetry of  $\rho(|\mathbf{v}|)$  around the initial velocity  $|\mathbf{v}_0| = 100$ . Particularly, particles with  $t_{esc} > 10$ , being originally exclusively librators, possess escape velocities  $|\mathbf{v}| < |\mathbf{v}_0|$ . The origin of this behavior are vertical processes which turn librators into rotators that subsequently escape. The latter process however takes place during the expanding

phase of the moving ellipse which leads exclusively to a decrease *i.e.* loss of energy [25,26]. The escape velocities of particles starting on rotator orbits lie on a serpentine band, whose oscillations match the periodic oscillations of the ellipse. We therefore conclude that the breathing ellipse acts as a decelerator for particles in the HVE regime.

**Conclusion.** – In conclusion, we have explored the dynamics of a driven elliptical billiard with a focus on the scattering mechanisms and their impact on the escape rate. Observed phenomena such as the long-time algebraic decay which can be tuned by varying the driving amplitude, a pulsed escape rate and lowered velocities of the escaping particles are all caused by projectiles emanating from libration orbits. They can be understood by means of two fundamental scattering processes that turn librating into rotating orbits. Consequently, these effects can be further enhanced and manipulated by preparing suitable initial ensembles. For example, a high velocity ensemble initially consisting exclusively of librators restrains particles with an escape time  $t < 10$  which do not show a pulsed decay and whose escape velocities are symmetrically distributed around the initial velocity. It is foreseeable that the application of different driving laws and driving modes (*e.g.* area-preserving oscillations) as well as the preparation of suitable initial (thermal) ensembles will further advance equally our understanding as well as the phenomenology of driven nonequilibrium systems. Driven billiards serve as prototype systems in this respect.

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